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# CONSTRUCTION OF REGULAR NON-ATOMIC STRICTLY-POSITIVE MEASURES IN SECOND-COUNTABLE NON-ATOMIC LOCALLY COMPACT HAUSDORFF SPACES

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**Abstract.** This paper presents a constructive proof of the existence of a regular non-atomic strictly-positive measure on any second-countable non-atomic locally compact Hausdorff space. This construction involves a sequence of finitely-additive set functions defined recursively on an ascending sequence of rings of subsets with a set function limit that is extendable to a measure with the desired properties. Non-atomicity of the space provides a meticulous way to ensure that the set function limit is $\sigma$-additive.

## 1. Introduction

The construction of the Lebesgue measure on the real line is a well-known result from measure theory. The Lebesgue measure can be formed as a Haar measure on the locally compact group $\mathbb{R}$ (see Theorem 9.2.2 in [1]), as a positive linear functional on the space $\mathcal{C}_c(\mathbb{R})$ of compactly-supported continuous functions on $\mathbb{R}$ via the Riesz representation theorem (see Theorem 2.14 in [4]), or as the extension of a $\sigma$-additive set function on a ring of sets (see [2]).

However, the following construction of the Lebesgue measure will be adopted for this paper. We begin with a set function $\lambda$ defined on the ring

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of finite unions of precompact intervals with rational endpoints which returns the total length. From $\lambda$, an outer measure $\lambda^*$ is formed using open coverings from the ring of sets. This construction of the Lebesgue measure has several crucial steps to show:

1. $\lambda$ has finite additivity on the ring.
2. $\lambda^* \leq \lambda$ on open sets in the ring.
3. $\lambda^* \geq \lambda$ on compact sets in the ring.
4. $\lambda^*$ is zero on the boundaries of open sets in the ring (subsets of $\mathbb{Q}$).
5. $\lambda = \lambda^*$ on the ring.
6. Sets from the ring are $\lambda^*$-measurable.
7. Form the Lebesgue measure $\lambda^\dagger$ by restricting $\lambda^*$ to the $\sigma$-algebra of $\lambda^*$-measurable sets.

The Lebesgue measure on Borel sets is known to be regular, non-atomic and strictly-positive. On which topological spaces can measures with such properties be guaranteed?

This paper answers this question by applying a similar construction to 2nd-countable non-atomic locally compact Hausdorff spaces. The choice of such spaces is motivated by the steps involved in the Lebesgue measure construction. In fact, steps 2 and 3 are guaranteed with the usual choice of outer measure, and once step 4 is established, steps 5, 6, and 7 immediately follow.

Steps 1 and 4 prove to be challenging for these spaces. Indeed, the space is not necessarily a topological group, the ring of sets is only described topologically (or via a metric), and the finitely additive set function is not as clearly or easily defined as the set function $\lambda$.

This paper provides a careful construction of a finitely-additive set function via the limit of a sequence of set functions defined recursively on a growing sequence of rings. The sequence of rings of sets and the sequence of set functions must be coupled together meticulously so that the outer measure satisfies step 4, and I thank Piotr Mikusiński for the comments which gave me that insight. Once the measure is formed, it can be shown to be regular, non-atomic and strictly-positive.

This paper is based on part of my PhD dissertation.

## 2. Notation

Let $(X, \tau_X)$ be a topological space and let $A \subseteq X$. Then $A^\circ$, $\overline{A}$, $\partial A$ and $A^e$ denote the interior, closure, boundary and exterior of $A$ respectively. For sets $A$ and $B$, the expression $A - B$ denotes the relative set complement: that is, $a \in A - B$ if $a \in A$ and $a \notin B$. Furthermore, the expression $A \uplus B$ denotes the union of pairwise-disjoint sets $A$ and $B$. In this paper, " $\subset$ " will always mean proper subset, and " $\subseteq$ " will always mean subset or equality. An open set $U$ is called *regular* if $U$ is the interior of $\overline{U}$ and a closed set $F$ is called *regular* if $F$ is the closure of $F^\circ$.

## 3. Non-atomic topologies

DEFINITION 3.1. A topological space $(X, \tau_X)$, or a topology $\tau_X$ is *non-atomic* if for all $x \in X$ and for every open $U$ containing $x$, there exists an open set $V$ with $x \in V \subset U$.

The following properties of non-atomic topological spaces can be easily verified by definition.

PROPOSITION 3.2.
(a) *Let $(X, \tau_X)$ be a non-atomic topological space. Then every $x \in X$ and any open neighborhood $U$ of $x$ yield infinitely many open neighborhoods of $x$ which are proper subsets of $U$.*
(b) *If $(X, \tau_X)$ is non-atomic, then every non-empty open set in $\tau_X$ must be infinite in cardinality.*
(c) *If $(X, \tau_X)$ is a $T_1$-space with all non-empty open sets having infinite cardinality, then $(X, \tau_X)$ is non-atomic.*

EXAMPLE 3.3. Any non-trivial normed space over the real or complex field is non-atomic. In particular, $(\mathbb{R}, |\cdot|)$ is non-atomic since $x \in (a, b)$ implies $x \in (a + \varepsilon, b - \varepsilon) \subset (a, b)$ for any $\varepsilon$ satisfying $0 < \varepsilon < \min\{x - a, b - x\}$.

Non-atomic spaces have a nice property pertaining to topological bases.

PROPOSITION 3.4. *Let $(X, \tau_X)$ be a non-atomic topological space, let $\{U_i : i \in I\}$ be a topological basis for $\tau_X$, and let $J$ be a cofinite subset of $I$. Then $\{U_j : j \in J\}$ is also a topological basis for $\tau_X$.*

For non-atomic locally compact Hausdorff spaces, the next proposition allows for open sets to be "bored" by compact closures of open subsets. This will be utilized in Lemma 4.5 later.

PROPOSITION 3.5. *Let $(X, \tau_X)$ be a non-atomic locally compact Hausdorff space. Then every open neighborhood $x \in U$ admits an open neighborhood $x \in V$ with compact closure and with $\overline{V} \subset U$. Additionally, $V$ can be chosen* *to be regular such that $V$ and $U - \overline{V}$ are disjoint non-empty open subsets of $U$ with $U = V \uplus \partial V \uplus (U - \overline{V})$.*

## 4. Constructing the desired measure

Before discussing the main result, some definitions are in order.

DEFINITION 4.1. Let $X$ be a nonempty set, and let $\Sigma$ be a $\sigma$-algebra of subsets of $X$. Then:

(a) $\mu\colon \Sigma \to [0, \infty]$ is a *measure* if $\mu$ is $\sigma$-additive on $\Sigma$ with $\mu(\varnothing) = 0$.
(b) $\nu\colon 2^X \to [0, \infty]$ is an *outer measure* if $\nu$ is $\sigma$-subadditive and monotonic with $\nu(\varnothing) = 0$.

THEOREM 4.2. *Let $(X, \Sigma_X)$ be a measurable space generated by a second-countable locally compact Hausdorff non-atomic space $(X, \tau_X)$. Then there exists a finite regular non-atomic strictly-positive measure on $(X, \Sigma_X)$.*

PROOF. First, we need a candidate for the sequence of rings of sets. Since $(X, \tau_X)$ is a second-countable locally compact regular space, there exists a countable basis of nonempty regular open sets in $X$ with compact closure. Let $(V_i) = (V_i)_{i=1}^{\infty}$ be one such basis expressed as a sequence: such sequences will be called *regular precompact basis sequences*. While $(V_i)_{i=1}^{\infty}$ is arbitrarily chosen, the remainder of the proof will keep $(V_i)_{i=1}^{\infty}$ fixed.

Regularity of the basis sequence grants several useful properties: $V_i$ being regular implies that $\overline{V_i}$ and $V_i^e$ are also regular; finite intersections of regular open sets are regular open sets; $A$ and $B$ being nonempty regular open sets with $A$ intersecting $\partial B$ implies that $A \cap B$ and $A \cap B^e$ are nonempty open sets.

The following lemma produces a sequence of rings of sets, denoted by $(\mathcal{D}_k)$, and a limit ring of subsets $\mathcal{D}$ which contains basis sets $V_i$ for $i \in \mathbb{N}$ and generates all Borel sets from $(X, \tau_X)$.

LEMMA 4.3. *Let $(X, \Sigma_X)$ be a second-countable non-atomic locally compact Hausdorff space with regular precompact basis sequence $(V_i)_{i=1}^{\infty}$. For each $k \in \mathbb{N}$, define*

$$\mathcal{A}_k := \{\cap_{i=1}^{k} R_i \mid R_i = V_i \textit{ or } R_i = V_i^e \textit{ for all } 1 \leq i \leq k\} - \{\varnothing, \cap_{i=1}^{k} V_i^e\},$$

$$\mathcal{B}_k := \{\uplus_{j=1}^{n} S_j \mid n \in \mathbb{N}_0, S_j \in \mathcal{A}_k\},$$

$$\mathcal{C}_k := \{C \mid C \subseteq \cup_{i=1}^{k} \partial V_i\},$$

$$\mathcal{D}_k := \{B \uplus C \mid B \in \mathcal{B}_k,\ C \in \mathcal{C}_k\}.$$

*Then: $\mathcal{A}_k$ consists of pairwise disjoint nonempty regular open sets; every $A \in \mathcal{A}_{k+1}$ with $A \subseteq \cup_{i=1}^k V_i$ has a unique $B \in \mathcal{A}_k$ with $A \subseteq B$; if $S, T \in \mathcal{B}_k$, then $S \cup T, S \cap T$, and $S \cap T^e \in \mathcal{B}_k$; $(\mathcal{D}_k)$ is an ascending sequence of rings of sets; $\mathcal{D} := \cup_{k=1}^\infty \mathcal{D}_k$ is a ring of sets that generates the $\sigma$-algebra $\Sigma_X$ and is insensitive to permutations on $(V_i)_{i=1}^\infty$.*

Proof. If $\varnothing \neq A = \cap_{i=1}^{k+1} R_i \in \mathcal{A}_{k+1}$ with $A \subseteq \cup_{i=1}^k V_i$, then $B := \cap_{i=1}^k R_i$ is the unique set in $\mathcal{A}_k$ containing $A$.

Let $S$ and $T$ be in $\mathcal{B}_k$. This means that $S := \uplus_{i=1}^m S_i$ and $T := \uplus_{j=1}^n T_j$, where $S_i$ and $T_j$ are in $\mathcal{A}_k$ for all $1 \leq i \leq m$ and for all $1 \leq j \leq n$. It is fairly straight-forward to show that $\mathcal{B}_k$ is closed under finite unions and intersections. We will show that $S \cap T^e \in \mathcal{B}_k$. If $m = 0$, then $S = \varnothing$, implying that $S \cap T^e = \varnothing \in \mathcal{B}_k$. If $n = 0$, then $T^e = X$, implying that $S \cap T^e = S \in \mathcal{B}_k$. If $m = n = 1$, then $S, T \in \mathcal{A}_k$. Since sets in $\mathcal{A}_k$ are contained in each other's exteriors, the open set $\cup\{A \in \mathcal{A}_k - \{T\}\} \in \mathcal{B}_k$ is a subset of $T^e$. The remaining possible points in $T^e$ are either in $\cup_{i=1}^k \partial V_i$ or in $\cap_{i=1}^k V_i^e$, which are both disjoint with $S$. Therefore, $S \cap T^e = S \cap (\cup\{A \in \mathcal{A}_k - \{T\}\}) \in \mathcal{B}_k$. This elementary case, in tandem with closure of $\mathcal{B}_k$ under finite unions and intersections, proves the general case since

$$S \cap T^e = (\uplus_{i=1}^m S_i) \cap (\uplus_{j=1}^n T_j)^e = (\uplus_{i=1}^m S_i) \cap (\cap_{j=1}^n T_j^e) = \uplus_{i=1}^m \cap_{j=1}^n (S_i \cap T_j^e).$$

It is clear that $\mathcal{C}_k$ is closed under finite unions.

If $D_1 = B_1 \uplus C_1$ and $D_2 = B_2 \uplus C_2$, where $B_1, B_2 \in \mathcal{B}_k$ and $C_1, C_2 \in \mathcal{C}_k$, then it follows that $D_1 \cup D_2 = (B_1 \cup B_2) \uplus (C_1 \cup C_2) \in \mathcal{D}_k$ via closure of $\mathcal{B}_k$ and $\mathcal{C}_k$ under finite unions. Furthermore,

$$\begin{aligned} D_1 - D_2 &= (B_1 \uplus C_1) \cap (B_2 \uplus C_2)^c = (B_1 \uplus C_1) \cap (B_2^c \cap C_2^c) \\ &= (B_1 \cap B_2^c \cap C_2^c) \uplus (C_1 \cap B_2^c \cap C_2^c) \\ &= (B_1 \cap B_2^c) \uplus (C_1 \cap B_2^c \cap C_2^c) \\ &= (B_1 \cap B_2^e) \uplus ((B_1 \cap \partial B_2) \cup (C_1 \cap B_2^c \cap C_2^c)) \in \mathcal{D}_k, \end{aligned}$$

since $B_1 \cap B_2^e \in \mathcal{B}_k$ and the second set is in $\mathcal{C}_k$. Therefore, $\mathcal{D}_k$ is a ring of sets.

If $D_k := B_k \uplus C_k \in \mathcal{D}_k$ with $B_k \in \mathcal{B}_k$ and $C_k \in \mathcal{C}_k$, then notice that $(B_k \cap V_{k+1}) \cup (B_k \cap V_{k+1}^e) \in \mathcal{B}_{k+1}$ and that $C_k \cup (B_k \cap \partial V_{k+1}) \in \mathcal{C}_{k+1}$ ensure that $D_k \in \mathcal{D}_{k+1}$. Therefore, $\mathcal{D}_k \subseteq \mathcal{D}_{k+1}$ for all natural $k$. It follows that $\mathcal{D} = \cup_{k=1}^\infty \mathcal{D}_k$ is also a ring of sets. Since $\mathcal{D}$ contains all basis sets and since $\tau_X$ is second-countable, $\mathcal{D}$ generates $\Sigma_X$.

Finally, let $\pi \colon \mathbb{N} \to \mathbb{N}$ be any permutation and let $\mathcal{A}'_k, \mathcal{B}'_k, \mathcal{C}'_k, \mathcal{D}'_k, \mathcal{D}'$ denote the collections discussed earlier for the permutated sequence $(V_{\pi(i)})_{i=1}^\infty$.

Let $K \in \mathcal{D}$. Then $K \in \mathcal{D}_n$ for some natural $n$. Let $m$ be the smallest natural number such that $\{V_{\pi(1)}, V_{\pi(2)}, \dots, V_{\pi(m)}\} \supseteq \{V_1, V_2, \dots, V_n\}$. It follows that $K \in \mathcal{D}'_m \subset \mathcal{D}'$. A similar argument suffices to show that $\mathcal{D}' \subseteq \mathcal{D}$. □

Second, we need a finitely additive set function defined on $\mathcal{D}$. Given a basis sequence $(V_i)_{i=1}^{\infty}$, we need to intuitively develop a sequence of set functions $\mu_m \colon \mathcal{A}_m \to [0,1]$ for $m \in \mathbb{N}$. The crucial idea is that when an open set $A \in \mathcal{A}_m$ intersects $\partial V_{m+1}$, regularity properties of $A$ and $V_{m+1}$ ensure that $A$ is fragmented by $\partial V_{m+1}$ into two nonempty open sets $A' := A \cap V_{m+1}$ and $A'' := A \cap V_{m+1}^e$ in $\mathcal{A}_{m+1}$. Hence, we can evenly divide the size $\mu_m(A)$ in half and distribute each to $A'$ and $A''$, meaning we insist that $\mu_{m+1}(A') = \mu_{m+1}(A'') = \frac{1}{2}\mu_m(A)$. Of course, we also need to insist that the next set function equals the previous set function for open sets in $\mathcal{A}_m$ that persist in $\mathcal{A}_{m+1}$. Finally, sets in $\mathcal{A}_{m+1}$ that are disjoint with all sets from $\mathcal{A}_m$ represent new regions that can be assigned any size, and we shall assign sizes to these new regions in a way to cause all set functions to have maximum size output less than 1. These components correspond to lines 3–5 in the construction of $(\mu_m)_{m=1}^{\infty}$ in Lemma 4.4. Figure 1 illustrates how the set functions on $(\mathcal{A}_k)$ behave.

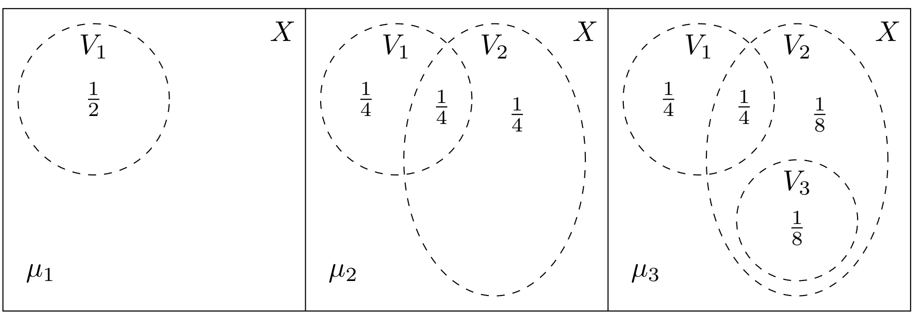


Figure 1. Behavior of the set functions $(\mu_k)$ on $(\mathcal{A}_k)$

Once $(\mu_m)_{m=1}^{\infty}$ is constructed, we can easily develop a sequence of finitely additive set functions $(\kappa_n)_{n=1}^{\infty}$ such that $\kappa_{n+1}$ is an extension of $\kappa_n$ for all $n \in \mathbb{N}$, then define a finitely additive set function $\kappa$ as the overall extension of $(\kappa_n)_{n=1}^{\infty}$ to $\mathcal{D}$.

Lemma 4.4. *Let $(X, \Sigma_X)$ be a second-countable non-atomic locally compact Hausdorff space with regular precompact basis sequence $(V_i)_{i=1}^{\infty}$, where $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, $\mathcal{D}_k$ and $\mathcal{D}$ are previously developed via Lemma 4.3 for all $k \in \mathbb{N}$. Recursively define a sequence of functions $\mu_m \colon \mathcal{A}_m \cup \{\varnothing\} \to [0,1]$ as follows:*

(1) $\mu_1(\varnothing) := 0;$

(2) $\mu_1(V_1) := 1/2;$

(3) $\mu_m(A) := 1/2 \cdot \mu_{m-1}(B)$ *if* $\varnothing \neq A \subset B \in \mathcal{A}_{m-1}$;

(4) $\mu_m(A) := \mu_{m-1}(A)$ *if* $A \in \mathcal{A}_{m-1} \cup \{\emptyset\}$;

(5) $\mu_m(V_m - \cup_{i=1}^{m-1}\overline{V_i}) := 1/2^m$ *if* $V_m - \cup_{i=1}^{m-1}\overline{V_i} \neq \varnothing$.

*Then there exists a finitely additive set function* $\kappa\colon \mathcal{D} \to [0,1]$ *such that* $\kappa(A) = \mu_m(A)$ *when* $A \in \mathcal{A}_m$.

Proof. Define a sequence of functions $\nu_k\colon \mathcal{B}_k \to [0,1]$ via $\nu_k(\uplus_{j=1}^n S_j) := \sum_{j=1}^n \mu_k(S_j)$, where $S_j \in \mathcal{A}_k$ for $1 \leq j \leq n$. Then define a sequence of functions $\kappa_n\colon \mathcal{D}_n \to [0,1]$ via $\kappa_n(S \cup T) = \nu_n(S)$, where $S \in \mathcal{B}_n$ and $T \in \mathcal{C}_n$.

Finite additivity of $(\nu_k)$ and $(\kappa_n)$ is easy to verify. It follows by the definitions of $(\nu_k)$ and $(\kappa_n)$ and by (3) and (4) that any set in $\mathcal{D}_N$ will have the same value under all functions $\kappa_n$ with $n \geq N$. Therefore, the set function $\kappa\colon \mathcal{D} \to [0,1]$ such that $\kappa(T) = \kappa_n(T)$ when $T \in \mathcal{D}_n$ is well defined. Finite additivity of $\kappa$ follows from the finite additivity of $(\kappa_n)$. □

Third, we need to show that the set-function $\kappa^*\colon 2^X \to [0,\infty]$ defined via

$$\kappa^*(A) := \inf\left\{\sum_{i=1}^{\infty} \kappa(A_i) : A \subseteq \cup_{i=1}^{\infty} A_i, \text{ and } A_i \in \mathcal{D} \text{ is open for all } i \in \mathbb{N}\right\},$$

is an outer measure and that $\kappa^*$ is zero on all subsets of $\cup_{k=1}^{\infty}\partial V_k$. One can apply Theorem 2.11.3 from [3] or otherwise verify that $\kappa^*$ is an outer measure. However, notice that the set function $\kappa$ (hence, the outer measure $\kappa^*$) we develop *depends on the order of the basis sequence* $(V_i)$. This is important, because without a careful choice made for the ordering of these basis sets, Step 4 in the outline may be difficult or impossible. What kind of sequence do we select? The next lemma serves two purposes: to provide the crucial properties needed to obtain step 4 in the introduction, and to help verify non-atomicity of the measure formed at the end.

Lemma 4.5. *Let* $(X, \tau_X)$ *be a second-countable non-atomic locally compact Hausdorff space with regular precompact basis sequence* $(V_i)_{i=1}^{\infty}$, *and let* $(\mathcal{A}_k)_{k=1}^{\infty}$ *be the sequence of collections of sets formed in Lemma* 4.3 *with respect to* $(V_i)_{i=1}^{\infty}$. *There exists a permutation* $\pi\colon \mathbb{N} \to \mathbb{N}$ *such that the collection* $(\mathcal{A}'_n)_{n=1}^{\infty}$ *from Lemma* 4.3 *and the set function* $\kappa$ *from Lemma* 4.4 *formed with respect to* $(V_{\pi(i)})_{i=1}^{\infty}$ *have the following properties:*

(a) $\kappa^*$ *is zero on every subset of* $\cup_{i=1}^{\infty}\partial V_i$.

(b) $\max\{\kappa(A) : A \in \mathcal{A}'_n\} \to 0$ *as* $n \to \infty$.

Proof. Note that the boundaries $(\partial V_k)_{k=1}^{\infty}$ are compact, so we can find covers of them using other basis open sets. The utility of the non-atomic locally compact Hausdorff space is that we can purposefully use the closures of other basis elements to bore "closed holes" into a given cover of a given boundary $\partial V_i$, then find a *better* cover of the same boundary that does not intersect these holes. The holes should cause the new cover to have outer measure at most half of the previous cover's outer measure when $\kappa$ is formed. This process is repeated for all $\partial V_i$ countably many times so that the outer measure for each must be zero. Consequently, the implementation below will also satisfy (b).

To form a permutation with the desired properties, we partition $\mathbb{N}$ into three double-indexed families $\{F_{i,j} : i,j \in \mathbb{N}\}$, $\{G_{i,j} : i,j \in \mathbb{N}\}$ and $\{H_{i,j} : i,j \in \mathbb{N}\}$ of finite sets with $g(i,j) := \max G_{i,j}$ for all $i,j \in \mathbb{N}$ and $f(i,j) := \max F_{i,j}$ for all $i \geq 1, j \geq 2$ such that the following holds:

(i) $\mathcal{G}_{i,j} := \{V_k \mid k \in G_{i,j}\}$ covers $\partial V_i$ for $i,j \in \mathbb{N}$; that is, $\partial V_i \subset \cup \mathcal{G}_{i,j}$.

(ii) $\mathcal{F}_{i,j} := \{\overline{V_k} \mid k \in F_{i,j}\}$ satisfies $\cup\mathcal{G}_{i,j} \subseteq \cup\mathcal{G}_{i,j-1} - \cup\mathcal{F}_{i,j}$ for all $i \geq 1$ and $j \geq 2$, and $F_{i,1} := \varnothing$ (hence $\mathcal{F}_{i,1} := \varnothing$) for all $i \geq 1$.

(iii) For all $i \geq 1, j \geq 2$, and for each $U \in \mathcal{A}_{g(i+1,j-1)}$, there exists a unique $K \in \mathcal{F}_{i,j}$ such that $K \subset U$.

(iv) $\max G_{i,j} < \min G_{i',j'}$ when $i+j < i'+j'$ or when $i+j = i'+j'$ and $j < j'$. The same is true for $\{F_{i,j}\}_{i,j=1}^{\infty}$ and $\{H_{i,j}\}_{i,j=1}^{\infty}$ whenever the compared sets are both nonempty.

(v) $\max G_{i+1,j-1} < \min F_{i,j} \leq \max F_{i,j} < \min G_{i,j}$ for all $i \geq 1, j \geq 2$.

(vi) $H_{i,j}$ are remainder sets; that is,
$H_{1,1} = \{1,\ldots,g(1,1)\} - G_{1,1}$,
$H_{i,1} = \{g(1,i-1)+1,\ldots,g(i,1)\} - G_{i,1}$ for all $i \geq 2$, and
$H_{i,j} = \{g(i+1,j-1)+1,\ldots,g(i,j)\} - (G_{i,j} \cup F_{i,j})$ for all $i \geq 1, j \geq 2$.

Next, we define the permutation $\pi\colon \mathbb{N} \to \mathbb{N}$, and hence the preferred sequence $(W_i)_{i=1}^{\infty} := (V_{\pi(i)})_{i=1}^{\infty}$ by making the arrangement $R_{i,j} := (F_{i,j}, G_{i,j}, H_{i,j})$ for each $i,j \geq 1$ and then stating that $R_{i,j} \leq R_{i',j'}$ exactly when $i+j < i'+j'$ or when $i+j = i'+j'$ and $j < j'$; that is, make the arrangement $R_{1,1}, R_{2,1}, R_{1,2}, R_{3,1}, R_{2,2}, R_{1,3}, \ldots$. The details are provided below.

$\{V_i\}_{i=1}^{\infty}$ covers the compact set $\partial V_1$, so there exists some finite subcover. Let $\mathcal{G}_{1,1}$ denote one possible choice, indexed by $G_{1,1} \subset \mathbb{N}$ with maximum index $g(1,1)$. Let $H_{1,1} := \{1,2,\ldots,g(1,1)\} - G_{1,1}$.

Now $\{V_i : i > g(1,1)\}$ forms a basis by Proposition 3.4, so perform the same procedure on $\partial V_2$ using the new basis, obtaining the collection $\mathcal{G}_{2,1}$, indexed by $G_{2,1}$ and number $g(2,1)$. To gather all remaining indices between $g(1,1)+1$ and $g(2,1)$, let $H_{2,1} := \{g(1,1)+1, g(1,1)+2, \ldots, g(2,1)\} - G_{2,1}$. Next to construct are $\mathcal{G}_{1,2}$, $G_{1,2}$, and $g(1,2)$. For each $A \in \mathcal{A}_{g(2,1)}$, there exists a basis set $V_{k(A)}$ such that $\overline{V_{k(A)}} \subset A$ and $k(A) > g(2,1)$. Doing this for every $A \in \mathcal{A}_{g(2,1)}$, let $F_{1,2} := \{k(A) : A \in \mathcal{A}_{g(2,1)}\}$ and let $\mathcal{F}_{1,2} :=$

$\{\overline{V_k} : k \in F_{1,2}\}$. The open set $\cup\mathcal{G}_{1,1} - \cup\mathcal{F}_{1,2}$ is a union of sets from the basis $\{V_i : i > f(1,2)\}$, so choose from that union a finite cover $\mathcal{G}_{1,2}$ of $\partial V_1$, indexed by $G_{1,2}$ with maximum index $g(1,2)$. Now denote $H_{1,2} := \{g(2,1)+1, g(2,1)+2, \ldots, g(1,2)\} - (G_{1,2} \cup F_{1,2})$.

Inductively, with $\mathcal{G}_{i,j}, G_{i,j}, g(i,j), F_{i,j}$, and $H_{i,j}$ determined when $i+j \leq m$ for some $m \geq 2$, construct $\mathcal{G}_{m,1}$, $G_{m,1}$, $g(m,1)$ and $H_{m,1}$ based on the $(1, m-1)$ step similar to how $\mathcal{G}_{2,1}$, $G_{2,1}$, $g(2,1)$ and $H_{2,1}$ were constructed based on the $(1,1)$ step. Then construct $\mathcal{F}_{m-1,2}$, $F_{m-1,2}$, $\mathcal{G}_{m-1,2}$, $G_{m-1,2}$, $g(m-1,2)$ and $H_{m-1,2}$ based on the $(m,1)$ step similar to how $\mathcal{F}_{1,2}$, $F_{1,2}$, $\mathcal{G}_{1,2}$, $G_{1,2}$, $g(1,2)$ and $H_{1,2}$ were constructed based on the $(2,1)$ step. Repeat this again by constructing $\mathcal{F}_{m-2,3}$, $F_{m-2,3}$, $\mathcal{G}_{m-2,3}$, $G_{m-2,3}$, $g(m-2,3)$ and $H_{m-2,3}$ based on the $(m-1,2)$ step. Continue in this manner until $\mathcal{F}_{1,m}, F_{1,m}, \mathcal{G}_{1,m}, G_{1,m}, g(1,m)$ and $H_{1,m}$ are constructed. Now all appropriate numbers and collections have been found for when $i + j = m + 1$. Complete this process via induction.

Now let $\kappa$ be the finitely-additive set function created in Lemma 4.4 with respect to $(V_{\pi(i)})$. It remains to verify the desired properties. For part (a), proving that $\kappa^*(\cup_{i=1}^{\infty}\partial V_i) = 0$ is sufficient by the monotonicity of $\kappa^*$. We show by induction that for any $\partial V_i$, the sequence of covers $\{\mathcal{G}_{i,j}\}_{j=1}^{\infty}$ satisfies (for all natural $j$) the inequality

$$\kappa(\cup\mathcal{G}_{i,j}) \leq \kappa(\cup\mathcal{G}_{i,1}) \left(\frac{1}{2}\right)^{j-1}.$$

The inequality is obvious when $j = 1$. Assume that the inequality is true for some natural $j$. Since each open subset of $\cup\mathcal{G}_{i,j}$ from $\mathcal{A}_{g(i+1,j)}$ is bored by a compact set from $\mathcal{F}_{i,j+1}$, Lemma 4.4(3) ensures that

$$\kappa(\cup\mathcal{G}_{i,j+1}) \leq \kappa(\cup\mathcal{G}_{i,j} - \cup\mathcal{F}_{i,j+1}) = \frac{1}{2}\kappa(\cup\mathcal{G}_{i,j}) \leq \frac{1}{2}\kappa(\cup\mathcal{G}_{i,1}) \left(\frac{1}{2}\right)^{j-1}.$$

Therefore, it follows that $\kappa^*(\partial V_i) \leq \kappa(\cup\mathcal{G}_{i,j}) \leq \kappa(\cup\mathcal{G}_{i,1}) \left(\frac{1}{2}\right)^{j-1}$ for all natural $j$, which implies that $\kappa^*(\partial V_i) = 0$. Since $i \in \mathbb{N}$ was arbitrary, it follows that $\cup_{i=1}^{\infty}\partial V_i$ has outer measure zero, showing (a).

Let $m \in \mathbb{N}$. The basis sets $\{V_k : k \in F_{1,m}\}$ fragment each of the sets in $\mathcal{A}'_{g(2,m-1)}$, resulting in

$$\max\{\kappa(A) : A \in \mathcal{A}'_{g(1,m)}\} \leq \frac{1}{2} \max\{\kappa(A) : A \in \mathcal{A}'_{g(2,m-1)}\}.$$

Since $\max\{\kappa(A) : A \in \mathcal{A}'_n\}$ is a non-increasing function of $n$ by the construction of $\kappa$ in Lemma 4.4, it follows from above that $\max\{\kappa(A) : A \in \mathcal{A}'_n\} \to 0$ as $n \to \infty$. □

Let $(\mathcal{A}'_k)$ and $(\mathcal{D}'_k)$ denote the collections of sets formed in Lemma 4.3 when applied to the permuted sequence $(V_{\pi(i)})$ (Recall that $\mathcal{D}' = \mathcal{D}$). Now we construct a measure on $\mathcal{D}$ using the steps from the introduction.

Step 1: Construct the set function $\kappa\colon \mathcal{D} \to [0,1]$ with respect to $(V_{\pi(i)})$ via Lemmas 4.4 and 4.5. Consequently, $\kappa$ is finitely additive.

Step 2: To show that $\kappa^* \leq \kappa$ on open sets from $\mathcal{D}$, let $U \in \mathcal{D}$ be open. Then $U \subseteq U \cup \varnothing \cup \varnothing \cup \dots$, so $\kappa^*(U) \leq \kappa(U) + \kappa(\varnothing) + \kappa(\varnothing) + \cdots = \kappa(U)$.

Step 3: The following argument shows that $\kappa \leq \kappa^*$ on compact sets from $\mathcal{D}$. Let $C \in \mathcal{D}$ be compact, and let $\varepsilon > 0$. Choose some sequence $(A_i)_{i=1}^\infty$ of open sets from $\mathcal{D}$ with $C \subseteq \cup_{i=1}^\infty A_i$ and such that $\sum_{i=1}^\infty \kappa(A_i) \leq \kappa^*(C) + \varepsilon$. Then there exists some finite subcover, meaning there exists $n \in \mathbb{N}$ with $C \subseteq \cup_{i=1}^n A_i$ and there exists some $N \in \mathbb{N}$ with $C, A_1, \dots, A_n \in \mathcal{D}'_N$. Therefore $\kappa(C) = \kappa_N(C) \leq \sum_{i=1}^n \kappa_N(A_i) = \sum_{i=1}^n \kappa(A_i) \leq \sum_{i=1}^\infty \kappa(A_i) \leq \kappa^*(C) + \varepsilon$. With $\varepsilon$ arbitrary, it follows that $\kappa \leq \kappa^*$ on compact sets from $\mathcal{D}$.

Step 4: It has been shown in Lemma 4.5 that $\kappa^*$ is zero on subsets of $\cup_{k=1}^\infty \partial V_k$.

Step 5: To show that $\kappa^* = \kappa$ on $\mathcal{D}$, let $A \in \mathcal{D}$. Then $A = B \uplus C$, where $B \in \cup_{k=1}^\infty \mathcal{B}_k \subseteq \mathcal{D}$ is open and $C \subseteq C \cup \partial A \subseteq \cup_{k=1}^\infty \partial V_k$. Note that $\overline{A} \in \mathcal{D}$ is compact. Then

$$\kappa(A) \leq \kappa\left(\overline{A}\right) \leq \kappa^*\left(\overline{A}\right) = \kappa^*(A) = \kappa^*(B) \leq \kappa(B) \leq \kappa(A).$$

Therefore, $\kappa^* = \kappa$ on $\mathcal{D}$.

Step 6: Since $\kappa^*$-measurable sets form a $\sigma$-algebra, and since the $\sigma$-algebra of Borel subsets of $X$ can be generated by the open sets from $\mathcal{D}$, it suffices to prove that the open sets from $\mathcal{D}$ are $\kappa^*$-measurable. Let $U \in \mathcal{D}$ be open, let $\varepsilon > 0$ and let $A \subseteq X$ with $\kappa^*(A) < \infty$. Then there exists an open covering $(A_i)_{i=1}^\infty$ from $\mathcal{D}$ of $A$ such that $\sum_{i=1}^\infty \kappa(A_i) \leq \kappa^*(A) + \varepsilon$. Then $(A_i \cap U)_{i=1}^\infty$ and $(A_i \cap U^e)_{i=1}^\infty$ are open coverings from $\mathcal{D}$ of $A \cap U$ and $A \cap U^e$ respectively. Also, take an open covering $(B_i)_{i=1}^\infty$ of $A \cap \partial U$ such that $\sum_{i=1}^\infty \kappa(B_i) \leq \varepsilon$. Thus, finite additivity of $\kappa$ and $\kappa(A \cap \partial U) = 0$ imply that

$$\begin{aligned}\kappa^*(A \cap U) + \kappa^*(A \cap U^c) &\leq \sum_{i=1}^\infty \kappa(A_i \cap U) + \sum_{i=1}^\infty (\kappa(B_i) + \kappa(A_i \cap U^e)) \\ &\leq \varepsilon + \sum_{i=1}^\infty \kappa(A_i) \leq \kappa^*(A) + 2\varepsilon.\end{aligned}$$

Since $\varepsilon$ was arbitrary, it follows that $\kappa^*(A \cap U) + \kappa^*(A \cap U^c) \leq \kappa^*(A)$. Since $\kappa^*$ is subadditive, we have that $\kappa^*(A \cap U) + \kappa^*(A \cap U^c) = \kappa^*(A)$. Thus, $U$ is $\kappa^*$-measurable.

Step 7: Now we apply a fundamental theorem (see Theorem 1.3.6 from [1] or Theorem 2.11.2 from [3]). The restriction of $\kappa^*$ to the $\sigma$-algebra $\Sigma_{\kappa^*}$ of all

$\kappa^*$-measurable sets is a measure. Since $\Sigma_X \subseteq \Sigma_{\kappa^*}$, restricting $\kappa^*$ to $\Sigma_X$ also forms a measure which we denote by $\kappa^\dagger$. It follows that $\kappa^\dagger$ is strictly-positive since each set in the basis sequence $(V_i)$ was assigned a positive measure by $\kappa$ on $\mathcal{D}$. Since $X$ is a Polish space, we automatically have that $\kappa^\dagger$ is regular (see Theorem 8.1.12 in [1]).

To show that $\kappa^\dagger$ is non-atomic, we apply the following lemma.

LEMMA 4.6. *Let $\mu$ be a finite measure on $(X, \Sigma)$. Then $\mu$ is non-atomic if and only if for every $\varepsilon > 0$, there exists a finite partition of $X$ into measurable sets with each set having $\mu$-measure less than $\varepsilon$.*

Using Lemma 4.6, it suffices to show that $X$ has a finite partition of measurable sets, each with $\kappa^\dagger$ measure arbitrarily small. For each $m \in \mathbb{N}$, consider the finite partition $\mathcal{A}'_m \cup \left\{\cup_{i=1}^m \partial V_{\pi(i)}, X - \cup_{i=1}^m \overline{V_{\pi(i)}}\right\}$ of $X$. According to Lemma 4.5(b),

$$\max\{\kappa^\dagger(A)\colon A \in \mathcal{A}'_m\} = \max\{\kappa(A)\colon A \in \mathcal{A}'_m\} \to 0$$

as $m \to \infty$. By Lemma 4.5(a), $\kappa^\dagger(\cup_{i=1}^m \partial V_{\pi(i)}) = 0$ for all $m \in \mathbb{N}$. Since $X - \cup_{i=1}^m \overline{V_{\pi(i)}}$ decreases to $\varnothing$ as $m \to \infty$, and since $\kappa^\dagger$ is a finite measure, it follows that

$$\kappa^\dagger\left(X - \cup_{i=1}^m \overline{V_{\pi(i)}}\right) \to 0$$

as $m \to \infty$. Thus, the largest $\kappa^\dagger$-measure among sets from the finite collection decreases to zero as $m \to \infty$, implying that $\kappa^\dagger$ is non-atomic.

At last, $\kappa^\dagger$ is a measure on $(X, \Sigma_X)$ with the sought properties. □

## References

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