sciendo
Annales Mathematicae Silesianae **36** (2022), no. 1, 26–29
DOI: 10.2478/amsil-2021-0013

A SEPARATELY CONTINUOUS FUNCTION NOT SOMEWHAT CONTINUOUS

Wojciech Bielas

Abstract. We construct a separately continuous function $f\colon \mathbb{Q}\times\mathbb{Q}\to[0,1]$ and a dense subset $D\subseteq\mathbb{Q}\times\mathbb{Q}$ such that $f[D]$ is not dense in $f[\mathbb{Q}\times\mathbb{Q}]$, in other words, f is separately continuous and not somewhat (feebly) continuous.

1. Introduction

The notion of a feebly continuous function was introduced and examined by Z. Frolík ([1]). Some authors say *somewhat continuous function* instead of feebly continuous function, see for example [2]. A function f is *somewhat continuous*, if the preimage $f^{-1}[U]$ of any non-empty open subset U has non-empty interior. By [2, Theorem 3], any surjection $f\colon P\to Q$ is somewhat continuous if and only if the image $f[M]$ of a dense subset $M\subseteq P$ is dense in Q.

If X is a Baire space, Y is of weight ω and Z is a metric space, then any separately continuous function $f\colon X\times Y\to Z$ is somewhat continuous, see [3]. Recall that a function $f\colon X\times Y\to Z$ is *separately continuous*, if functions $f(x,\cdot)$ and $f(\cdot,y)$ are continuous for each $(x,y)\in X\times Y$. T. Neubrunn observed that the assumption of separate continuity can be weakened, see [4, Theorem 2 and Theorem 3]. Additionally, he presented counterexamples,

Received: 08.07.2021. Accepted: 26.09.2021. Published online: 11.10.2021.
(2020) Mathematics Subject Classification: 54C08, 54C30, 54G20.
Key words and phrases: separate continuity, somewhat continuous function.

which witness that both conditions: X being Baire and Y being second countable, are necessary, see [4, Example 3 and Example 4]. In [4, Example 3], it is defined a function $f\colon (0,1)\times Y \to \{0,1\}$ such that all functions $f(\cdot,y)$ are quasicontinuous and all functions $f(x,\cdot)$ are somewhat continuous, but f is not somewhat continuous. But [4, Example 4] shows that there is a function $f\colon (\mathbb{Q}\cap(0,1))\times(1,\infty)\to\{0,1\}$ such that all sections $f(\cdot,y)$ are quasicontinuous and all sections $f(x,\cdot)$ are somewhat continuous, but f is not somewhat continuous.

We construct a separately continuous function $f\colon \mathbb{Q}\times\mathbb{Q}\to[0,1]$ and a dense subset $D\subseteq\mathbb{Q}\times\mathbb{Q}$ such that the image $f[\mathbb{Q}\times\mathbb{Q}]\subseteq[0,1]$ is dense and $f[D]$ is a singleton, hence f is not somewhat continuous. The construction relies on the following observation. If $(x_0,y_0),\ldots,(x_n,y_n)$ are such that $x_i\neq x_j$ and $y_i\neq y_j$ for $i<j\leqslant n$, then the intersection

$$(\{x_n\}\times\mathbb{Q})\cup(\mathbb{Q}\times\{y_n\})\cap\bigcup_{i<n}(\{x_i\}\times\mathbb{Q})\cup(\mathbb{Q}\times\{y_i\})$$

is finite. Thus, having defined a continuous function on the set $\bigcup_{i<n}(\{x_i\}\times\mathbb{Q})\cup(\mathbb{Q}\times\{y_i\})$, we can easily extend it to a continuous function on the set $\bigcup_{i\leqslant n}(\{x_i\}\times\mathbb{Q})\cup(\mathbb{Q}\times\{y_i\})$. It remains to observe that there exists a dense subset $\{(x_n,y_n)\colon n\in\mathbb{N}\}\subseteq\mathbb{Q}\times\mathbb{Q}$ such that

$$\mathbb{Q}\times\mathbb{Q}=\bigcup_{n\in\mathbb{N}}((\{x_n\}\times\mathbb{Q})\cup(\mathbb{Q}\times\{y_n\}))$$

with $x_i\neq x_j$ and $y_i\neq y_j$ for $i<j$.

2. The construction

We proceed to establish the following lemma.

LEMMA 1. *If $(x_0,y_0),\ldots,(x_n,y_n)$ in $\mathbb{Q}\times\mathbb{Q}$ are different points and $\xi_i,\eta_i\in[0,1)$ for $0\leqslant i<n$, then there exists a continuous function $f\colon(\{x_n\}\times\mathbb{Q})\cup(\mathbb{Q}\times\{y_n\})\to[0,1]$ such that*

- *$f(x,y)=1$ iff $(x,y)=(x_n,y_n)$;*
- *$f(x_n,y_i)=\xi_i$ and $f(x_i,y_n)=\eta_i$, for any $0\leqslant i<n$;*
- $\operatorname{cl} f[\{x_n\}\times\mathbb{Q}]=[0,1]$.

Proof. The set $A = \{(x_n, y_i) : i \leqslant n\} \cup \{(x_i, y_n) : i \leqslant n\}$ is finite, hence there exists a continuous function $g\colon \mathbb{R} \times \mathbb{R} \to [0,1]$ such that $g(x_n, y_n) = 1$, $g(x_n, y_i) = \xi_i$ and $g(x_i, y_n) = \eta_i$, for any $i < n$. Let $h\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by the formula

$$h(x,y) = \max\{0, 1 - \operatorname{dist}((x,y), A)\}$$

for any $(x,y) \in \mathbb{R} \times \mathbb{R}$. The restriction

$$(h \cdot g)|_{(\{x_n\}\times\mathbb{Q})\cup(\mathbb{Q}\times\{y_n\})} = f$$

is the desired function. □

If $A \subseteq X \times Y$, then the sets $A_x = \{y \in Y : (x,y) \in A\}$ and $A^y = \{x \in X : (x,y) \in A\}$ are called *sections*. Fix a subset

$$A = \{(x_n, y_n) : n \in \mathbb{N}\} \subseteq \mathbb{Q} \times \mathbb{Q}$$

such that all sections A_{x_n}, A^{y_n} are singletons and $\{x_n : n \in \mathbb{N}\} = \{y_n : n \in \mathbb{N}\} = \mathbb{Q}$.

Theorem 1. *If A is defined as above, then there exists a separately continuous function $f\colon \mathbb{Q}\times\mathbb{Q} \to [0,1]$ such that $f[A]$ is a singleton, but the image $f[\mathbb{Q}\times\mathbb{Q}]$ is dense.*

Proof. Let $f_0\colon (\{x_0\}\times\mathbb{Q})\cup(\mathbb{Q}\times\{y_0\}) \to [0,1]$ be any continuous function such that $f_0(x_0, y_0) = 1$ and $\operatorname{cl} f_0[\{x_0\} \times \mathbb{Q}] = [0,1]$. Assume that we have defined functions $f_0, \ldots, f_{n-1}$ such that if $i < k < n$, then

(1) $f_k\colon (\{x_k\} \times \mathbb{Q}) \cup (\mathbb{Q} \times \{y_k\}) \to [0,1]$ is continuous,
(2) $f_k(x_k, y_k) = 1$,
(3) $f_k(x_k, y_i) = f_i(x_k, y_i)$ and $f_k(x_i, y_k) = f_i(x_i, y_k)$.

Using Lemma 1 with parameters $\xi_i = f_i(x_n, y_i)$ and $\eta_i = f_i(x_i, y_n)$ for any $i < n$, we obtain a continuous function

$$f_n\colon (\{x_n\} \times \mathbb{Q}) \cup (\mathbb{Q} \times \{y_n\}) \to [0,1]$$

such that conditions (1)–(3) are satisfied for $i < k \leqslant n$. Finally, let $f\colon \mathbb{Q}\times\mathbb{Q} \to [0,1]$ be given by the formula $f(x_m, y_n) = f_m(x_m, y_n)$ for $m, n \in \mathbb{N}$.

Fix $(x_m, y_n) \in \mathbb{Q} \times \mathbb{Q}$. For each $y_k \in \mathbb{Q}$, we have $f(x_m, y_k) = f_m(x_m, y_k)$, which implies that $f|_{\{x_m\}\times\mathbb{Q}}$ is continuous. By condition (3), $f_k(x_k, y_n) = f_n(x_k, y_n)$, hence $f(x_k, y_n) = f_k(x_k, y_n) = f_n(x_k, y_n)$, which implies that $f|_{\mathbb{Q}\times\{y_n\}}$ is continuous.

Functions f and f_0 agree on the set $\{x_0\} \times \mathbb{Q}$, hence $\operatorname{cl} f[\{x_0\} \times \mathbb{Q}] = [0, 1]$. Clearly, we have $f[A] = \{1\}$. □

Corollary 1. *There exists a separately continuous function* $f\colon \mathbb{Q} \times \mathbb{Q} \to [0, 1]$ *which is not somewhat continuous.*

Proof. It suffices to assume that the set A in Theorem 1 is also dense in $\mathbb{Q} \times \mathbb{Q}$. □

References

[1] Z. Frolík, *Remarks concerning the invariance of Baire spaces under mappings*, Czechoslovak Math. J. **11(86)** (1961), no. 3, 381–385.

[2] K.R. Gentry, H.B. Hoyle III, *Somewhat continuous functions*, Czechoslovak Math. J. **21(96)** (1971), no. 1, 5–12.

[3] N.F.G. Martin, *Quasi-continuous functions on product spaces*, Duke Math. J. **28** (1961), no. 1, 39–43.

[4] T. Neubrunn, *Generalized continuity and separate continuity*, Math. Slovaca **27** (1977), no. 3, 307–314.

Institute of Mathematics
University of Silesia in Katowice
Bankowa 14
40-007 Katowice
Poland
e-mail: wojciech.bielas@us.edu.pl