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FAMILIES OF COMMUTING FORMAL POWER SERIES AND FORMAL FUNCTIONAL EQUATIONS

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Dedicated to Zygfryd Kominek on the occasion of his 75th birthday

Abstract. In this paper we describe families of commuting invertible formal power series in one indeterminate over $\mathbb{C}$, using the method of *formal functional* equations. We give a characterization of such families where the set of multipliers (first coefficients) σ of its members $F(x) = \sigma x + \dots$ is infinite, in particular of such families which are maximal with respect to inclusion, so called families of *type I*. The description of these families is based on Aczél–Jabotinsky differential equations, iteration groups, and on some results on normal forms of invertible series with respect to conjugation.

1. Introduction: Basic concepts and formulation of the main problem

In this paper we want to describe families of commuting invertible formal power series in one indeterminate over $\mathbb{C}$, using the method of *formal functional* equations. We give a characterization of such families where the set of

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multipliers (first coefficients) σ of its members $F(x) = \sigma x + \dots$ is infinite, in particular of such families which are maximal with respect to inclusion, so called families of *type I*. The description of these families is based on Aczél–Jabotinsky differential equations, iteration groups, and on some results on normal forms of invertible series with respect to conjugation. The study of families of commuting formal series started with [21].

The method of formal functional equations was so far applied to the construction of iteration groups of formal series and for solving the (associated) system of cocycle equations (see [3, 4, 5, 6, 7]). The basic idea of this method is to replace the coefficient functions by certain polynomials in one ore more variables.

The situation of maximal families of commuting power series where the set of multipliers is finite, is technically more complicate and will be considered in a separate paper, also using the method of formal functional equations.

Now we proceed by giving basic notations and results, in some places with indications of proofs, for the convenience of the reader.

Let $\mathbb{C}[\![x]\!]$ be the ring of formal power series in one indeterminate x over $\mathbb{C}$. For a detailed introduction to formal power series we refer the reader to [1] and [8]. Together with addition $+$ and multiplication $\cdot$ the set $\mathbb{C}[\![x]\!]$ forms a commutative ring. If $F \neq 0$, then the order of $F(x) = \sum_{n\geq 0} c_n x^n \in \mathbb{C}[\![x]\!]$, $c_n \in \mathbb{C}$, is defined as $\operatorname{ord}(F) = \min\{n \geq 0 \mid c_n \neq 0\}$. Moreover, $\operatorname{ord}(0) = \infty$. According to the order topology in $\mathbb{C}[\![x]\!]$, the series $F, G \in \mathbb{C}[\![x]\!]$ are "close" if $\operatorname{ord}(F - G)$ is "large". For $F, G \in \mathbb{C}[\![x]\!]$, $F(x) = \sum_{n\geq 0} c_n x^n$, $\operatorname{ord}(G) \geq 1$, the *composition* $\circ$ of formal series is defined as

$$(F \circ G)(x) = F(G(x)) = \lim_{n\to\infty} \sum_{\nu=0}^{n} c_\nu [G(x)]^\nu$$

which exists in the order topology. Substitution of a fixed $G \in \mathbb{C}[\![x]\!]$ yields a homomorphism $\mathbb{C}[\![x]\!] \to \mathbb{C}[\![x]\!]$. The composition of formal series is associative, $F \circ (G \circ H) = (F \circ G) \circ H$, $F, G, H \in \mathbb{C}[\![x]\!]$, $\operatorname{ord}(G), \operatorname{ord}(H) \geq 1$. With $\Gamma = (\Gamma, \circ)$ we denote the group of invertible formal power series with respect to composition, thus

$$\Gamma = \{F \in \mathbb{C}[\![x]\!] \mid \operatorname{ord}(F) = 1\} = \{F \in \mathbb{C}[\![x]\!] \mid F(x) = c_1 x + \dots,\ c_1 \neq 0\}.$$

Let Γ_1 be the subgroup of Γ with $c_1 = 1$,

$$\Gamma_1 = \{F \in \mathbb{C}[\![x]\!] \mid F(x) = x + c_2 x^2 + \dots\}.$$

The group Γ is not abelian, e.g., for $F(x) = -x$ and $G(x) = x + x^2$ we have $(F \circ G)(x) = -x - x^2$ and $(G \circ F)(x) = -x + x^2$, thus $F \circ G \neq G \circ F$.

Let $\mathbb{C}[\![x,y]\!] := (\mathbb{C}[\![x]\!])[\![y]\!]$ be the ring of formal power series in two indeterminates over $\mathbb{C}$, then $\mathbb{C}[\![x,y]\!] = \mathbb{C}[\![y,x]\!]$. An element of $\mathbb{C}[\![x,y]\!]$ can be written as

$$F(x,y) = \sum_{\nu,\mu\geq 0} c_{\nu\mu}x^\nu y^\mu = \sum_{\mu\geq 0}\Big(\sum_{\nu\geq 0} c_{\nu\mu}x^\nu\Big)y^\mu.$$

We will also need the ring $\mathbb{C}[\![x,y,z]\!]$ in three indeterminates, and use the fact that $(\mathbb{C}[y])[\![x]\!] < \mathbb{C}[\![x,y]\!]$.

Differentiation is a formal process in $\mathbb{C}[\![x]\!]$. Let $F(x) = \sum_{n\geq 0} c_n x^n$, then

$$\frac{d}{dx}F(x) = F'(x) = \sum_{n\geq 1} nc_n x^{n-1} = \sum_{n\geq 0}(n+1)c_{n+1}x^n.$$

Differentiation satisfies the chain rule $(F\circ G)'(x) = F'(G(x))\cdot G'(x)$.

Considering a family $(F(t,x))_{t\in\mathbb{C}}$ of formal power series

$$F(t,x) = \sum_{n\geq 0} c_n(t)x^n, \qquad t\in\mathbb{C},$$

where all the coefficient functions c_n are holomorphic in a neighborhood $U(0)$ of 0, we define

$$\frac{\partial}{\partial t}F(t,x) = \sum_{n\geq 0} c_n'(t)x^n, \quad t\in U(0).$$

In this situation the mixed chain rule

$$\frac{\partial}{\partial t}F(G(t,x)) = F'(G(t,x))\cdot\frac{\partial}{\partial t}G(t,x), \quad t\in U(0),$$

holds true.

We also consider formal partial differentiation in $\mathbb{C}[\![x,y]\!]$. Each $F(x,y)\in\mathbb{C}[\![x,y]\!]$ can be written as

$$F(x,y) = \sum_{n\geq 0} P_n(y)x^n \in (\mathbb{C}[\![y]\!])[\![x]\!]$$

and

$$\frac{\partial}{\partial y}F(x,y) = \sum_{n\geq 0} P_n'(y)x^n.$$

Two formal power series $F, G \in \Gamma$ are conjugate if there exists some $S \in \Gamma$ such that

$$G = S^{-1} \circ F \circ S.$$

Let $\mathcal{E}$ be the group of complex roots of 1. It is a subgroup of the multiplicative group $\mathbb{C}^*$.

Remark 1. Let $F(x) = \rho x + c_2 x^2 + \ldots \in \Gamma$, $\rho \notin \mathcal{E}$. Then there exists a unique series $S \in \Gamma_1$ such that

$$(S^{-1} \circ F \circ S)(x) = \rho x =: L_\rho(x).$$

We say that F can be linearized, or F is linearizable.

Proof. We determine all solutions $S \in \Gamma_1$ of Schröder's equation

$$F(S(x)) = S(\rho x). \tag{S}$$

The left hand side can be expanded as

$$\begin{aligned} F(S(x)) &= \rho S(x) + \sum_{n \geq 2} c_n [S(x)]^n \\ &= \rho x + (\rho s_2 + c_2) x^2 + (\rho s_3 + 2 c_2 s_2 \rho + c_3) x^3 + \ldots \\ &= \rho x + \sum_{n \geq 2} (\rho s_n + p_n(\rho, s_2, \ldots, s_{n-1}, c_2, \ldots, c_n)) x^n \end{aligned}$$

with polynomials $p_n(t, u_2, \ldots, u_{n-1}, v_2, \ldots, v_n) \in \mathbb{C}[t, u_2, \ldots, u_{n-1}, v_2, \ldots, v_n]$, $n \geq 2$. The right hand side equals $\rho x + \sum_{n \geq 2} s_n \rho^n x^n$. Since ρ is not a complex root of unity, $\rho^n \neq \rho$ for all $n \geq 2$, and comparison of coefficients yields: $(\rho^2 - \rho) s_2 = c_2$, whence s_2 is uniquely determined. In general for $n \geq 2$ we obtain inductively

$$(\rho^n - \rho) s_n = p_n(\rho, s_2, \ldots, s_{n-1}, c_2, \ldots, c_n)$$

and s_n is uniquely determined. □

Remark 2. Assume that $\rho \notin \mathcal{E}$, $\hat{F}(x) = \sigma x + \ldots \in \Gamma$, and $L_\rho \circ \hat{F} = \hat{F} \circ L_\rho$. Then $\hat{F}(x) = \sigma x$. If $F(x) = \rho x + \ldots$, $F(x) = S(\rho S^{-1}(x))$ for some $S \in \Gamma$, $\hat{F}(x) = \sigma x + \ldots \in \Gamma$, and $F \circ \hat{F} = \hat{F} \circ F$, then $\hat{F}(x) = S(\sigma S^{-1}(x))$.

Proof. The first assertion is obtained by direct calculation. F commutes with $\hat{F}$ if and only if $S \circ L_\rho \circ S^{-1} \circ \hat{F} = \hat{F} \circ S \circ L_\rho \circ S^{-1}$ which is equivalent to $L_\rho \circ S^{-1} \circ \hat{F} \circ S = S^{-1} \circ \hat{F} \circ S \circ L_\rho$ and the assertion follows from the first part. $\square$

Consider some $\sigma = \mathrm{e}^{2\pi \mathrm{i}\ell/m} \in \mathcal{E}$, $\gcd(\ell, m) = 1$, $m \geq 1$. Then

$$N(x) = \sigma x + \sum_{n \geq 1} d_{nm+1} x^{nm+1}$$

is a *semicanonical form with respect to* σ.

Remark 3. Let $\sigma \in \mathcal{E}$ be primitive of order $m > 1$, and $F(x) = \sigma x + \ldots \in \Gamma$. There exists some $S \in \Gamma_1$ such that

$$(S^{-1} \circ F \circ S)(x) = \sigma x + \sum_{n \geq 1} d_{nm+1} x^{nm+1},$$

i.e. a semicanonical form with respect to σ. In general, S is not uniquely determined. (Cf. [20, Section 2].)

Proof. When σ is a root of unity primitive of order $m > 1$, then in general (S) cannot be satisfied by choosing coefficients in a suitable way, since in the comparison of coefficients of x^n for $n \equiv 1 \bmod m$ the term $\sigma^n - \sigma$ vanishes, which means that $p_n(\sigma, s_2, \ldots, s_{n-1}, c_2, \ldots, c_n)$ must vanish, and s_n is not determined by this equation. Therefore, we study the equation

$$F(S(x)) = S(\sigma x + \sum_{n \geq 1} d_{nm+1} x^{nm+1}).$$

Its left hand side is $\sigma x + \sum_{n \geq 2} (\sigma s_n + p_n(\sigma, s_2, \ldots, s_{n-1}, c_2, \ldots, c_n)) x^n$ as in the proof of Remark 1. The right hand side can be expanded as

$$\sigma x + \sum_{n \geq 2} (\sigma^n s_n + q_n(\sigma, s_2, \ldots, s_{n-1}, d_{m+1}, \ldots, d_{\lfloor (n-1)/m \rfloor m+1})) x^n$$

with polynomials $q_n(t, u_2, \ldots, u_{n-1}, v_1, \ldots, v_{\lfloor (n-1)/m \rfloor}) \in \mathbb{C}[t, u_2, \ldots, u_{n-1}, v_1, \ldots, v_{\lfloor (n-1)/m \rfloor}]$, $n \geq 2$. In particular $q_{\nu m+1}(t, u_2, \ldots, u_{\nu m}, v_1, \ldots, v_\nu) = v_\nu + \tilde{q}_{\nu m+1}(t, u_2, \ldots, u_{\nu m}, v_1, \ldots, v_{\nu-1})$, $\nu \geq 1$. Comparison of coefficients yields

$$(*) \qquad (\sigma^n - \sigma) s_n = p_n(\ldots) - q_n(\ldots), \quad n \geq 2.$$

Inductively for $n \geq 2$, either s_n is uniquely determined by (*), or (in the case that $n \equiv 1 \bmod m$) the coefficient s_n is not determined, whence it can be arbitrarily chosen, and equation (*) determines d_n via the condition $p_n(\dots) - \tilde{q}_n(\dots) = d_n$. □

If $F \neq L_\sigma$, then in a semicanonical form $(S^{-1} \circ F \circ S)(x) = \sigma x + d_{n_0 m+1} x^{n_0 m+1} + \dots$ with $d_{n_0 m+1} \neq 0$ the index n_0 is unique. According to [22, Lemma 3] n_0 is an invariant.

Remark 4. Let $\sigma \in \mathcal{E}$ be primitive of order $m > 1$, and assume that $N(x) = \sigma x + \dots$ is a semicanonical form with respect to σ. Let $G \in \Gamma$ and assume that $G \circ N = N \circ G$. Then G has the structure similar to a semicanonical form with respect to σ, i.e.

$$G(x) = \rho x + \sum_{n \geq 1} d_{nm+1} x^{nm+1}, \quad \rho \in \mathbb{C}^*.$$

This class of formal power series is characterized by

$$G(\sigma x) = \sigma G(x), \quad G \in \Gamma.$$

Comparing coefficients of x^m proves

Remark 5. Let $F(x) = \rho x + \dots \in \Gamma$, $\rho \notin \mathcal{E}$, $G(x) = x + d_m x^m + \dots$, $m \geq 2$, $d_m \neq 0$. Then

$$F \circ G \neq G \circ F.$$

The main problem: Describe the families of commuting formal power series $(F(t,x))_{t \in \mathbb{C}}$, $F(t,x) \in \Gamma$ for all $t \in \mathbb{C}$, that is the families such that

$$\text{(C)} \qquad F(t, F(s,x)) = F(s, F(t,x)), \quad s,t \in \mathbb{C}.$$

Without loss of generality we always take the index set $\mathbb{C}$.

(C) is a "weak functional equation", weak because of the freedom in choosing the index set and the parameterization.

If we write $F_t(x) = F(t,x) = \sum_{n \geq 1} c_n(t) x^n$, $c_1(t) \neq 0$ for all $t \in \mathbb{C}$, then (C) is rewritten as

$$\text{(C)} \qquad F_t \circ F_s = F_s \circ F_t, \quad s,t \in \mathbb{C}.$$

Important special cases are:

1. Finite subgroups of Γ. In fact, in Remark 7 we prove that each finite subgroup of Γ is cyclic whence abelian.
2. Cyclic subgroups $\langle F\rangle$, $F \in \Gamma$.
3. One-parameter groups (iteration groups).
4. $\{F^n \mid n \geq 1\}$, $F \in \Gamma$, $F^1 = F$, $F^n = F \circ F^{n-1}$, $n \geq 2$.
5. Maximal families of commuting formal power series (maximal with respect to inclusion). These families are maximal abelian subgroups of Γ. Each maximal abelian subgroup of Γ can be written as a maximal family of commuting formal power series. (Since $\mathbb{C}$ has the cardinality $\aleph^1$ also $\mathbb{C}[\![x]\!]$ has this cardinality and each subset of $\mathbb{C}[\![x]\!]$ can be parameterized as a family of formal power series of the form $(F(t,x))_{t\in\mathbb{C}}$.)

Here we comment briefly on case 3. Let $(G,+)$ be an abelian group. A one-parameter group (iteration group) is a homomorphism $\theta\colon (G,+) \to (\Gamma,\circ)$. For $t \in G$, $\theta(t)(x) \in \Gamma$, $\theta(t)(x) = \theta(t,x) = \theta_t(x)$. As a homomorphism it satisfies the translation equation

$$\text{(T)} \qquad \theta(t+s,x) = \theta(t,\theta(s,x)), \quad s,t \in G.$$

Here we discuss $(G,+) = (\mathbb{C},+)$. The case of general G is more complicate. We distinguish the following types of one-parameter groups.

type 0: $\theta(t,x) = x$, $t \in \mathbb{C}$.

type I: $\theta(t,x) = c_1(t)x + \ldots$, $t \in \mathbb{C}$, where $c_1(t) \neq 0$ for all $t \in \mathbb{C}$ and $c_1 \neq 1$. Then

$$c_1(s+t) = c_1(s)c_1(t), \quad s,t \in \mathbb{C},$$

thus c_1 is a generalized exponential function. If $c_1 \neq 1$, then it takes infinitely many values.

type II: $\theta(t,x) = x + c_k(t)x^k + \ldots$, $t \in \mathbb{C}$, $k \geq 2$, $c_k \neq 0$. Then

$$c_k(t+s) = c_k(t) + c_k(s), \quad s,t \in \mathbb{C},$$

and c_k is a non-trivial additive function, thus it takes infinitely many values.

Iteration groups were studied e.g. in D. N. Lewis [16], S. Sternberg [26], K.-T. Chen [2], St. Scheinberg [25] E. Peschl and L. Reich [18], L. Reich and J. Schwaiger [23] or [24], G. Mehring [17], C. Praagman [19], H. Fripertinger and L. Reich [3, 4], and W. Jabłoński [12, 13, 11, 14].

A possible approach for obtaining results concerning the structure of these iteration groups and the explicit form of the coefficient functions is the method of formal functional equations [7].

From Remark 2 we deduce

Remark 6. The set $\{L_\rho \mid \rho \in \mathbb{C}^*\}$ is a maximal abelian subgroup of Γ. For each $S \in \Gamma$ the set $\{S \circ L_\rho \circ S^{-1} \mid \rho \in \mathbb{C}^*\}$ is a maximal abelian subgroup of Γ.

A consequence of properties of normal forms is:

Remark 7. A finite subgroup $\mathcal{F}$ of Γ of order $m \geq 2$ has the form $\langle S^{-1}(\sigma S(x))\rangle$ where $\sigma = \mathrm{e}^{2\pi\mathrm{i}/m}$, $S \in \Gamma$. Conversely, each group $\langle S^{-1}(\sigma S(x))\rangle$, $S \in \Gamma_1$, $\sigma = \mathrm{e}^{2\pi\mathrm{i}/m}$, $m \geq 1$, is a finite subgroup of Γ.

Proof. Define a mapping $\pi\colon \mathcal{F} \to \mathbb{C}^*$, $F(x) = \rho x + \ldots \mapsto \rho$. Then π is a homomorphism and $\pi(\mathcal{F})$ is a finite subgroup of $\mathbb{C}^*$, whence $\pi(\mathcal{F})$ is cyclic, thus of the form $\{1, \sigma, \ldots, \sigma^{m-1}\}$ and $\sigma = \mathrm{e}^{2\pi\mathrm{i}/m}$. The mapping π is injective, hence it is an isomorphism between $\mathcal{F}$ and $\pi(\mathcal{F})$. In order to show that π is injective assume that there are power series $F_1 \neq F_2$ in $\mathcal{F}$ such that $\pi(F_1) = \pi(F_2) = \rho$. Thus $F_1(x) = \rho x + c_2^{(1)} x^2 + \ldots$ and $F_2(x) = \rho x + c_2^{(2)} x^2 + \ldots$. Then $F_1^{-1} \circ F_2 \in \mathcal{F}$ and $(F_1^{-1} \circ F_2)(x) = x + d_n x^n + \ldots$, $n \geq 2$, $d_n \neq 0$. But then $(F_1^{-1} \circ F_2)^\ell(x) = x + \ell d_n x^{\ell-1+n} + \ldots \in \mathcal{F}$ for $\ell \in \mathbb{N}$, hence $\mathcal{F}$ is infinite, which is a contradiction. Thus $\mathcal{F}$ is isomorphic to $\pi(\mathcal{F})$, a finite cyclic group, hence abelian.

Assume that $\mathcal{F} = \langle F(x)\rangle$ with $F(x) = \sigma x + \ldots$, then by Remark 3 there is a semicanonical form $F = S^{-1} \circ N \circ S$, where $N(x) = \sigma x + \sum_{n\geq 1} d_{nm+1} x^{nm+1}$. If $N(x) \neq x$, then there exists some $n_0 \geq 1$ such that $d_{n_0 m+1} \neq 0$ and $N(x) = \sigma x + d_{n_0 m+1} x^{n_0 m+1} + \ldots$. But then $N^\ell(x) = \sigma^\ell x^\ell + \ell \sigma^{\ell-1} d_{n_0 m+1} x^{\ell + n_0 m} + \ldots$ for $\ell \in \mathbb{N}$, which gives infinitely many N^ℓ and infinitely many elements $F^\ell \in \mathcal{F}$, a contradiction. □

Remark 8. A maximal family $\mathcal{F}$ of commuting power series in Γ is infinite.

Proof. Assume that $\mathcal{F}$ is finite. Since it is a group by Remark 7 it is a finite cyclic group $\langle F(x)\rangle$, where $F(x) = \sigma x + \ldots$ can be linearized, $\sigma = \mathrm{e}^{2\pi\mathrm{i}/m}$. Thus $F(x) = S^{-1}(\sigma S(x))$ which is contained in the abelian group $\{S^{-1}(\rho S(x)) \mid \rho \in \mathbb{C}^*\}$. The latter is already maximal. □

In order to investigate maximal abelian subgroups of Γ we consider **two types of families of commuting formal power series.** We consider only families $\mathcal{F} = (F(t,x))_{t\in\mathbb{C}}$, $F(t,x) = c_1(t)x + \ldots$, $t \in \mathbb{C}$, of formal power series in Γ satisfying (C) which are, as sets, infinite. If c_1 takes infinitely many values, then $\mathcal{F}$ is of *type I*, otherwise of *type II*. Iteration groups of type I or type II are families of commuting formal power series of type I resp. type II. In the present paper we study families of type I.

We finish the Introduction by describing our results briefly. In Proposition 9 we associate, applying the method of formal functional equations, with each family $\mathcal{F} = (F(t,x))_{t\in\mathbb{C}}$ of commuting formal power series of type I, $F(t,x) = c_1(t)x + c_2(t)x^2 + \dots$ a formal object $G(y,x) = yx + \sum_{n\geq 2} P_n(y)x^n \in (\mathbb{C}[y])[\![x]\!]$ such that $F(t,x) = G(c_1(t),x)$, $t \in \mathbb{C}$, $G(1,x) = x$, and both

$$G(y, G(z,x)) = G(z, G(y,x)) \tag{$\mathrm{C}_{\mathrm{formal}}$}$$

and

$$G(y, G(z,x)) = G(yz, x) \tag{$\mathrm{T}_{\mathrm{formal}}$}$$

hold true in $(\mathbb{C}[y,z])[\![x]\!]$. In order to determine $G(y,x)$ and to describe it in detail we introduce in Proposition 10 the generator $H(x) = \frac{\partial}{\partial y}\, G(y,x)|_{y=1}$ of $\mathcal{F}$ (and of $G(y,x)$) and we show that each $F(t,x)$ satisfies the Aczél–Jabotinsky differential equation

$$H(x)\frac{\partial}{\partial x}\, F(t,x) = H(F(t,x)), \tag{AJ,H}$$

and also the formal Aczél–Jabotinsky differential equation

$$H(x)\frac{\partial}{\partial x}\, G(y,x) = H(G(y,x)) \tag{$\mathrm{AJ}_{\mathrm{formal}}$,H}$$

and the formal partial differential equation

$$H(x)\frac{\partial}{\partial x}\, G(y,x) = y\frac{\partial}{\partial y}\, G(y,x) \tag{$\mathrm{PD}_{\mathrm{formal}}$}$$

hold true. It is known from previous papers (e.g. [3]) that for given $H(x) = x + h_2x^2 + \dots$ the formal Aczél–Jabotinsky differential equation ($\mathrm{AJ}_{\mathrm{formal}}$,H) has a unique solution $G(y,x) = yx + \sum_{n\geq 2} P_n(y)x^n \in (\mathbb{C}[y])[\![x]\!]$ with $G(1,x) = x$ which satisfies ($\mathrm{C}_{\mathrm{formal}}$). This is recalled in Proposition 11. In Theorem 12 we show that for a given series $H(x) = x + h_2x^2 + \dots$ the set of all solutions of the Aczél–Jabotinsky equation (AJ,H) in Γ is a maximal abelian subgroup of Γ whose generator is $H(x)$ by Proposition 10. Theorem 12 is proved by applying the method of formal functional equations. In Theorem 13 we show the standard form of a family of type I. To each family $\mathcal{F} = (F(t,x))_{t\in\mathbb{C}}$ of type I there exists a unique $S \in \Gamma_1$ such that $F(t,x) = S^{-1}(c_1(t)S(x))$, $t \in \mathbb{C}$. If $G(y,x)$ is the formal object associated with $\mathcal{F}$ by Proposition 9, then $G(y,x) = S^{-1}(yS(x))$.

In Proposition 15 and Proposition 16 we deal with the problem whether a given $F(x) = \rho x + c_2x^2 + \dots \in \Gamma$ is contained in a maximal family of type I. If

$\rho \notin \mathcal{E}$, there is exactly one maximal abelian subgroup of Γ of type I containing F which can be constructed as an analytic embedding of F. If $\rho \in \mathcal{E}$, $\rho \neq 1$, then F is contained in a family of commuting power series of type I if and only if it is linearizable. We describe the maximal abelian subgroups of Γ containing F in terms of the generators and of $T \in \Gamma_1$ for which $F(x) = T^{-1}(\rho T(x))$.

Proposition 17 shows, in the same context, that a family of commuting formal power series of type I is contained in a unique maximal abelian subgroup of Γ of type I, and Proposition 18 gives a further description of maximal families of type I as iteration groups and as families $(F(t,x))_{t\in\mathbb{C}}$, $F(t,x) = c_1(t)x + \dots$, $t \in \mathbb{C}$, of type I for which $\mathrm{Im}(c_1) = \mathbb{C}^*$. Theorem 19 deals with a universal representation of the formal objects $G(y,x)$ associated with families of type I in terms of the coefficients of their generators. In Theorem 20 we summarize the results of the paper.

2. Families of commuting power series of type I and maximal abelian subgroups Γ of type I

A family $(F(t,x))_{t\in\mathbb{C}}$ of commuting formal power series of type I, consists of power series of the form $F(t,x) = c_1(t)x + \ldots \in \Gamma$, $t \in \mathbb{C}$, satisfying (C), where c_1 takes infinitely many values.

Proposition 9. *Let $(F(t,x))_{t\in\mathbb{C}}$ be a family of commuting formal power series of type I.*

1. *Then there exists a unique*

$$G(y,x) \in (\mathbb{C}[y])[\![x]\!], \qquad G(y,x) = yx + \sum_{n\geq 2} P_n(y)x^n,$$

with polynomials P_n such that

$$G(y,G(z,x)) = G(z,G(y,x)) \text{ in } (\mathbb{C}[y,z])[\![x]\!] \tag{$\mathrm{C}_{\mathrm{formal}}$}$$

is satisfied and $F(t,x) = c_1(t)x + \sum_{n\geq 2} P_n(c_1(t))x^n = G(c_1(t),x)$, $t \in \mathbb{C}$. Moreover, $P_n(1) = 0$ for $n \geq 2$.

2. *If a series $\psi(x) = \rho x + d_2x^2 + \ldots \in \Gamma$ commutes with each $F(t,x)$, then $\psi(x) = G(\rho,x)$. Hence $\psi(G(y,x)) = G(y,\psi(x))$ in $(\mathbb{C}[y])[\![x]\!]$ and $\psi(G(\sigma,x)) = G(\sigma,\psi(x))$ for all $\sigma \in \mathbb{C}^*$.*

3. *$G(y,x)$ satisfies*

$$(\mathrm{T}_{\mathrm{formal}}) \qquad G(y,G(z,x)) = G(yz,x),$$

and $G(\rho,G(\sigma,x)) = G(\rho\sigma,x)$ for all $\rho,\sigma\in\mathbb{C}^*$.

Proof. 1. Consider a family $(F(t,x))_{t\in\mathbb{C}}$ of formal power series $F(t,x) = \sum_{n\ge 1} c_n(t)x^n$, $t\in\mathbb{C}$. Comparison of coefficients of x^n in (C) yields for $n=2$

$$c_1(t)c_2(s) + c_2(t)c_1(s)^2 = c_1(s)c_2(t) + c_2(s)c_1(t)^2, \quad t,s\in\mathbb{C},$$

and in general for $n\ge 2$

$$\begin{aligned} c_1(t)&c_n(s) + c_n(t)c_1(s)^n \\ &+ \psi_n(c_2(t),\dots,c_{n-1}(t);c_1(s),\dots,c_{n-1}(s)) = c_1(s)c_n(t) + c_n(s)c_1(t)^n \\ &\qquad\qquad + \psi_n(c_2(s),\dots,c_{n-1}(s);c_1(t),\dots,c_{n-1}(t)), \quad t,s\in\mathbb{C}, \end{aligned}$$

with polynomials ψ_n. Therefore, $(F(t,x))_{t\in\mathbb{C}}$ satisfies (C), if and only if

$$\begin{aligned} (1)\quad & A_n(c_1(t),\dots,c_n(t);c_1(s),\dots,c_n(s)) \\ &\qquad = A_n(c_1(s),\dots,c_n(s);c_1(t),\dots,c_n(t)),\ t,s\in\mathbb{C}, \end{aligned}$$

for $n\ge 2$, where A_n are polynomials over $\mathbb{C}$. Due to the above form of A_n this is equivalent to

$$\begin{aligned} (2)\quad & (c_1(s)^n - c_1(s))c_n(t) \\ &\qquad = \Phi_n(c_1(t),\dots,c_{n-1}(t);c_1(s),\dots,c_n(s)), \quad s,t\in\mathbb{C}, \end{aligned}$$

for $n\ge 2$, where Φ_n are polynomials over $\mathbb{C}$.

If there exists some $s^*\in\mathbb{C}$ such that $c_1(s^*)\notin\mathcal{E}$, then $c_1(s^*)^n - c_1(s^*)\neq 0$ for $n\ge 2$. If for each $s\in\mathbb{C}$, $c_1(s)\in\mathcal{E}$, then $\operatorname{ord}(c_1(s))$ is not bounded from above since there are infinitely many different values of $c_1(s)$. Hence for each n there exists $s_n\in\mathbb{C}$ such that $c_1(s_n)^n - c_1(s_n)\neq 0$. This is also true in the first case for $s_n = s^*$.

From (2) it follows that

$$(3)\quad c_n(t) = (c_1(s_n)^n - c_1(s_n))^{-1}\Phi_n(c_1(t),\dots,c_{n-1}(t);c_1(s_n),\dots,c_n(s_n)),$$

$t\in\mathbb{C}$, for $n\ge 2$.

From (3) it follows by induction that there exists exactly one family $(P_n(y))_{n\geq 1}$ of polynomials $P_n \in \mathbb{C}[y]$, $P_1(y) = y$, with $c_n(t) = P_n(c_1(t))$, $t \in \mathbb{C}$, $n \geq 1$, thus

$$F(t,x) = c_1(t)x + \sum_{n\geq 2} P_n(c_1(t))x^n, \quad t \in \mathbb{C}.$$

Reformulation of (1) yields

$$\begin{aligned}(4)\quad & A_n(c_1(t), P_2(c_1(t)), \dots, P_n(c_1(t)); c_1(s), P_2(c_1(s)), \dots, P_n(c_1(s))) \\ &= A_n(c_1(s), P_2(c_1(s)), \dots, P_n(c_1(s)); c_1(t), P_2(c_1(t)), \dots, P_n(c_1(t))), \quad t, s \in \mathbb{C}.\end{aligned}$$

Define

$$G(y,x) := yx + \sum_{n\geq 2} P_n(y)x^n \in (\mathbb{C}[y])[\![x]\!].$$

Since in (4) $c_1(t)$ and $c_1(s)$ run independently through infinitely many values from (4) we obtain

$$\begin{aligned}(5)\quad & A_n(y, P_2(y), \dots, P_n(y); z, P_2(z), \dots, P_n(z)) \\ &\qquad = A_n(z, P_2(z), \dots, P_n(z); y, P_2(y), \dots, P_n(y))\end{aligned}$$

in $\mathbb{C}[y,z]$. By the meaning of A_n from (1) we get from (5) that G satisfies ($\mathrm{C}_{\mathrm{formal}}$). Substituting $y = 1$ and $z = \rho \notin \mathcal{E}$ in ($\mathrm{C}_{\mathrm{formal}}$) we get $G(1, G(\rho,x)) = G(\rho, G(1,x))$, thus by Remark 2 $G(1,x) = x$, and equivalently $P_n(1) = 0$, $n \geq 2$.

2. The series $\psi(x) = \rho x + d_2x^2 + \dots$ commutes with $F(t,x) = c_1(t)x + c_2(t)x^2 + \dots$ for all $t \in \mathbb{C}$ if and only if, analogously to (4),

$$\begin{aligned}& A_n(c_1(t), P_2(c_1(t)), \dots, P_n(c_1(t)); \rho, d_2, \dots, d_n) \\ &\qquad = A_n(\rho, d_2, \dots, d_n; c_1(t), P_2(c_1(t)), \dots, P_n(c_1(t))), \quad t \in \mathbb{C},\end{aligned}$$

holds true, where the polynomials A_n have the same meaning as in (4). By the method of formal functional equations we get

$$A_n(y, P_2(y), \dots, P_n(y); \rho, d_2, \dots, d_n) = A_n(\rho, d_2, \dots, d_n; y, P_2(y), \dots, P_n(y)),$$

which is the same as $G(y, \psi(x)) = \psi(G(y,x))$ and consequently

$$G(\sigma, \psi(x)) = \psi(G(\sigma, x))$$

for all $\sigma \in \mathbb{C}^*$. Therefore, also $\psi^{-1}(G(\sigma,x)) = G(\sigma,\psi^{-1}(x))$ for all $\sigma \in \mathbb{C}^*$. We choose a fixed $\sigma \notin \mathcal{E}$. Since $G(\sigma,G(\rho,x)) = G(\rho,G(\sigma,x))$ we have

$$G(\sigma,G(\rho,\psi^{-1}(x))) = G(\rho,G(\sigma,\psi^{-1}(x))) = G(\rho,\psi^{-1}(G(\sigma,x))),$$

i.e. $G(\rho,\psi^{-1}(x))$ commutes with $G(\sigma,x)$. But $G(\rho,\psi^{-1}(x)) = x + \ldots \in \Gamma_1$, hence by Remark 5 $G(\rho,\psi^{-1}(x)) = x$, thus $\psi(x) = G(\rho,x)$ which we have to prove.

3. Let $\rho,\sigma \in \mathbb{C}^*$. Then the series $G(\rho,x)$ and $G(\sigma,x)$ commute with each $G(\tau,x)$, $\tau \in \mathbb{C}^*$, and hence also $G(\rho,G(\sigma,x))$ commutes with each $G(\tau,x)$, $\tau \in \mathbb{C}^*$, or with each series $G(c_1(t),x)$, i.e. with $\mathcal{F}$. Hence by 2. there exists some $\omega \in \mathbb{C}^*$ such that $G(\omega,x) = G(\rho,G(\sigma,x))$. But $G(\rho,G(\sigma,x)) = \rho\sigma x + \sum_{n\geq 2} P_n(\omega)x^n$, and $G(\omega,x) = \omega x + \sum_{n\geq 2} P_n(\omega)x^n$, whence, $\omega = \rho\sigma$. This is true for all $\rho,\sigma \in \mathbb{C}^*$, therefore, by the method of formal functional equations ($\mathrm{T}_{\text{formal}}$) is true. (It is the formal translation equation which has been introduced in [3] to study iteration groups in Γ.) □

Proposition 10. *Let $\mathcal{F} = (F(t,x))_{t\in\mathbb{C}}$ be a family of commuting formal power series of type I and let $G(y,x) \in (\mathbb{C}[y])[\![x]\!]$ be associated with $\mathcal{F}$ according to Proposition 9. Then the following is true:*

1. *There exists a unique $H(x) = x + h_2x^2 + \ldots \in \mathbb{C}[\![x]\!]$ such that*

$$H(F(t,x)) = H(x)\frac{\partial}{\partial x} F(t,x), \quad t \in \mathbb{C}, \tag{AJ,H}$$

and in $(\mathbb{C}[y])[\![x]\!]$ we have

$$H(G(y,x)) = H(x)\frac{\partial}{\partial x} G(y,x). \tag{$\mathrm{AJ}_{\text{formal}}$,H}$$

These are the Aczél–Jabotinsky equations. Here

$$H(x) = \frac{\partial}{\partial y} G(y,x)|_{y=1}$$

is called the generator of both $(F(t,x))_{t\in\mathbb{C}}$ and $G(y,x)$.

2. *$G(y,x)$ and $H(x)$ also satisfy*

$$H(x)\frac{\partial}{\partial x} G(y,x) = y\frac{\partial}{\partial y} G(y,x). \tag{$\mathrm{PD}_{\text{formal}}$}$$

It is possible that (AJ,H) has more solutions in Γ than $(F(t,x))_{t\in\mathbb{C}}$.

Proof. 1. Since $G(y,x)$ satisfies ($\mathrm{C}_{\mathrm{formal}}$) we obtain

$$\frac{\partial}{\partial y} G(y,G(z,x)) = \frac{\partial}{\partial x} G(z,x)|_{x=G(y,x)} \frac{\partial}{\partial y} G(y,x).$$

Substituting $y = 1$ we have $H(G(z,x)) = H(x)\frac{\partial}{\partial x}G(z,x)$ for $H(x) = \frac{\partial}{\partial y} G(y,x)|_{y=1}$. This $H(x) = x + \ldots$ satisfies ($\mathrm{AJ}_{\mathrm{formal}}$,H). Consider $\tilde{H}(x) = x + \ldots$ satisfying $\tilde{H}(G(z,x)) = \tilde{H}(x)\frac{\partial}{\partial x} G(z,x)$. Then

$$\frac{\tilde{H}(G(z,x))}{H(G(z,x))} = \frac{\tilde{H}(x)\frac{\partial}{\partial x} G(z,x)}{H(x)\frac{\partial}{\partial x} G(z,x)},$$

thus $K(G(z,x)) = K(x)$ for $K(x) = \tilde{H}(x)/H(x) = 1+\ldots$. We claim $K(x) = 1$, since otherwise $K(x) = 1 + k_n x^n + \ldots$ with $n \geq 1$ and $k_n \neq 0$. But then $K(G(z,x))$ contains the summand $k_n z^n x^n$ which does not occur in $K(x)$. Consequently, $K = 1$, and $H = \tilde{H}$ is uniquely determined.

Moreover, $F(t,x)$ satisfies (AJ,H) for all $t \in \mathbb{C}$ if and only if

$$H(G(c_1(t),x)) = H(x)\frac{\partial}{\partial x} G(c_1(t),x)$$

holds true for all $t \in \mathbb{C}$, which means $H(G(y,x)) = H(x)\frac{\partial}{\partial x} G(y,x)$ in $(\mathbb{C}[y])[\![x]\!]$.

2. From ($\mathrm{T}_{\mathrm{formal}}$) we get, by differentiating with respect to z and putting $z = 1$, the linear partial differential equation ($\mathrm{PD}_{\mathrm{formal}}$). It was discussed in [3, Section 2.1]. □

In a similar way it can be shown that under the assumptions of Proposition 10 the series $G(y,x)$ and $H(x)$ satisfy

$$(\mathrm{D}_{\mathrm{formal}}) \qquad y\frac{\partial}{\partial y} G(y,x) = H(G(y,x)).$$

It was discussed in [3, Section 2.2].

A deeper study of families $\mathcal{F} = (F(t,x))_{t\in\mathbb{C}}$ of commuting power series of type I and in particular of maximal abelian subgroups of type I, is based on the above two Aczél–Jabotinsky equations.

Proposition 11. *Let $H(x) = x + h_2x^2 + \ldots \in \mathbb{C}[\![x]\!]$. Then the formal Aczél–Jabotinsky equation* ($\mathrm{AJ}_{\mathrm{formal}}$,H) *has exactly one solution of the form $G(y,x) = yx + \ldots \in (\mathbb{C}[y])[\![x]\!]$. It satisfies* ($\mathrm{C}_{\mathrm{formal}}$).

Moreover, the solutions $\Phi \in \Gamma$ of the Aczél–Jabotinsky equation

$$(\mathrm{AJ,H}) \qquad H(\Phi(x)) = H(x)\Phi'(x)$$

are given by $\Phi(x) = G(\rho,x)$, $\rho \in \mathbb{C}^*$, where H is the generator of G.

Proof. For the first assertion see [3, Theorem 12]. Introducing coefficients for $\Phi(x) = \sum_{n\geq 1} c_n x^n$, it is easy to deduce, that for any $c_1 \neq 0$ we can find uniquely determined c_n, $n \geq 2$, depending on c_1, which satisfy $H(\Phi(x)) = H(x)\Phi'(x)$. Since $G(c_1, x)$ satisfies (AJ,H), $\Phi(x) = G(c_1, x)$, $c_1 \in \mathbb{C}^*$. □

Theorem 12. *Let* $H(x) = x + h_2x^2 + \ldots \in \mathbb{C}[\![x]\!]$. *The set* $\mathcal{F}_H$ *of all solutions of* (AJ,H) *in* Γ *is a maximal abelian subgroup of* Γ.

Proof. The set of solutions of (AJ,H) is $\mathcal{F}_H = \{G(\rho, x) \mid \rho \in \mathbb{C}^*\}$ and the mapping $\mathcal{F}_H \to \mathbb{C}^*$, $G(\rho, x) \mapsto \rho$, $\rho \in \mathbb{C}^*$, is an isomorphism. This proves that $\mathcal{F}_H$ is an abelian subgroup of Γ. In order to show that it is a maximal abelian subgroup, consider some $\psi(x) = \sum_{n\geq 1} c_n x^n \in \Gamma$ commuting with all $\Phi \in \mathcal{F}_H$. Then by Proposition 11

$$G(\rho, \psi(x)) = \psi(G(\rho, x)), \quad \rho \in \mathbb{C}^*.$$

Comparing coefficients, this is equivalent to $B_n(\rho, c_1, \ldots, c_n) = 0$, $n \geq 1$, for certain polynomials B_n over $\mathbb{C}$. Since these polynomial relations hold true for infinitely many ρ they can be replaced by $B_n(y, c_1, \ldots, c_n) = 0$, $n \geq 1$, y an indeterminate. This is the method of formal functional equations, and we obtain

$$G(y, \psi(x)) = \psi(G(y, x)) \text{ in } (\mathbb{C}[y])[\![x]\!].$$

Therefore

$$\frac{\partial}{\partial y} G(y, \psi(x)) = \psi'(G(y, x)) \frac{\partial}{\partial y} G(y, x)$$

and by substituting $y = 1$ we obtain the Aczél–Jabotinsky equation

$$H(\psi(x)) = H(x)\psi'(x),$$

whence $\psi \in \mathcal{F}_H$. □

By conjugation we obtain the standard form of a family of commuting formal power series of type I:

Theorem 13. *To each family* $\mathcal{F}$ *of commuting formal power series of type I, in particular to each one-parameter group* $(F(t, x))_{t\in\mathbb{C}}$ *of type I,* $F(t, x) = c_1(t)x + \ldots$, $t \in \mathbb{C}$, *there exists a unique* $S \in \Gamma_1$ *such that*

$$F(t, x) = S^{-1}(c_1(t)S(x)), \quad t \in \mathbb{C}. \tag{ST}$$

For the associated formal series $G(y,x) \in (\mathbb{C}[y])[\![x]\!]$ *we also have*

$$G(y,x) = S^{-1}(yS(x)).$$

(ST) *is the standard form of the family* $\mathcal{F}$.

PROOF. Let $G(y,x) = yx + \sum_{n\geq 2} P_n(y)x^n$ be the formal object associated with $\mathcal{F}$ according to Proposition 9. Let $\rho \in \mathbb{C}^* \setminus \mathcal{E}$. Then by Remark 1, there exists a unique $S \in \Gamma_1$ such that $G(\rho,x) = S^{-1}(\rho S(X))$. By Proposition 9 ($\mathrm{C}_{\mathrm{formal}}$) holds true and substituting ρ for y and an arbitrary $\sigma \in \mathbb{C}^*$ for z we deduce $G(\rho, G(\sigma,x)) = G(\sigma, G(\rho,x))$, hence by Remark 2 $G(\sigma,x) = S^{-1}(\sigma S(x))$ for all $\sigma \in \mathbb{C}^*$, in particular $F(t,x) = S^{-1}(c_1(t)S(x))$, $t \in \mathbb{C}$, the standard form for $\mathcal{F}$. $G(\sigma,x) = S^{-1}(\sigma S(x))$ leads by performing the substitutions to $G(\sigma,x) = \sigma x + \sum_{n\geq 2} Q_n(\sigma)x^n$, $\sigma \in \mathbb{C}^*$, with polynomials Q_n. Hence we have $P_n(y) = Q_n(y)$, $n \geq 2$, in $\mathbb{C}[\![y]\!]$, which gives by the meaning of the polynomials Q_n that also $G(y,x) = S^{-1}(yS(x))$ in $(\mathbb{C}[y])[\![x]\!]$. □

REMARK 14. Since the standard form is often used we present two further possibilities to prove Theorem 13. If $G(y,x)$ is the formal series associated with $\mathcal{F}$, then we determine $S \in \Gamma_1$ such that the generator of $S(G(y, S^{-1}(x)))$ is x which is achieved by solving a Briot–Bouquet differential equation (see [15, Section 5.2], [9, Section 11.1], and [10, Section 12.6]). This implies that $S(G(y,S^{-1}(x))) = yx$, the standard form of $G(y,x)$.

Another possibility to prove Theorem 13 is based on the linear partial differential equation ($\mathrm{PD}_{\mathrm{formal}}$) for $G(y,x)$. If we substitute $G(y,x) = S^{-1}(yS(x))$ into ($\mathrm{PD}_{\mathrm{formal}}$), then we arrive at a Briot–Bouquet differential equation for S. Together with a uniqueness result for ($\mathrm{PD}_{\mathrm{formal}}$) this leads to the standard form for $G(y,x)$.

If the multiplier ρ of a formal series $F \in \Gamma$ is not a complex root of one, then F can be linearized by Remark 1, i.e. there exists some $S \in \Gamma_1$ such that $F(x) = S(\rho S^{-1}(x))$. Consider some $\lambda \in \mathbb{C}$ such that $\mathrm{e}^{\lambda} = \rho$, then $(\theta(t,x))_{t\in\mathbb{C}}$, $\theta(t,x) = S(\mathrm{e}^{\lambda t}S^{-1}(x))$, $t \in \mathbb{C}$, is an analytic iteration group of type I, satisfying $\theta(0,x) = x$ and $\theta(1,x) = F(x)$. We call it an embedding of F into an analytic iteration group.

PROPOSITION 15. *Each series* $F(x) = \rho x + c_2x^2 + \dots$, $\rho \neq 0$, $\rho \notin \mathcal{E}$, *is contained in exactly one maximal abelian subgroup of* Γ *of type I. It can be constructed as the embedding of* F *into an analytic iteration group* $(\theta(t,x))_{t\in\mathbb{C}}$ *such that* $\theta(t,x) = \mathrm{e}^{\lambda t}x + \dots$ *where* $\mathrm{e}^{\lambda} = \rho$.

PROOF. There exists exactly one $S \in \Gamma_1$ such that $S \circ F \circ S^{-1} = L_{\rho}$. According to Remark 2 the series $\hat{F}(x) = \sigma x + \dots \in \Gamma$ commutes with F if and only if $\hat{F} = S \circ L_\sigma \circ S^{-1}$. Thus $\{S \circ L_\sigma \circ S^{-1} \mid \sigma \in \mathbb{C}^*\}$ is the unique maximal abelian subgroup of Γ containing F. It can be parameterized as an analytic embedding $(\theta(t,x))_{t\in\mathbb{C}}$ of F. □

PROPOSITION 16. *Let $\Phi(x) = \sigma x + \dots$, $\sigma = \mathrm{e}^{2\pi\mathrm{i}\ell/m} \in \mathcal{E}$, $\gcd(\ell,m) = 1$, $m \geq 2$. Then the following holds true:*

1. *Φ is contained in a family of commuting series of type I if and only if Φ is linearizable, i.e. there exists $S \in \Gamma_1$ such that $\Phi(x) = S^{-1}(\sigma S(x))$.*
2. *If $\Phi(x) = S^{-1}(\sigma S(x))$, then the families of type I containing Φ are given by $\mathcal{F} = \{T^{-1}(\rho T(x)) \mid \rho \in I\}$ where I is an infinite subset of $\mathbb{C}^*$ containing σ and $T = U \circ S$, with*
$$U(x) = x + \sum_{n\geq 1} u_{nm+1}x^{nm+1}.$$
$\mathcal{F}$ is maximal if and only if $I = \mathbb{C}^*$.
3. *Let $\Phi(x) = S^{-1}(\sigma S(x))$, $S \in \Gamma_1$. Then Φ belongs to a family of type I if the generator H of $\mathcal{F}$ has the form*
$$H(x) = \frac{H^*(S(x))}{S'(x)}, \quad \text{where } H^*(x) = x + \sum_{n\geq 1} h^*_{nm+1}x^{nm+1} \in \Gamma_1.$$
If the family $\mathcal{F}$ has the generator $H(x) = \frac{H^*(S(x))}{S'(x)}$ with $H^*(x)$ as above and if $\mathcal{F}$ contains a series with multiplier σ (which is true if $\mathcal{F}$ is maximal) then Φ belongs to $\mathcal{F}$.

This means that if Φ is linearizable, then it can be embedded into a continuum of maximal abelian subgroups of type I, namely into each group $\{S^{-1}(\rho S(x)) \mid \rho \in \mathbb{C}^*\}$, if $\Phi(x) = S^{-1}(\sigma S(x))$.

PROOF. 1. If $\Phi(x) = S^{-1}(\sigma S(x))$, $S \in \Gamma_1$, then $\Phi \in \{S^{-1}(\rho S(x)) \mid \rho \in \mathbb{C}^*\} =: \mathcal{F}$ which is a maximal family of commuting formal power series of type I.

Assume that $\Phi \in \mathcal{F}$, where $\mathcal{F}$ is a family of commuting formal power series of type I. Then by Theorem 13 $\mathcal{F}$ has the standard form $\{S^{-1}(c_1(t)S(x)) \mid t \in \mathbb{C}\}$ with $S \in \Gamma_1$, whence, $\Phi(x) = S^{-1}(c_1(t_0)S(x))$ for some $t_0 \in \mathbb{C}$ such that $c_1(t_0) = \sigma$.

2. Assume that $\Phi(x) = S^{-1}(\sigma S(x))$, $S \in \Gamma_1$, $\sigma = \mathrm{e}^{2\pi\mathrm{i}\ell/m}$, $\gcd(\ell,m) = 1$, $m \geq 2$, and $U(x) = x + \sum_{n\geq 2} u_n x^n$. Then simple computations show that $U(\sigma x) = \sigma U(x)$ if and only if $U(x) = x + \sum_{n\geq 1} u_{nm+1}x^{nm+1}$. Assume that

U is of this form, then $U(\sigma x) = \sigma U(x)$, or $L_\sigma = U^{-1} \circ L_\sigma \circ U$, hence with $T = U \circ S$ we obtain

$$T^{-1}(\sigma T(x)) = (S^{-1} \circ U^{-1} \circ L_\sigma \circ U \circ S)(x) = S^{-1}(\sigma S(x)) = \Phi(x).$$

Hence $\mathcal{F} = \{T^{-1}(\rho T(x)) \mid \rho \in I\}$ with infinite $I \subseteq \mathbb{C}^*$ and $\sigma \in I$ is a family of commuting formal power series of type I containing Φ.

Assume conversely, that $\mathcal{F}$ is a family of commuting formal power series of type I containing $\Phi(x) = S^{-1}(\sigma S(x))$. By Theorem 13 $\mathcal{F}$ has the standard form $\mathcal{F} = \{T^{-1}(\rho T(x)) \mid \rho \in I\}$ for some $T \in \Gamma_1$. The family $\mathcal{F}$ contains a series with multiplier σ, namely Φ, and this is exactly the series $T^{-1}(\sigma T(x))$. So we have $T^{-1}(\sigma T(x)) = S^{-1}(\sigma S(x))$, or equivalently, $(S \circ T^{-1}) \circ L_\sigma \circ (S \circ T^{-1})^{-1} = L_\sigma$, hence $T = U \circ S$ for some $U(x) = x + \sum_{n\geq 1} u_{nm+1}x^{nm+1}$.

3. Let $\Phi(x) = S^{-1}(\sigma S(x))$ for some $S \in \Gamma_1$. If Φ belongs to a family of commuting formal power series of type I with generator H, then the Aczél–Jabotinsky equation reads as $H(S^{-1}(\sigma S(x))) = H(x)\frac{d}{dx}S^{-1}(\sigma S(x))$ which is the same as $H(S^{-1}(\sigma S(x))) = H(x)(S^{-1})'(\sigma S(x))\sigma S'(x)$. Substituting $S^{-1}(x)$ for x we obtain

$$H(S^{-1}(\sigma x)) = H(S^{-1}(x))(S^{-1})'(\sigma x)\sigma S'(S^{-1}(x))$$

or

$$\frac{H(S^{-1}(\sigma x))}{(S^{-1})'(\sigma x)} = \sigma \frac{H(S^{-1}(x))}{(S^{-1})'(x)}, \quad \text{i.e.} \quad \frac{H \circ S^{-1}}{(S^{-1})'}(\sigma x) = \sigma \frac{H \circ S^{-1}}{(S^{-1})'}(x),$$

since $S'(S^{-1}(x)) \cdot (S^{-1})'(x) = 1$. Because of the fact that $\hat{H}(x) := \frac{H \circ S^{-1}}{(S^{-1})'}(x) \in \Gamma_1$ satisfies $\hat{H}(\sigma x) = \sigma \hat{H}(x)$, it is of the form $H^*(x) = x + \sum_{n\geq 1} h^*_{nm+1}x^{nm+1}$ or, equivalently, $H(x) = \frac{H^*(S(x))}{S'(x)}$. □

Any $\Phi(x) = \sigma x + \dots$, $\sigma \in \mathcal{E}$, which cannot be linearized does not belong to a maximal family of commuting power series of type I.

A direct consequence of Proposition 10 and Theorem 12 is

PROPOSITION 17. *A family $(F(t,x))_{t\in\mathbb{C}}$ of commuting formal power series of type I is contained in exactly one maximal abelian subgroup $\mathcal{F}$ of Γ of type I. If $H(x) = x + h_2x^2 + \dots$ is the generator of $(F(t,x))_{t\in\mathbb{C}}$, then $\mathcal{F}$ is the set of solutions $\Phi \in \Gamma$ of* (AJ,H).

In other words, the maximal abelian subgroup of Γ containing the family $\mathcal{F} = (F(t,x))_{t\in\mathbb{C}}$ of commuting formal power series of type I is $\{G(\rho,x) \mid \rho \in \mathbb{C}^*\}$ where $G(y,x) = yx + \ldots \in (\mathbb{C}[y])[\![x]\!]$ is associated with $\mathcal{F}$ as in Proposition 9. Since analytic iteration groups of type I have a coefficient function $c_1(s) = \mathrm{e}^{\lambda s}$ for some $\lambda \neq 0$ we obtain the second assertion of

Proposition 18. 1. *A family* $(F(t,x))_{t\in\mathbb{C}}$, $F(t,x) = c_1(t)x + \ldots$, $t \in \mathbb{C}$, *of type I is a maximal abelian subgroup of* Γ *of type I if and only if the image* $\mathrm{Im}(c_1) = \mathbb{C}^*$.

2. *Each analytic iteration group of type I is a maximal family of commuting power series.*

There exists a universal representation of the coefficients of a solution of $(\mathrm{AJ}_{\mathrm{formal}},\mathrm{H})$. From the second part of [3, Theorem 12] we deduce

Theorem 19. *For each* $n \geq 2$ *there is a polynomial* $\tilde{P}_n(y,u_2,\ldots,u_n) \in \mathbb{C}[y,u_2,\ldots,u_n]$ *such that the solution* $G(y,x) \in (\mathbb{C}[y])[\![x]\!]$ *of* $(\mathrm{AJ}_{\mathrm{formal}},\mathrm{H})$ *with* $H(x) = x + h_2x^2 + \ldots$ *is given by*

$$G(y,x) = yx + \sum_{n\geq 2} \tilde{P}_n(y,h_2,\ldots,h_n)x^n,$$

i.e. the polynomials from Proposition 9 can be written as $P_n(y) = \tilde{P}_n(y,h_2,\ldots,h_n)$, $n \geq 2$.

We summarize the main results concerning maximal abelian subgroups of Γ in

Theorem 20. 1. *To each maximal abelian subgroup* $\mathcal{F}$ *of* Γ *of type I there exists a unique* $H(x) = x + h_2x^2 + \ldots \in \mathbb{C}[\![x]\!]$ *(the generator of* $\mathcal{F}$*) such that* $\mathcal{F}$ *is the set of solutions* $\Phi \in \Gamma$ *of* (AJ,H), *i.e.*

$$H(x)\Phi'(x) = H(\Phi(x)). \qquad \text{(AJ, H)}$$

2. *Conversely, if* $H(x) = x + h_2x^2 + \ldots \in \mathbb{C}[\![x]\!]$ *is given, then the set of solutions* $\Phi \in \Gamma$ *of* (AJ,H) *is a maximal abelian subgroup of* Γ *of type I. The generator of* $\mathcal{F}$ *is* H.
3. *To each maximal abelian subgroup* $\mathcal{F}$ *of* Γ *of type I there exists a unique* $S \in \Gamma_1$ *such that*

$$\mathcal{F} = \{S^{-1}(\rho S(x)) \mid \rho \in \mathbb{C}^*\}.$$

The generator of $\{\rho x \mid \rho \in \mathbb{C}^*\}$ *is* x.

4. *Each maximal abelian subgroup $\mathcal{F}$ of Γ of type I can be parameterized as an iteration group of type I, i.e. it has the form $\mathcal{F} = (F(t,x))_{t\in\mathbb{C}}$, where*

$$F(t,x) = c_1(t) + \sum_{n\geq 2} \tilde{P}_n(c_1(t), h_2, \ldots, h_n)x^n,$$

where c_1 is a generalized exponential function with $c_1(\mathbb{C}) = \mathbb{C}^*$, and for $n \geq 2$ the $\tilde{P}_n(y, u_2, \ldots, u_n) \in \mathbb{C}[y, u_2, \ldots, u_n]$, are (universal) polynomials. The series $H(x) = x + h_2x^2 + \ldots$ is the generator of $\mathcal{F}$.
5. *Each series $F(x) = \rho x + c_2x^2 + \ldots$, $\rho \neq 0$, $\rho \notin \mathcal{E}$, belongs to exactly one maximal abelian subgroup of Γ of type I. It can be constructed as the embedding of F into an analytic iteration group.*
6. *A series $F(x) = \sigma x + \ldots$, $\sigma \in \mathcal{E}$, belongs to a maximal abelian subgroup of Γ of type I if and only if it can be linearized, i.e. $F(x) = S^{-1}(\sigma S(x))$ for some $S \in \Gamma$. If so, there is a continuum of maximal abelian subgroups of Γ of type I containing F, each one determined by the conjugating series S as $\{S^{-1}(\rho S(x)) \mid \rho \in \mathbb{C}^*\}$.*
7. *Each family $(F(t,x))_{t\in\mathbb{C}}$ of commuting formal power series of type I is contained in exactly one maximal abelian subgroup $\mathcal{F}$ of Γ of type I. If $H(x) = x + h_2x^2 + \ldots$ is the generator of $(F(t,x))_{t\in\mathbb{C}}$, then $\mathcal{F}$ is the set of solutions $\Phi \in \Gamma$ of* (AJ,H).
8. *To each family $(F(t,x))_{t\in\mathbb{C}}$, $F(t,x) = \sum_{n\geq 1} c_n(t)x^n$, $t \in \mathbb{C}$, of commuting formal power series of type I there exists a power series $G(y,x) = yx + \sum_{n\geq 2} P_n(y)x^n \in (\mathbb{C}[y])[\![x]\!]$ such that $c_n(t) = P_n(c_1(t))$, $t \in \mathbb{C}$, $n \geq 2$, and both*

$$G(y, G(z,x)) = G(z, G(y,x)) \tag{$\mathrm{C}_{\mathrm{formal}}$}$$

and

$$G(y, G(z,x)) = G(yz, x) \tag{$\mathrm{T}_{\mathrm{formal}}$}$$

hold true in $(\mathbb{C}[y,z])[\![x]\!]$. Furthermore, if $H(x) = \frac{\partial}{\partial y} G(y,x)|_{y=1}$, then we have in $(\mathbb{C}[y])[\![x]\!]$

$$H(G(y,x)) = H(x) \cdot \frac{\partial}{\partial x} G(y,x) \tag{$\mathrm{AJ}_{\mathrm{formal}}, \mathrm{H}$}$$

and

$$H(x) \frac{\partial}{\partial x} G(y,x) = y \frac{\partial}{\partial y} G(y,x) \tag{$\mathrm{PD}_{\mathrm{formal}}$}$$

hold true. The series $G(y,x)$ is the unique solution of $(\mathrm{AJ}_{\mathrm{formal}},\mathrm{H})$ *under the assumption that $G(y,x) = yx + \ldots$, and it is the unique solution of* $(\mathrm{PD}_{\mathrm{formal}})$ *under the assumption that $G(1,x) = x$.*

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