

A VARIANT OF D’ALEMBERT’S MATRIX FUNCTIONAL EQUATION

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Abstract. The aim of this paper is to characterize the solutions $\Phi: G \rightarrow M_2(\mathbb{C})$ of the following matrix functional equations

$$\frac{\Phi(xy) + \Phi(\sigma(y)x)}{2} = \Phi(x)\Phi(y), \quad x, y \in G,$$

and

$$\frac{\Phi(xy) - \Phi(\sigma(y)x)}{2} = \Phi(x)\Phi(y), \quad x, y \in G,$$

where G is a group that need not be abelian, and $\sigma: G \rightarrow G$ is an involutive automorphism of G . Our considerations are inspired by the papers [13, 14] in which the continuous solutions of the first equation on abelian topological groups were determined.

1. Introduction

Throughout this paper, let G be a group with neutral element e , and $\sigma: G \rightarrow G$ be a homomorphism such that $\sigma \circ \sigma = id$. Let $M_2(\mathbb{C})$ denote the algebra of complex 2×2 matrices. It will represent the range space of the

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solutions in this paper. The purpose of this paper is to solve the following matrix functional equation

$$(1.1) \quad \frac{\Phi(xy) + \Phi(\sigma(y)x)}{2} = \Phi(x)\Phi(y), \quad x, y \in G,$$

where $\Phi: G \rightarrow M_2(\mathbb{C})$ is the unknown function. The contribution of the present paper to the theory of matrix d'Alembert's functional equations lies in the study of (1.1) on groups that need not be abelian. On abelian groups the solutions of Eq. (1.1) are known: the matrix or even operator version (1.1) of d'Alembert's functional equation with $\sigma = -id$ has for $\Phi(e) = I$ been treated by Fattorini ([7]), Kurepa ([9]), Baker and Davidson ([2]), Kisiński ([8]), Székelyhidi ([17]), Chojnacki ([3]), Sinopoulos ([12]) under various conditions like $G = 2G$ or the solution being bounded on G . For a general involutive automorphism σ not just $\sigma = -id$, Stetkaer ([14]) determined the general solution $\Phi: G \rightarrow M_2(\mathbb{C})$ of (1.1). He did not need extra assumptions on the abelian topological group G and also found the solutions of (1.1) when $\Phi(e) \neq I$.

In the present paper, we extend the setting of Φ from an abelian topological group to a group that need not be abelian. So, our main contribution is a natural extension of the previous works [13, 14] to d'Alembert's matrix functional equation. This knowledge in turn enables us to solve the symmetrized matrix multiplicative Cauchy equation

$$(1.2) \quad \Phi(xy) + \Phi(yx) = 2\Phi(x)\Phi(y), \quad x, y \in G,$$

and the following matrix functional equation

$$(1.3) \quad \frac{\Phi(xy) + \Phi(\sigma(y)x)}{2} = \Phi(x) + \Phi(y) + \Phi(x)\Phi(y), \quad x, y \in G.$$

The 2×2 matrix valued solutions of (1.2) and (1.3) are given in Corollaries 6.1 and 6.2, respectively. Example 5.5 shows that solutions of (1.2) are not generally abelian (see Notation). This is in contrast to the complex valued solutions of (1.2) which are multiplicative ([15, Theorem 3.21]). We also show that any continuous solution of (1.1) on a compact group is abelian.

Another main result of this paper is the solution of the functional equation

$$(1.4) \quad \frac{\Phi(xy) - \Phi(\sigma(y)x)}{2} = \Phi(x)\Phi(y), \quad x, y \in M,$$

where M is a monoid and $\Phi: M \rightarrow M_2(\mathbb{C})$ is the unknown function. This is the subject of the last section. The functional equation (1.4) differs from the

functional equation (1.1) by having a minus on the left hand side instead of a plus.

Finally, we note that results about the scalar functional equations

$$(1.5) \quad f(xy) + f(\sigma(y)x) = 2f(x)f(y), \quad x, y \in G,$$

$$(1.6) \quad f(xy) + f(\sigma(y)x) = 2f(x)g(y), \quad x, y \in G,$$

$$(1.7) \quad f(xy) + f(\sigma(y)x) = 2f(x) + 2f(y), \quad x, y \in G,$$

the sine addition law

$$(1.8) \quad f(xy) = f(x)g(y) + f(y)g(x), \quad x, y \in G,$$

and the symmetrized additive Cauchy equation

$$(1.9) \quad f(xy) + f(yx) = 2f(x) + 2f(y) \quad x, y \in G,$$

play important roles in finding the solutions of the functional equation (1.1). The complex-valued solutions, where G is a semigroup, of (1.5), (1.8), and (1.9) were studied by Stetkær in [16], [15, Chapter 4], and [15, Chapter 2], respectively, while the complex-valued solutions, where G is a possibly non-abelian group or monoid, of (1.6) and (1.7) were obtained by Fadli, Zeglami and Kabbaj in [5] and [6], respectively. General results about similar scalar functional equations on abelian groups are summarized in the monograph by Aczél and Dhombres [1] that contains many references.

Notation. Throughout this paper we work in the following framework and with the following notation and terminology. We use it without explicit mentioning. G is a group that need not be abelian with neutral element e . Let $id: G \rightarrow G$ denote the identity map, and $\sigma: G \rightarrow G$ a homomorphism of G such that $\sigma \circ \sigma = id$. We let $M_2(\mathbb{C})$ the algebra of all complex 2×2 matrices, I its identity matrix and $GL_2(\mathbb{C})$ the group of its invertible matrices. We use the notation $\mathcal{A}(G)$ for the vector space of all additive maps from G to \mathbb{C} , and put $\mathcal{A}^\pm(G) := \{a \in \mathcal{A}(G) : a \circ \sigma = \pm a\}$.

By $\mathcal{N}(G, \sigma)$ we mean the set of the solutions $\theta: G \rightarrow \mathbb{C}$ of the homogeneous equation, namely

$$\theta(xy) - \theta(\sigma(x)y) = 0, \quad x, y \in G.$$

Let S be a semigroup and \mathbb{X} be a groupoid. A function $f: S \rightarrow \mathbb{X}$ is multiplicative on S if $f(xy) = f(x)f(y)$ for all $x, y \in S$. A character of G is a multiplicative function from G into \mathbb{C}^* . A function $f: S \rightarrow \mathbb{X}$ is abelian, if

$$f(x_{\pi(1)}x_{\pi(2)} \cdots x_{\pi(k)}) = f(x_1x_2 \cdots x_k),$$

for all $x_1, x_2, \dots, x_k \in S$, all permutations π of k elements and all $k = 2, 3, \dots$. Any abelian function f is central, meaning $f(xy) = f(yx)$ for all $x, y \in S$.

2. Auxiliary results

The following lemma presents some results that are essential for the proof of our first main result (Theorem 5.1).

LEMMA 2.1. *If the pair $X, Z: G \rightarrow \mathbb{C}$ satisfies the functional equation*

$$(2.1) \quad X(xy) = X(x) + X(y) + (\gamma(x) - 1)Z(y) + (\gamma(y) - 1)Z(x), \quad x, y \in G$$

where $\gamma: G \rightarrow \mathbb{C}$ is a multiplicative function such that $\gamma \neq 1$, then X and Z are abelian functions.

PROOF. For all $x, y, z \in G$ we have

$$\begin{aligned} X((xy)z) &= X(xy) + X(z) + (\gamma(xy) - 1)Z(z) + (\gamma(z) - 1)Z(xy) \\ &= X(x) + X(y) + X(z) + (\gamma(x) - 1)Z(y) + (\gamma(y) - 1)Z(x) \\ &\quad + (\gamma(xy) - 1)Z(z) + (\gamma(z) - 1)Z(xy), \end{aligned}$$

and

$$\begin{aligned} X(x(yz)) &= X(x) + X(yz) + (\gamma(x) - 1)Z(yz) + (\gamma(yz) - 1)Z(x) \\ &= X(x) + X(y) + X(z) + (\gamma(y) - 1)Z(z) + (\gamma(z) - 1)Z(y) \\ &\quad + (\gamma(x) - 1)Z(yz) + (\gamma(yz) - 1)Z(x). \end{aligned}$$

Subtracting the two previous identities we get

$$\begin{aligned} 0 &= (\gamma(x) - 1)[Z(y) - Z(yz)] + (\gamma(y) - 1)[Z(x) - Z(z)] \\ &\quad + (\gamma(z) - 1)[Z(xy) - Z(y)] + (\gamma(xy) - 1)Z(z) \\ (2.2) \quad &\quad - (\gamma(yz) - 1)Z(x), \end{aligned}$$

for all $x, y, z \in G$. Putting $x = z$ in (2.2) we obtain

$$(2.3) \quad (\gamma(z) - 1)[Z(z) - Z(yz)] = 0 \quad \text{for all } y, z \in G.$$

Let $z_0 \in G$ first satisfy that $\gamma(z_0) \neq 1$, then we get by (2.3) that

$$Z(z_0y) = Z(yz_0) \quad \text{for all } y \in G.$$

Let $z_0 \in G$ next satisfy that $\gamma(z_0) = 1$. Then the identity (2.2) gives

$$\begin{aligned} 0 &= (\gamma(x) - 1)[Z(y) - Z(yz_0)] + (\gamma(y) - 1)[Z(x) - Z(z_0)] \\ &\quad + (\gamma(xy) - 1)Z(z_0) - (\gamma(yz_0) - 1)Z(x), \end{aligned}$$

which implies after some computations that

$$(\gamma(x) - 1)Z(yz_0) = (\gamma(x) - 1)Z(y) + \gamma(y)(\gamma(x) - 1)Z(z_0).$$

Since $\gamma \neq 1$, then we can deduce easily that

$$(2.4) \quad Z(yz_0) = Z(y) + \gamma(y)Z(z_0).$$

If we put $x = z_0$ in (2.2) we obtain

$$(\gamma(z) - 1)Z(z_0y) = (\gamma(z) - 1)Z(y) + \gamma(y)(\gamma(z) - 1)Z(z_0).$$

Since $\gamma \neq 1$ we get

$$(2.5) \quad Z(z_0y) = Z(y) + \gamma(y)Z(z_0) \quad \text{for all } y \in G.$$

Thus, by (2.4) and (2.5), we have also $Z(z_0y) = Z(yz_0)$ for all $y \in G$. Therefore we get

$$Z(zy) = Z(yz) \quad \text{for all } y, z \in G.$$

Next, we show that X is abelian. Indeed, making the substitutions (x, yz) and (x, zy) in (2.1), we get respectively

$$\begin{aligned} X(xyz) &= X(x) + X(yz) + (\gamma(x) - 1)Z(yz) + (\gamma(yz) - 1)Z(x), \\ X(xzy) &= X(x) + X(zy) + (\gamma(x) - 1)Z(zy) + (\gamma(zy) - 1)Z(x). \end{aligned}$$

Subtracting the two previous identities, we get

$$X(xyz) - X(xzy) = [X(yz) - X(zy)] + (\gamma(x) - 1)[Z(yz) - Z(zy)].$$

Changing x and y in (2.1) we see that the function X is central. Since X and Z are central functions, then X is abelian. From the equation (2.1) and since $\gamma \neq 1$ we can prove that Z is also abelian. Hence we get the claimed result. \square

3. A connection to the sine addition law

The following lemma lists pertinent basic properties of any solution $\Phi: G \rightarrow M_2(\mathbb{C})$ of (1.1) satisfying $\Phi(e) = I$.

LEMMA 3.1. *Let G be a group. If $\Phi: G \rightarrow M_2(\mathbb{C})$ satisfies (1.1) and $\Phi(e) = I$, then*

- (1) $\Phi \circ \sigma = \Phi$.
- (2) $\Phi(x)\Phi(y) = \Phi(y)\Phi(x)$ for all $x, y \in G$.
- (3) For any invertible matrix $P \in M_2(\mathbb{C})$ the function $x \mapsto P\Phi(x)P^{-1}$, $x \in G$ is also a solution of (1.1).

PROOF. (1) Setting $x = e$ in (1.1) gives us

$$\Phi(y) + \Phi(\sigma(y)) = 2\Phi(y), \quad y \in G,$$

which implies that $\Phi(\sigma(y)) = \Phi(y)$ for all $y \in G$.

(2) Interchanging x and y in (1.1) we get

$$\frac{\Phi(yx) + \Phi(\sigma(x)y)}{2} = \Phi(y)\Phi(x), \quad x, y \in G,$$

and then replacing y by $\sigma(y)$ in the last equation, we obtain by using (1) that

$$\frac{\Phi(\sigma(y)x) + \Phi(xy)}{2} = \Phi(y)\Phi(x), \quad x, y \in G.$$

So $\Phi(x)\Phi(y) = \Phi(y)\Phi(x)$ for all $x, y \in G$.

(3) can be trivially shown. □

Lemma 3.2 below derives an interesting connection between (1.1) and the sine addition matrix functional equation, viz.

LEMMA 3.2. *Let $\Phi: G \rightarrow M_2(\mathbb{C})$ be a solution of the (1.1) such that $\Phi(e) = I$. Then Φ satisfies the sine addition matrix functional equation*

$$\Phi_a(xy) = \Phi_a(x)\Phi(y) + \Phi_a(y)\Phi(x), \quad a, x, y \in G,$$

where $\Phi_a(x) := \Phi(ax) - \Phi(a)\Phi(x)$.

PROOF. Making the substitutions (ax, y) , $(\sigma(y)a, x)$ and (a, xy) in (1.1) we get respectively

$$\begin{aligned}\Phi(axy) + \Phi(\sigma(y)ax) &= 2\Phi(ax)\Phi(y), \\ \Phi(\sigma(y)ax) + \Phi(\sigma(xy)a) &= 2\Phi(\sigma(y)a)\Phi(x) \\ &= 2[2\Phi(a)\Phi(y) - \Phi(ay)]\Phi(x) \\ &= 4\Phi(a)\Phi(y)\Phi(x) - 2\Phi(ay)\Phi(x), \\ \Phi(axy) + \Phi(\sigma(xy)a) &= 2\Phi(a)\Phi(xy).\end{aligned}$$

Subtracting the middle identity from the sum of the two others we get after some simplifications that

$$\begin{aligned}\Phi(axy) - \Phi(a)\Phi(xy) &= [\Phi(ax) - \Phi(a)\Phi(x)]\Phi(y) \\ &\quad + [\Phi(ay) - \Phi(a)\Phi(y)]\Phi(x) \\ &\quad + \Phi(a)[\Phi(x)\Phi(y) - \Phi(y)\Phi(x)].\end{aligned}$$

Using Lemma 3.1 (2) we get

$$\Phi(axy) - \Phi(a)\Phi(xy) = [\Phi(ax) - \Phi(a)\Phi(x)]\Phi(y) + [\Phi(ay) - \Phi(a)\Phi(y)]\Phi(x),$$

which, with the notation $\Phi_a(x) := \Phi(ax) - \Phi(a)\Phi(x)$, shows that the functional equation (1.1) is connected with the sine addition matrix functional equation as follows:

$$\Phi_a(xy) = \Phi_a(x)\Phi(y) + \Phi_a(y)\Phi(x) \text{ for all } a, x, y \in G. \quad \square$$

4. Simultaneous triangularization

To set the stage let $\Phi: G \rightarrow M_2(\mathbb{C})$ be a solution of the functional equation (1.1), namely

$$\frac{\Phi(xy) + \Phi(\sigma(y)x)}{2} = \Phi(x)\Phi(y), \quad x, y \in G.$$

Suppose that $\Phi(e) = I$. In view of Lemma 3.1 (2) the elements of the set $\{\Phi(x), x \in G\}$ commute pairwise. Then it is easy to verify after some computations that the elements of the following bigger set $E = \{\Phi(x), \Phi_a(x) \mid x, a \in G\}$ also commute pairwise, so by linear algebra all elements $\Phi(x), \Phi_a(x)$ of E can be brought into upper triangular form. Therefore there exist six functions $\phi_1, \phi_2, \psi_1, l_{1,a}, l_{2,a}, l_{3,a}: G \rightarrow \mathbb{C}$, and a matrix $P \in GL_2(\mathbb{C})$ such that

$$C(x) := P^{-1}\Phi(x)P = \begin{pmatrix} \phi_1(x) & \psi_1(x) \\ 0 & \phi_2(x) \end{pmatrix} \quad \text{for all } x \in G,$$

and

$$P^{-1}\Phi_a(x)P = \begin{pmatrix} l_{1,a}(x) & l_{3,a}(x) \\ 0 & l_{2,a}(x) \end{pmatrix} \quad \text{for all } a, x \in G.$$

According to Lemma 3.1 the function $x \mapsto C(x) = P^{-1}\Phi(x)P$, $x \in G$ is also a solution of (1.1), so its components satisfy the following system of functional equations

$$(4.1) \quad \begin{cases} \phi_1(xy) + \phi_1(\sigma(y)x) = 2\phi_1(x)\phi_1(y), \\ \phi_2(xy) + \phi_2(\sigma(y)x) = 2\phi_2(x)\phi_2(y), \\ \psi_1(xy) + \psi_1(\sigma(y)x) = 2\phi_1(x)\psi_1(y) + 2\psi_1(x)\phi_2(y). \end{cases}$$

Likewise, the component functions of Φ_a , $a \in G$ satisfy the following system of equations

$$(4.2) \quad \begin{cases} l_{1,a}(xy) = l_{1,a}(x)\phi_1(y) + l_{1,a}(y)\phi_1(x), \\ l_{2,a}(xy) = l_{2,a}(x)\phi_2(y) + l_{2,a}(y)\phi_2(x), \\ l_{3,a}(xy) = l_{1,a}(x)\psi_1(y) + l_{3,a}(x)\phi_2(y) \\ \quad + l_{1,a}(y)\psi_1(x) + l_{3,a}(y)\phi_2(x). \end{cases}$$

By the definition of Φ_a , the functions $l_{1,a}$, $l_{2,a}$ and $l_{3,a}$ can be expressed in terms of ϕ_1 , ϕ_2 and ψ_1 as follows:

$$(4.3) \quad \begin{aligned} l_{1,a}(x) &= \phi_1(ax) - \phi_1(a)\phi_1(x), \\ l_{2,a}(x) &= \phi_2(ax) - \phi_2(a)\phi_2(x), \\ l_{3,a}(x) &= \psi_1(ax) - \phi_1(a)\psi_1(x) - \psi_1(a)\phi_2(x), \quad a, x \in G. \end{aligned}$$

Furthermore, if $\phi_1 \neq \phi_2$ then there is $x_0 \in G$ such that $\Phi(x_0)$ is diagonalizable. Since $\Phi(x)\Phi(x_0) = \Phi(x_0)\Phi(x)$ for all $x \in G$ then the elements of the set

$\{\Phi(x) \mid x \in G\}$ can be simultaneously diagonalized and so we may assume that $\psi_1 = 0$. Thus the system (4.2) becomes as follows:

$$\begin{cases} l_{1,a}(xy) = l_{1,a}(x)\phi_1(y) + l_{1,a}(y)\phi_1(x), \\ l_{2,a}(xy) = l_{2,a}(x)\phi_2(y) + l_{2,a}(y)\phi_2(x), \\ l_{3,a}(xy) = l_{3,a}(x)\phi_2(y) + l_{3,a}(y)\phi_2(x). \end{cases}$$

Otherwise we have $\phi := \phi_1 = \phi_2$ and $l_{0,a} := l_{1,a} = l_{2,a}$ where $a \in G$. Then by (4.2) combined with (4.3) we get:

$$(4.4) \quad \begin{cases} l_{0,a}(xy) = l_{0,a}(x)\phi(y) + l_{0,a}(y)\phi(x), \\ l_{3,a}(xy) = l_{0,a}(x)\psi_1(y) + l_{3,a}(x)\phi(y) + l_{0,a}(y)\psi_1(x) + l_{3,a}(y)\phi(x), \end{cases}$$

for all $a, x, y \in G$.

5. Main results

Putting $x = y = e$ in (1.1) we get $\Phi(e)^2 = \Phi(e)$, from which we see that $\Phi(e): \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is a projection, so there are only the following three cases: $\Phi(e) = 0$, $\Phi(e) = I$ or $\Phi(e)$ is a 1-dimensional projection.

The first case implies that

$$\Phi(x) = \Phi(x)\Phi(e) = 0 \quad \text{for all } x \in G.$$

So from now on we are going to focus only on the other two cases.

The first main theorem of the present paper concerns the second case: it highlights the form of the solutions Φ of the matrix functional equation (1.1) for which $\Phi(e) = I$. It reads as follows:

THEOREM 5.1. *The solutions $\Phi: G \rightarrow M_2(\mathbb{C})$ of the matrix functional equation (1.1) satisfying $\Phi(e) = I$ are the matrix valued functions of the three forms below in which P ranges over $GL_2(\mathbb{C})$:*

$$(1) \quad (5.1) \quad \Phi = P \begin{pmatrix} \frac{\chi_1 + \chi_1 \circ \sigma}{2} & 0 \\ 0 & \frac{\chi_2 + \chi_2 \circ \sigma}{2} \end{pmatrix} P^{-1},$$

where χ_1 and χ_2 are characters of G .

$$(2) \quad (5.2) \quad \Phi = P \begin{pmatrix} \frac{\chi + \chi \circ \sigma}{2} & \frac{\chi + \chi \circ \sigma}{2} a^+ + \frac{\chi - \chi \circ \sigma}{2} a^- \\ 0 & \frac{\chi + \chi \circ \sigma}{2} \end{pmatrix} P^{-1},$$

where χ is a character of G such that $\chi \neq \chi \circ \sigma$ and $a^\pm \in \mathcal{A}^\pm(G)$.

$$(3) \quad (5.3) \quad \Phi = \chi P \begin{pmatrix} 1 & S + \psi \\ 0 & 1 \end{pmatrix} P^{-1},$$

where χ is a character of G such that $\chi = \chi \circ \sigma$, ψ is a solution of the symmetrized additive Cauchy equation (1.9) such that $\psi \in \mathcal{N}(G, \sigma)$ and $S: G \rightarrow \mathbb{C}$ is a map of the form $S(x) = B(x, x)$, $x \in G$, where $B: G \times G \rightarrow \mathbb{C}$ is a bi-additive function of G such that $B(x, \sigma(y)) = -B(y, x)$.

PROOF. It is easy to verify with simple computations that all formulas above for Φ define solutions of (1.1). So it remains to show the other direction. So we assume that $\Phi: G \rightarrow M_2(\mathbb{C})$ is a solution of (1.1) such that $\Phi(e) = I$. With the notation from Section 4, we have two cases:

Case 1: Suppose that $\phi_1 \neq \phi_2$. Since there is $x_0 \in G$ such that $\Phi(x_0)$ is diagonalizable and $\Phi(x)\Phi(x_0) = \Phi(x_0)\Phi(x)$ for all $x \in G$ then the elements of the set $\{\Phi(x) \mid x \in G\}$ can be simultaneously diagonalized. So we may assume that $\psi_1 = 0$. According to (4.1) the functions ϕ_1 and ϕ_2 are solutions of the variant of d'Alembert's functional equation (1.5) with $\phi_1(e) = \phi_2(e) = 1$, so by [16, Theorem 2.1] there exist characters χ_1 and χ_2 of G such that

$$\phi_1 = \frac{\chi_1 + \chi_1 \circ \sigma}{2} \quad \text{and} \quad \phi_2 = \frac{\chi_2 + \chi_2 \circ \sigma}{2}.$$

So we are in case (1) of our statement.

Case 2: Suppose that $\phi_1 = \phi_2 = \phi$, then for every $a \in G$ we have $l_{1,a} = l_{2,a} =: l_{0,a}$. Since ϕ is a solution of (1.5), then from [16, Theorem 2.1] there exists a character χ of G such that

$$\phi = \frac{\chi + \chi \circ \sigma}{2}.$$

Now, we are going to distinguish between two subcases:

Subcase 2.1: If $\chi = \chi \circ \sigma$, then we get $\phi = \chi$. From (4.1) ψ_1 is a solution of the following equation:

$$(5.4) \quad \psi_1(xy) + \psi_1(\sigma(y)x) = 2\chi(x)\psi_1(y) + 2\psi_1(x)\chi(y), \quad x, y \in G.$$

Dividing (5.4) by $\chi(x)\chi(y)$ and putting $\Gamma := \psi_1/\chi$, then we see that Γ is a solution of the variant of the quadratic functional equation

$$\Gamma(xy) + \Gamma(\sigma(y)x) = 2\Gamma(x) + 2\Gamma(y), \quad x, y \in G,$$

which shows, according to [6, Theorem 5.4], that

$$\Gamma(x) = B(x, x) + \psi(x), \quad x \in G,$$

where $B: G \times G \rightarrow \mathbb{C}$ is a bi-additive function of G such that $B(x, \sigma(y)) = -B(y, x)$ for all $x, y \in G$, and ψ is a solution of the symmetrized additive Cauchy equation (1.9) such that $\psi \in \mathcal{N}(G, \sigma)$. Hence we get

$$\psi_1(x) = \chi(x)(B(x, x) + \psi(x)), \quad x \in G.$$

So we are in case (3) of our statement.

Subcase 2.2: Here $\chi \neq \chi \circ \sigma$. We will start by showing that Φ is abelian. According to (4.4) $(l_{0,a}, \phi)$, $a \in G$ is a solution of the sine addition law, then from [15, Theorem 4.1] there exist $\alpha_a \in \mathbb{C}^*$ such that

$$l_{0,a} = \frac{\chi - \chi \circ \sigma}{2\alpha_a}, \quad a \in G.$$

Replacing ϕ and $l_{0,a}$ into (4.4), then we get

$$(5.5) \quad \begin{aligned} l_{3,a}(xy) &= \chi(x)H(y) + H(x)\chi(y) + \chi \circ \sigma(x)L(y) \\ &\quad + L(x)\chi \circ \sigma(y), \quad x, y \in G, \end{aligned}$$

where

$$H(x) = \frac{l_{3,a}(x)}{2} + \frac{\psi_1(x)}{2\alpha_a} \quad \text{and} \quad L(x) = \frac{l_{3,a}(x)}{2} - \frac{\psi_1(x)}{2\alpha_a}, \quad a, x \in G.$$

Dividing (5.5) by $\chi(x)\chi(y)$ gives us

$$(5.6) \quad X(xy) = Y(x) + Y(y) + \gamma(x)Z(y) + \gamma(y)Z(x), \quad x, y \in G,$$

where

$$X := \frac{l_{3,a}}{\chi}, \quad Y := \frac{H}{\chi}, \quad Z := \frac{L}{\chi}, \quad \text{and} \quad \gamma := \frac{\chi \circ \sigma}{\chi}.$$

Since $\Phi(e) = I$, then we have $\phi(e) = 1$ and $\psi_1(e) = 0$ which imply that

$$Y(e) = \frac{H(e)}{\chi(e)} = H(e) = \frac{l_{3,a}(e)}{2} + \frac{\psi_1(e)}{2\alpha_a} = 0,$$

because by (4.3)

$$l_{3,a}(e) = \psi_1(a) - \phi(a)\psi_1(e) - \psi_1(a)\phi(e) = 0.$$

Similarly we can deduce easily that $Z(e) = 0$. Putting $y = e$ in (5.6), we obtain

$$X(x) = Y(x) + Z(x) \quad \text{for all } x \in G.$$

So the functional equation (5.6) becomes

$$X(xy) = X(x) + X(y) + (\gamma(x) - 1)Z(y) + (\gamma(y) - 1)Z(x), \quad x, y \in G,$$

where $\gamma \neq 1$, because $\chi \neq \chi \circ \sigma$. From Lemma 2.1 we get that X and Z are abelian. Then so are $l_{3,a} = \chi X$ and $L = \chi Z$. From

$$L = \frac{l_{3,a}}{2} - \frac{\psi_1}{2\alpha_a},$$

we infer that ψ_1 is also abelian. Therefore

$$\Phi = PCP^{-1} = P \begin{pmatrix} \phi & \psi_1 \\ 0 & \phi \end{pmatrix} P^{-1}$$

is abelian.

Furthermore, since $\chi \neq \chi \circ \sigma$, there exists $x_0 \in G$ such that $\chi(x_0) - \chi \circ \sigma(x_0) \neq 0$. As $\phi(x_0^2) - \phi(\sigma(x_0)x_0)$ are the diagonal elements of $C(x_0^2) - C(\sigma(x_0)x_0)$, and

$$\begin{aligned} \phi(x_0^2) - \phi(\sigma(x_0)x_0) &= \frac{\chi(x_0^2) + \chi \circ \sigma(x_0^2)}{2} - \frac{\chi(\sigma(x_0)x_0) + \chi \circ \sigma(\sigma(x_0)x_0)}{2} \\ &= \frac{\chi(x_0)^2 + \chi \circ \sigma(x_0)^2}{2} - \chi(x_0)\chi \circ \sigma(x_0) \\ &= \frac{(\chi(x_0) - \chi \circ \sigma(x_0))^2}{2} \neq 0, \end{aligned}$$

then the matrix $\Omega := C(x_0^2) - C(\sigma(x_0)x_0)$ is invertible. Since the matrix $\frac{1}{2}(C(x_0^2) - C(\sigma(x_0)x_0))^{-1}$ is invertible, it has a square root K , which is

a polynomial in Ω (see, e.g., [4, Chapter VII, Section 1]). Now $C(x)$ commutes with Ω , so $C(x)$ commutes with any polynomial in Ω , and in particular it commutes with K . Since $C(x)$ for any $x \in G$ is an upper triangular matrix, so is Ω . It follows that K , being a polynomial in Ω , is also upper triangular. In a similar fashion as in the case of an abelian topological group ([13]), we introduce another function, this time N , as

$$N(x) := C(x_0x) - C(\sigma(x_0)x), \quad x \in G,$$

in order that the function M , defined by

$$M(x) := C(x) + KN(x), \quad x \in G,$$

satisfies $M(xy) = M(x)M(y)$. Indeed, as in the computations on pp. 220-221 of [1] we can prove that

$$\begin{aligned} & M(x)M(y) - M(xy) \\ &= [C(xy) - C(\sigma(y)x)] \left(K^2[C(x_0^2) - C(\sigma(x_0)x_0)] - \frac{1}{2}I \right) = 0, \end{aligned}$$

which means that the function M is multiplicative. Moreover, using Lemma 3.1, we get

$$(5.7) \quad C(x) = \frac{M(x) + M \circ \sigma(x)}{2}, \quad x \in G.$$

Since the matrix-valued functions $C(x)$, K and $N(x)$, $x \in G$ are upper triangular, where the diagonal elements of each function are equal, then by using the definition of M we may put $M = \begin{pmatrix} m & m_{12} \\ 0 & m \end{pmatrix}$. From (5.7) we get $m + m \circ \sigma = \chi + \chi \circ \sigma$, which implies by the linear independence of group homomorphisms from G into \mathbb{C}^* that

$$m = \chi \quad \text{or} \quad m = \chi \circ \sigma.$$

As it is possible to exchange χ and $\chi \circ \sigma$ then we may assume that $m = \chi$. Since M is a multiplicative function, then we get

$$\begin{aligned} \chi(xy) \begin{pmatrix} 1 & \frac{m_{12}}{\chi}(xy) \\ 0 & 1 \end{pmatrix} &= M(xy) = M(x)M(y) \\ &= \chi(x)\chi(y) \begin{pmatrix} 1 & \frac{m_{12}}{\chi}(x) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{m_{12}}{\chi}(y) \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$= \chi(xy) \begin{pmatrix} 1 & \frac{m_{12}}{\chi}(x) + \frac{m_{12}}{\chi}(y) \\ 0 & 1 \end{pmatrix}.$$

Hence, $a := m_{12}/\chi$ is an additive function. By using (5.7) we obtain

$$\psi_1(x) = \frac{\chi(x)a(x) + \chi \circ \sigma(x)a \circ \sigma(x)}{2},$$

which is equivalent to

$$\psi_1(x) = \frac{\chi(x) + \chi \circ \sigma(x)}{2} a^+(x) + \frac{\chi(x) - \chi \circ \sigma(x)}{2} a^-(x), \quad x \in G,$$

where $a^\pm := \frac{a \pm a \circ \sigma}{2} \in \mathcal{A}^\pm(G)$. So we are in case (2) of our statement, which completes the proof. \square

REMARK 5.2. If we assume that G is a topological group and that the function $\Phi: G \rightarrow M_2(\mathbb{C})$ is a continuous solution of (1.1) then the functions $\chi, \chi_1, \chi_2, a^+, a^-, S$ and ψ in Theorem 5.1 are continuous. Indeed, using [15, Theorem 3.18 (d)], it is easy to see that the characters in Theorem 5.1 are continuous. For the case (3) of Theorem 5.1, we have that $g_1 := S + \psi$ by assumption is continuous. Hence so is $g_2(x) := g_1(x^2)$, $x \in G$. But $g_2 - 2g_1 = 2S$, so S is continuous. ψ is also continuous, because $\psi = g_1 - S$. If we are in case (2) of Theorem 5.1, we can prove that a^+ and a^- are continuous. In fact, we have $x \mapsto N(x) = C(x_0x) - C(\sigma(x_0)x)$ and $N \circ \sigma = -N$ are continuous. These yield that $M = C + KN$ and $M \circ \sigma = C \circ \sigma + N \circ \sigma = C - KN$ are continuous. Since $a = m_{12}/\chi$, we can deduce easily that a^+ and a^- are continuous.

The second main theorem of the present paper concerns the third case: It describes the complete solutions Φ of (1.1) when $\Phi(e)$ is a 1-dimensional projection. It reads as follows:

THEOREM 5.3. *The solutions $\Phi: G \rightarrow M_2(\mathbb{C})$ of (1.1), such that $\Phi(e)$ is a 1-dimensional projection, are the matrix valued functions of the two forms below in which $P \in GL_2(\mathbb{C})$:*

$$(5.8) \quad \Phi = P \begin{pmatrix} \frac{\chi + \chi \circ \sigma}{2} & 0 \\ \beta \frac{\chi - \chi \circ \sigma}{2} & 0 \end{pmatrix} P^{-1} \quad \text{if } \chi \neq \chi \circ \sigma,$$

$$(5.9) \quad \Phi = \chi P \begin{pmatrix} 1 & 0 \\ a^- & 0 \end{pmatrix} P^{-1} \quad \text{if } \chi = \chi \circ \sigma,$$

where χ is a character, $\beta \in \mathbb{C}$ and $a^- \in \mathcal{A}^-(G)$.

PROOF. Let $\Phi: G \rightarrow M_2(\mathbb{C})$ be a solution of (1.1) such that $\Phi(e)$ is a 1-dimensional projection. Then there exists $P \in GL_2(\mathbb{C})$ such that $P^{-1}\Phi(e)P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. We write $\begin{pmatrix} \phi_1 & \phi_3 \\ \phi_2 & \phi_4 \end{pmatrix} := P^{-1}\Phi P$. If we put $y = e$ in (1.1), then we get that

$$(5.10) \quad \Phi(x) = \Phi(x)\Phi(e), \quad x \in G.$$

From (5.10) it is easy to show that $\phi_3 = \phi_4 = 0$, so that we have $\Phi = P \begin{pmatrix} \phi_1 & 0 \\ \phi_2 & 0 \end{pmatrix} P^{-1}$. Then simple computations show that ϕ_1 and ϕ_2 satisfy the following system of functional equations

$$(5.11) \quad \begin{cases} \phi_1(xy) + \phi_1(\sigma(y)x) = 2\phi_1(x)\phi_1(y), \\ \phi_2(xy) + \phi_2(\sigma(y)x) = 2\phi_2(x)\phi_1(y). \end{cases}$$

Thus from [5, Theorem 3.6] there exists a character χ of G such that

$$\phi_1 = \frac{\chi + \chi \circ \sigma}{2} \quad \text{and} \quad \begin{cases} \phi_2 = \alpha \frac{\chi + \chi \circ \sigma}{2} + \beta \frac{\chi - \chi \circ \sigma}{2}, & \text{if } \chi \neq \chi \circ \sigma, \\ \phi_2 = (\alpha + a^-)\chi, & \text{if } \chi = \chi \circ \sigma, \end{cases}$$

where $\alpha, \beta \in \mathbb{C}$ and $a^- \in \mathcal{A}^-(G)$. Since $\phi_2(e) = 0$, then we get

$$\phi_1 = \frac{\chi + \chi \circ \sigma}{2} \quad \text{and} \quad \begin{cases} \phi_2 = \beta \frac{\chi - \chi \circ \sigma}{2}, & \text{if } \chi \neq \chi \circ \sigma, \\ \phi_2 = a^- \chi, & \text{if } \chi = \chi \circ \sigma. \end{cases}$$

And so we get the desired result. Conversely, it is easy to verify that any function Φ of the form (5.8) or (5.9) is a solution of (1.1) such that $\Phi(e)$ is a 1-dimensional projection. \square

REMARK 5.4. It is known that the scalar valued solutions $\Phi: S \rightarrow \mathbb{C}$ of the variant of d'Alembert's functional equation (1.5), where S is a semigroup, are always abelian [16, Theorem 2.1]. While here, we see that the solutions $\Phi: G \rightarrow M_2(\mathbb{C})$ of (1.1) presented in case (3) of Theorem 5.1 are not always central, and therefore not abelian as shown in Example 5.5.

EXAMPLE 5.5. For non-abelian continuous solutions of the equation (1.1) on a topological group, we consider the 3-dimensional Heisenberg group

$$G = H_3(\mathbb{R}) := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\},$$

and we take the identity map $id: G \rightarrow G$ as the involutive automorphism σ in (1.1). It is known that the continuous characters on $H_3(\mathbb{R})$ are the functions

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \rightarrow e^{ax+by}, \quad x, y, z \in \mathbb{R},$$

where a and b range over \mathbb{C} (see e.g., [15, Example 3.14]). We consider the functions of the form

$$\Phi \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = e^{ax+by} P \begin{pmatrix} 1 & c(2z - xy) \\ 0 & 1 \end{pmatrix} P^{-1}, \quad x, y, z \in \mathbb{R},$$

where $a, b, c \in \mathbb{C}$, $c \neq 0$, and $P \in GL_2(\mathbb{C})$. It is elementary to check that these functions are non-abelian solutions of (1.1) on $H_3(\mathbb{R})$ in which $\sigma = id$ because the complex-valued function

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto 2z - xy,$$

is a solution of the symmetrized additive Cauchy equation (1.9) on $H_3(\mathbb{R})$ and is not even central (see [15, Example 12.4]).

6. Applications

By applying Theorems 5.1 and 5.3 we describe the matrix valued solutions of the symmetrized multiplicative Cauchy equation on groups.

COROLLARY 6.1. *The non-zero solutions $\Phi: G \rightarrow M_2(\mathbb{C})$ of the matrix functional equation*

$$(6.1) \quad \Phi(xy) + \Phi(yx) = 2\Phi(x)\Phi(y), \quad x, y \in G,$$

are the matrix valued functions of the three forms below in which P ranges over $GL_2(\mathbb{C})$:

$$\Phi = P \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix} P^{-1}, \quad \Phi = \chi P \begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix} P^{-1} \quad \text{or} \quad \Phi = \chi P \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} P^{-1},$$

where χ_1, χ_2, χ are characters of G , and ψ is a solution of the symmetrized additive Cauchy equation (1.9).

PROOF. The proof follows from Theorems 5.1 and 5.3. □

As another application of our results we give, in the following corollary, a complete description of the solutions of the equation (1.3), that is

$$\frac{\Phi(xy) + \Phi(\sigma(y)x)}{2} = \Phi(x) + \Phi(y) + \Phi(x)\Phi(y), \quad x, y \in G,$$

where the unknown function takes its values in the complex 2×2 matrices. Setting $x = y = e$ in (1.3), we get $\Phi^2(e) = -\Phi(e)$, which means that $-\Phi(e)$ (or equivalently $I + \Phi(e)$) is a projection.

COROLLARY 6.2. *The solutions $\Phi: G \rightarrow M_2(\mathbb{C})$ of (1.3) are the functions of the following forms:*

- (1) *If $\Phi(e) = -I$, then $\Phi = -I$.*
- (2) *If $\Phi(e) = 0$, then Φ has one of the following three forms below in which P ranges over $GL_2(\mathbb{C})$:*

$$\Phi = P \begin{pmatrix} \frac{\chi_1 + \chi_1 \circ \sigma}{2} - 1 & 0 \\ 0 & \frac{\chi_2 + \chi_2 \circ \sigma}{2} - 1 \end{pmatrix} P^{-1},$$

where χ_1 and χ_2 are characters of G .

$$\Phi = P \begin{pmatrix} \frac{\chi + \chi \circ \sigma}{2} - 1 & \frac{\chi + \chi \circ \sigma}{2} a^+ + \frac{\chi - \chi \circ \sigma}{2} a^- \\ 0 & \frac{\chi + \chi \circ \sigma}{2} - 1 \end{pmatrix} P^{-1},$$

where χ is a character of G such that $\chi \neq \chi \circ \sigma$ and $a^\pm \in \mathcal{A}^\pm(G)$.

$$\Phi = P \begin{pmatrix} \chi - 1 & \chi(S + \psi) \\ 0 & \chi - 1 \end{pmatrix} P^{-1},$$

where χ is a character of G such that $\chi = \chi \circ \sigma$, ψ is a solution of the symmetrized additive Cauchy equation (1.9) such that $\psi \in \mathcal{N}(G, \sigma)$ and $S: G \rightarrow \mathbb{C}$ is a map of the form $S(x) = B(x, x)$, $x \in G$, where $B: G \times G \rightarrow \mathbb{C}$ is a bi-additive function of G such that $B(x, \sigma(y)) = -B(y, x)$.

(3) If $I + \Phi(e)$ is a 1-dimensional projection, then Φ has one of the two forms:

$$\Phi = P \begin{pmatrix} \frac{\chi + \chi \circ \sigma}{2} - 1 & 0 \\ \beta \frac{\chi - \chi \circ \sigma}{2} & -1 \end{pmatrix} P^{-1}, \quad \text{if } \chi \neq \chi \circ \sigma,$$

$$\Phi = P \begin{pmatrix} \chi - 1 & 0 \\ \chi a^- & -1 \end{pmatrix} P^{-1}, \quad \text{if } \chi = \chi \circ \sigma,$$

where χ is a character of G , $P \in GL_2(\mathbb{C})$, $\beta \in \mathbb{C}$ and $a^- \in \mathcal{A}^-(G)$.

PROOF. Let $\Phi: G \rightarrow M_2(\mathbb{C})$ be a solution of (1.3). If we add the identity matrix in the two sides of (1.3), we get that

$$\frac{\Psi(xy) + \Psi(\sigma(y)x)}{2} = \Psi(x)\Psi(y), \quad x, y \in G,$$

where $\Psi := \Phi + I$. So, by applying Theorems 5.1 and 5.3 we obtain the claimed result.

Conversely, simple computations show that the above forms of Φ are solutions of (1.3). \square

Now, we derive formulas for the continuous solutions of (1.1) on compact groups.

COROLLARY 6.3. *The non-zero continuous solutions $\Phi: G \rightarrow M_2(\mathbb{C})$ of (1.1), on a compact group, are the functions of the following two forms:*

$$\Phi = P \begin{pmatrix} \frac{\chi_1 + \chi_1 \circ \sigma}{2} & 0 \\ 0 & \frac{\chi_2 + \chi_2 \circ \sigma}{2} \end{pmatrix} P^{-1},$$

$$\Phi = P \begin{pmatrix} \frac{\chi + \chi \circ \sigma}{2} & 0 \\ \beta \frac{\chi - \chi \circ \sigma}{2} & 0 \end{pmatrix} P^{-1},$$

where $P \in GL_2(\mathbb{C})$, χ, χ_1, χ_2 are continuous characters of G and $\beta \in \mathbb{C}$.

PROOF. Let $\Phi: G \rightarrow M_2(\mathbb{C})$ be a non-zero continuous solution of (1.1) on a compact group. It is easy to see that the functions a^-, χ in Theorem 5.3 are continuous and in view of Remark 5.2 the functions a^+, a^-, S, ψ and the characters in Theorem 5.1 are also continuous. Hence a^+, a^- and S are bounded because G is compact. So by [15, Exercise 2.5] we deduce that $a^\pm \equiv 0$.

We may use the same argument as in [15, Exercise 2.5] to show that $S \equiv 0$. From [15, Proposition 2.17] and [15, Corollary 12.6] we can prove that any continuous solution of (1.9) on a compact group will vanish. So the first direction deduces easily from Theorems 5.1 and 5.3.

Conversely, it is elementary to show that the above forms of Φ are solutions of (1.1). \square

REMARK 6.4. Corollary 6.3 above implies that any continuous solution $\Phi: G \rightarrow M_2(\mathbb{C})$ of (1.1) on a compact group is abelian.

REMARK 6.5. On a compact group if $\Phi: G \rightarrow M_2(\mathbb{C})$ is a continuous solution of (6.1), then it is a multiplicative function. Example 5.5 shows that this result is not generally true in any group.

The following corollary writes down the non-zero solutions of the matrix functional equation

$$(6.2) \quad \frac{\Phi(x\sigma(y)) + \Phi(yx)}{2} = \Phi(x)\Phi(y), \quad x, y \in G.$$

COROLLARY 6.6. *The non-zero solutions $\Phi: G \rightarrow M_2(\mathbb{C})$ of (6.2) are the functions of the following five forms (5.1), (5.2), (5.3), (5.8) and (5.9).*

PROOF. Putting $x = y = e$ in (6.2), we get $\Phi(e)^2 = \Phi(e)$ from which we see that $\Phi(e) = 0$, $\Phi(e) = I$ or $\Phi(e)$ is a 1-dimensional projection. If $\Phi(e) = 0$ then $\Phi \equiv 0$. This case is excluded by assumption.

Suppose now that $\Phi(e) = I$. Putting $x = e$ in (6.2) then we get

$$\Phi(\sigma(y)) + \Phi(y) = 2\Phi(y) \quad \text{for all } y \in G$$

which implies that $\Phi \circ \sigma = \Phi$. Replacing y by $\sigma(y)$ in (6.2) we obtain the equation (1.1). So the desired result can be deduced by using Theorem 5.1.

Finally, suppose that $\Phi(e)$ is a 1-dimensional projection and there exists $P \in GL_2(\mathbb{C})$ such that $P^{-1}\Phi P =: \begin{pmatrix} \phi_1 & \phi_3 \\ \phi_2 & \phi_4 \end{pmatrix}$. We can follow the same procedure as in the proof of Theorem 5.3 to show that $\phi_3 = \phi_4 = 0$, and ϕ_1, ϕ_2 satisfy the following system of equations

$$(6.3) \quad \begin{cases} \phi_1(x\sigma(y)) + \phi_1(yx) = 2\phi_1(x)\phi_1(y), \\ \phi_2(x\sigma(y)) + \phi_2(yx) = 2\phi_2(x)\phi_1(y), \end{cases} \quad x, y \in G.$$

Since $\phi_1(e) = 1$ (otherwise we find $\phi_1 = \phi_2 = 0$ i.e. $\Phi \equiv 0$ and this case does not occur here) then we obtain by putting $x = e$ in (6.3) that $\phi_1 \circ \sigma = \phi_1$.

And so if we replace y by $\sigma(y)$ in (6.3), then we get (5.11). Consequently we can deduce the desired result by using the proof of Theorem 5.3. Conversely the formulas (5.1), (5.2), (5.3), (5.8) and (5.9) define solutions of (6.2). \square

7. Solution of Eq. (1.4)

As another main result of this paper, we solve the matrix functional equation

$$(7.1) \quad \frac{\Phi(xy) - \Phi(\sigma(y)x)}{2} = \Phi(x)\Phi(y), \quad x, y \in M,$$

where M is a monoid, the function Φ to be determined takes its values in $M_2(\mathbb{C})$, and $\sigma: M \rightarrow M$ is a homomorphism such that $\sigma \circ \sigma = id$.

Putting $x = y = e$ in (7.1), we get that $\Phi(e)$ is nilpotent with index less than 2, then we have only the two possibilities: $\Phi(e) = 0$ or $\Phi(e)$ is a nilpotent matrix with index 2.

The following lemma will be useful to find the solutions of (7.1) satisfying $\Phi(e) = 0$.

LEMMA 7.1. *Let M be a monoid. If $\Phi: M \rightarrow M_2(\mathbb{C})$ satisfies (7.1) such that $\Phi(e) = 0$, then*

- (1) $\Phi \circ \sigma = \Phi$.
- (2) $\Phi(x)\Phi(y) = -\Phi(y)\Phi(x)$ for all $x, y \in M$.
- (3) $\Phi(x)^2 = 0$ for all $x \in M$.

PROOF. (1) When we set $x = e$ in (7.1), then we get $\Phi \circ \sigma = \Phi$.

(2) Interchanging x and y in (7.1) gives us

$$\frac{\Phi(yx) - \Phi(\sigma(x)y)}{2} = \Phi(y)\Phi(x), \quad x, y \in M,$$

and then replacing y by $\sigma(y)$ and using (1), thus we get (2).

(3) Putting $x = y$ in (2) we obtain easily (3). \square

In the following theorem we express the solutions of (7.1) in terms of the complex-valued solutions of the variant of the homogeneous equation, namely

$$(7.2) \quad \theta(xy) - \theta(\sigma(y)x) = 0, \quad x, y \in M.$$

THEOREM 7.2. *The solutions $\Phi: M \rightarrow M_2(\mathbb{C})$ of the matrix functional equation (7.1), are the matrix valued functions of the form*

$$(7.3) \quad \Phi = P \begin{pmatrix} 0 & \theta \\ 0 & 0 \end{pmatrix} P^{-1},$$

where P ranges over $GL_2(\mathbb{C})$ and θ is a solution of (7.2).

PROOF. It is easy to prove with simple computations that the above formula for Φ defines solutions of (7.1). So it remains to show the other direction. For that we are going to distinguish between two cases:

Case 1: If $\Phi(e) = 0$, then we can prove that each commutator of the form $\Phi(x)\Phi(y) - \Phi(y)\Phi(x)$, $x, y \in M$ is nilpotent. Indeed, by using Lemma 7.1 (2) and (3), we get

$$(\Phi(x)\Phi(y) - \Phi(y)\Phi(x))^2 = 0 \quad \text{for all } x, y \in M.$$

So from [10, Theorem 1.3.2] the matrices $\Phi(x)$, $x \in M$ can be simultaneously trigonalized, thus there exist $P \in GL_2(\mathbb{C})$ and a function $\psi: M \rightarrow \mathbb{C}$ such that

$$P^{-1}\Phi(x)P = \begin{pmatrix} 0 & \psi(x) \\ 0 & 0 \end{pmatrix}, \quad x \in M.$$

We can prove after some computations that the function ψ is a solution of (7.2). And so we get the desired form.

Case 2: If $\Phi(e)$ is a nilpotent matrix with index 2. So up to similarity we may assume that $\Phi(e)$ has the form $\Phi(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Writing $\Phi = \begin{pmatrix} \phi_1 & \phi_3 \\ \phi_2 & \phi_4 \end{pmatrix}$ and so putting $y = e$ in (7.1) we get $\Phi(x)\Phi(e) = 0$ for all $x \in M$, which implies that $\phi_1 = \phi_2 = 0$. So $\Phi = \begin{pmatrix} 0 & \phi_3 \\ 0 & \phi_4 \end{pmatrix}$ and the functions ϕ_3 and ϕ_4 satisfy the following system of equations

$$\begin{cases} \phi_3(xy) - \phi_3(\sigma(y)x) = 2\phi_3(x)\phi_4(y), \\ \phi_4(xy) - \phi_4(\sigma(y)x) = 2\phi_4(x)\phi_4(y), \quad x, y \in M. \end{cases}$$

From [11, Theorem 3.1] we get that $\phi_4 = 0$ and so ϕ_3 is a solution of (7.2). Finally we have the desired form. \square

By the same procedure as in the proof of Theorem 7.2 we can prove the following result

REMARK 7.3. The solutions $\Phi: M \rightarrow M_2(\mathbb{C})$ of the following matrix functional equation

$$\frac{\Phi(x\sigma(y)) - \Phi(yx)}{2} = \Phi(x)\Phi(y), \quad x, y \in M,$$

are the matrix valued functions of the form (7.3).

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