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ON THE RADON-NIKODYM PROPERTY FOR VECTOR MEASURES AND EXTENSIONS OF TRANSFUNCTIONS

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Abstract. If $(\mu_n)_{n=1}^{\infty}$ are positive measures on a measurable space (X, Σ) and $(v_n)_{n=1}^{\infty}$ are elements of a Banach space $\mathbb{E}$ such that $\sum_{n=1}^{\infty} \|v_n\| \mu_n(X) < \infty$, then $\omega(S) = \sum_{n=1}^{\infty} v_n \mu_n(S)$ defines a vector measure of bounded variation on (X, Σ). We show $\mathbb{E}$ has the Radon-Nikodym property if and only if every $\mathbb{E}$-valued measure of bounded variation on (X, Σ) is of this form. This characterization of the Radon-Nikodym property leads to a new proof of the Lewis-Stegall theorem.

We also use this result to show that under natural conditions an operator defined on positive measures has a unique extension to an operator defined on $\mathbb{E}$-valued measures for any Banach space $\mathbb{E}$ that has the Radon-Nikodym property.

1. Introduction

A Banach space $\mathbb{E}$ has the *Radon-Nikodym property* with respect to a measure space (X, Σ, μ) if for every $\mathbb{E}$-valued measure ω on (X, Σ) of bounded variation that is absolutely continuous with respect to μ there exists a Bochner integrable function f on (X, Σ, μ) such that $\omega(S) = \int_S f d\mu$ for every $S \in \Sigma$. We say that $\mathbb{E}$ has the *Radon-Nikodym property* if $\mathbb{E}$ has the Radon-Nikodym property with respect to every finite measure space. The Radon-Nikodym property plays an important role in the theory of Banach spaces [3, 4], and note that not every Banach space has this property (see [4]).

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If $(\mu_n)_{n=1}^{\infty}$ are positive measures on a measurable space (X, Σ) and $(v_n)_{n=1}^{\infty} \in \mathbb{E}$ are such that $\sum_{n=1}^{\infty} \|v_n\| \mu_n(X) < \infty$, then $\omega(S) = \sum_{n=1}^{\infty} v_n \mu_n(S)$ defines a vector measure of bounded variation on (X, Σ). We will show that every $\mathbb{E}$-valued measure of bounded variation is of this form if and only if $\mathbb{E}$ has the Radon-Nikodym property.

The proposed characterization of the Radon-Nikodym property is closely related to the one in the Lewis-Stegall theorem [4, 5, 9], which shows that the Radon-Nikodym property for $\mathbb{E}$ with respect to (X, Σ, μ) is equivalent to every bounded linear operator from $L_1(\mu)$ to $\mathbb{E}$ having a particular factorization.

The decomposition of measures in our characterization of the Radon-Nikodym property leads us to an extension theorem for transfunctions. By a transfunction (see [7] and [1]) we mean a map between sets of measures on measurable spaces. More precisely, if (X, Σ_X) and (Y, Σ_Y) are measurable spaces and $\mathcal{M}(X, \Sigma_X, \mathbb{R}^+)$ and $\mathcal{M}(Y, \Sigma_Y, \mathbb{R}^+)$ are the sets of finite positive measures on Σ_X and on Σ_Y, respectively, by a *transfunction* from (X, Σ_X) to (Y, Σ_Y) we mean a map $\Phi \colon \mathcal{M}(X, \Sigma_X, \mathbb{R}^+) \to \mathcal{M}(Y, \Sigma_Y, \mathbb{R}^+)$. If $f \colon (X, \Sigma_X) \to (Y, \Sigma_Y)$ is a measurable function, then $\Phi_f(\mu)(B) = \mu(f^{-1}(B))$, for $\mu \in \mathcal{M}(X, \Sigma_X, \mathbb{R}^+)$ and $B \in \Sigma_Y$, defines a transfunction from $\mathcal{M}(X, \Sigma_X, \mathbb{R}^+)$ to $\mathcal{M}(Y, \Sigma_Y, \mathbb{R}^+)$. Properties of transfunctions related to functions are discussed in [1].

We are interested in transfunctions as a generalization of a function from X to Y. Instead of mapping a point $x \in X$ to a point $y \in Y$ a transfunction can be thought of as mapping a probability distribution of the input to a probability distribution of the output. Connections of transfunctions to plans, Markov operators and optimal transport are discussed in [2]. Another interpretation of a transfunction could be a change in a population distributed in X. The total population can increase or decrease and its distribution in X can change. A transfunction captures all these changes.

The definition of transfunctions makes sense if finite positive measures are replaced by vector valued measures of bounded variation. In the last section of this note we define extensions of transfunctions to vector measures and discuss the question of uniqueness of such extensions.

We have two reasons to consider extensions of transfunctions to signed measures and vector measures. First, for some applications it is more natural or even necessary to use signed measures or vector measures. Second, by extending the domain of a transfunction to a vector space we are able to use tools from functional analysis.

2. Measures with values in a Banach space with the Radon-Nikodym property

In this note we use the same symbol to denote a subset of X and the characteristic function of that set, that is, if $A \subset X$ we will write

$$A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

By μ_A we denote the restriction of μ to A, that is, $\mu_A(S) = \mu(S \cap A)$.

Let X be a nonempty set, Σ a σ-algebra of subsets of X, and let $(\mathbb{E}, \|\cdot\|)$ be a Banach space. By the *variation* of a set function $\mu\colon \Sigma \to \mathbb{E}$ we mean the set function $|\mu|\colon \Sigma \to [0,\infty]$ defined by

$$|\mu|(A) = \sup \left\{ \sum_{B\in\pi} ||\mu(B)|| : \pi \subset \Sigma \text{ is a finite partition of } A \right\}.$$

Note that $|\mu|(A) \geq \|\mu(A)\|$ for any set $A \in \Sigma$. If $\|\mu\| = |\mu|(X) < \infty$, then we say that μ is of *bounded variation.* A σ-additive set function of bounded variation will be called an $\mathbb{E}$-valued measure or simply a vector measure.

Let (X, Σ, μ) be a measure space and let $\mathbb{E}$ be a Banach space. We use the following definition of Bochner integrable functions (see [6] or [8]):

Definition 2.1. A function $f\colon X \to \mathbb{E}$ is called *Bochner integrable* if there are sets $(A_n)_{n=1}^{\infty} \in \Sigma$ and vectors $(v_n)_{n=1}^{\infty} \in \mathbb{E}$ such that $\sum_{n=1}^{\infty} \|v_n\|\, \mu(A_n) < \infty$ and $f(x) = \sum_{n=1}^{\infty} v_n A_n(x)$ for every $x \in X$ for which $\sum_{n=1}^{\infty} \|v_n\| A_n(x) < \infty$.

If for some $f\colon X \to \mathbb{E}$, $(A_n)_{n=1}^{\infty} \in \Sigma$, and $(v_n)_{n=1}^{\infty} \in \mathbb{E}$, both conditions are satisfied, we write $f \simeq \sum_{n=1}^{\infty} v_n A_n$. If $f \simeq \sum_{n=1}^{\infty} v_n A_n$, then we define $\int f d\mu = \sum_{n=1}^{\infty} v_n \mu(A_n)$.

Note that we are using a nonstandard definition of Bochner integrability. However, assuming our definition, the partial sums $\sum_{n=1}^{N} v_n A_n(x)$ are simple functions, and

$$\int_X \|f\|\, d\mu \leq \int_X \sum_{n=1}^{\infty} \|v_n\|\, A_n(\cdot) d\mu \leq \sum_{n=1}^{\infty} \|v_n\|\, \mu(A_n) < \infty,$$

so f is Bochner integrable in the standard sense [4].

Conversely, every Bochner integrable function can be approximated by simple functions giving the sequences $(A_n)_{n=1}^{\infty}$, $(v_n)_{n=1}^{\infty}$ in our definition.

If $f\colon X \to \mathbb{E}$ is a Bochner integrable function on a measure space (X, Σ, μ), then $\omega(S) = \int_S f d\mu$ defines a vector measure of bounded variation on (X, Σ). Not every vector measure is of this form (see [4]).

THEOREM 2.2. *A Banach space $\mathbb{E}$ has the Radon-Nikodym property with respect to a measure space (X, Σ, μ) if and only if every $\mathbb{E}$-valued measure ω on (X, Σ) of bounded variation that is absolutely continuous with respect to μ is of the form*

$$\omega(S) = \sum_{n=1}^{\infty} v_n \mu_n(S)$$

where $(\mu_n)_{n=1}^{\infty}$ are positive measures on (X, Σ) that are absolutely continuous with respect to μ and $(v_n)_{n=1}^{\infty} \in \mathbb{E}$ are such that $\sum_{n=1}^{\infty} \|v_n\| \mu_n(X) < \infty$.

PROOF. Let ω be a $\mathbb{E}$-valued measure of bounded variation on (X, Σ) that is absolutely continuous with respect to μ. If $\mathbb{E}$ has the Radon-Nikodym property, there exists a Bochner integrable function f on (X, Σ, μ) such that $\omega(S) = \int_S f d\mu$ for every $S \in \Sigma$. Let $(A_n)_{n=1}^{\infty} \in \Sigma$ and $(v_n)_{n=1}^{\infty} \in \mathbb{E}$ be such that $f \simeq \sum_{n=1}^{\infty} v_n A_n$ in (X, Σ, μ). Then

$$\int_S f d\mu = \sum_{n=1}^{\infty} v_n \mu(S \cap A_n).$$

If we define $\mu_n(S) = \mu(S \cap A_n)$, then

$$\omega(S) = \int_S f d\mu = \sum_{n=1}^{\infty} v_n \mu_n(S).$$

Now let ω be an $\mathbb{E}$-valued measure on (X, Σ) of bounded variation that is absolutely continuous with respect to μ. If

$$\omega = \sum_{n=1}^{\infty} v_n \mu_n$$

where $(\mu_n)_{n=1}^{\infty}$ are positive measures on (X, Σ) that are absolutely continuous with respect to μ and $(v_n)_{n=1}^{\infty} \in \mathbb{E}$ are such that $\sum_{n=1}^{\infty} \|v_n\| \mu_n(X) < \infty$, we define $f_n = \frac{d\mu_n}{d\mu}$ for $n \in \mathbb{N}$. Since

$$\sum_{n=1}^{\infty} \|v_n f_n\|_1 = \sum_{n=1}^{\infty} \|v_n\| \|f_n\|_1 = \sum_{n=1}^{\infty} \|v_n\| \mu_n(X) < \infty,$$

where $\|\cdot\|_1$ denotes the L^1-norm with respect to μ, the series $\sum_{n=1}^{\infty} v_n f_n$ converges to a Bochner integrable function f on (X, Σ, μ) and we have

$$\omega(S) = \sum_{n=1}^{\infty} v_n \mu_n(S) = \sum_{n=1}^{\infty} v_n \int_S f_n d\mu = \int_S \sum_{n=1}^{\infty} v_n f_n d\mu = \int_S f d\mu,$$

for every $S \in \Sigma$. □

REMARK 2.3. The previous theorem can also be derived using tensor product techniques. We refer the interested reader to [10] for the necessary tools.

Corollary 2.4. *A Banach space $\mathbb{E}$ has the Radon-Nikodym property if and only if every $\mathbb{E}$-valued measure ω of bounded variation on any measurable space (X, Σ) is of the form $\omega = \sum_{n=1}^{\infty} v_n \mu_n$, where $(\mu_n)_{n=1}^{\infty}$ are positive measures on (X, Σ) that are absolutely continuous with respect to $|\omega|$ and $(v_n)_{n=1}^{\infty} \in \mathbb{E}$ are such that $\sum_{n=1}^{\infty} \|v_n\| \mu_n(X) < \infty$.*

From Theorem 2.2 we can obtain a simple proof of the Lewis-Stegall theorem.

THEOREM 2.5 (Lewis-Stegall). *A Banach space $\mathbb{E}$ has the Radon-Nikodym property with respect to (X, Σ, μ) if and only if every bounded linear operator $T\colon L_1(\mu) \to \mathbb{E}$ admits a factorization $T = T_1 T_2$, where $T_1\colon \ell_1 \to \mathbb{E}$, $T_2\colon L_1(\mu) \to \ell_1$ are continuous linear operators and T_2 is positive.*

PROOF. Suppose $\mathbb{E}$ has the Radon-Nikodym property with respect to (X, Σ, μ), and let $T\colon L_1(\mu) \to \mathbb{E}$ be a bounded linear operator. By the Riesz Representation Theorem [4], there is a $g \in L_\infty(\mu, \mathbb{E})$ such that $T(f) = \int f g d\mu$ for every $f \in L_1(\mu)$. We then define the measure $\omega(S) = \int_S g d\mu$ so that $TS(\cdot) = \int S(\cdot) g d\mu = \omega(S)$ for every chracteristic function $S(\cdot)$. Then by Theorem 2.2, $\omega = \sum_{n=1}^{\infty} v_n \mu_n$ where $(\mu_n)_{n=1}^{\infty}$ are positive measures on (X, Σ) that are absolutely continuous with respect to μ and $(v_n)_{n=1}^{\infty} \in \mathbb{E}$ are such that $\sum_{n=1}^{\infty} \|v_n\| \mu_n(X) < \infty$. Then for any simple function $f = \sum_{k=1}^{K} \alpha_k S_k(\cdot)$

$$\begin{aligned} Tf &= \sum_{k=1}^{K} \alpha_k \omega(S_k) = \sum_{k=1}^{K} \alpha_k \sum_{n=1}^{\infty} v_n \mu_n(S_k) \\ &= \sum_{n=1}^{\infty} v_n \sum_{k=1}^{K} \alpha_k \mu_n(S_k) = \sum_{n=1}^{\infty} \frac{v_n}{\|v_n\|} \left(\|v_n\| \int f d\mu_n \right), \end{aligned}$$

so we have the required factorization by defining

$$T_2(f) = \left(\|v_n\| \int f d\mu_n \right)_{n=1}^{\infty} \quad \text{and} \quad T_1((\alpha_n)_{n=1}^{\infty}) = \sum_{n=1}^{\infty} \alpha_n \frac{v_n}{\|v_n\|}.$$

Since simple functions are dense in $L_1(\mu)$, the result follows.

Conversely, suppose we have a factorization of the stated form for every bounded linear operator, and let ω be an $\mathbb{E}$-valued measure of bounded variation on (X, Σ) that is absolutely continuous with respect to μ. Then we have the factorized bounded linear operator $T = T_1 T_2$ that is defined by its action on characteristic functions: $TS(\cdot) = \omega(S)$, and

$$\omega(S) = T_1 T_2 S(\cdot) = \sum_{n=1}^{\infty} e_n^*(T_2(S(\cdot)))T_1(e_n),$$

where $(e_n)_{n=1}^{\infty}$ is the standard basis of ℓ_1 and $(e_n^*)_{n=1}^{\infty}$ are the coordinate functionals. Using the boundedness of T_2, the positive measures $\mu_n(S) = e_n^*(T_2(S(\cdot)))$ satisfy

$$\sum_{n=1}^{\infty} \mu_n(X) < |\omega|(X) < \infty.$$

Combining this with the boundedness of T_1 and letting $v_n = T_1(e_n)$, we have

$$\sum_{n=1}^{\infty} \|v_n\| \mu_n(X) < \infty$$

which verifies the conditions of Theorem 2.2. Hence $\mathbb{E}$ has the Radon-Nikodym property. □

3. Extension of transfunctions to vector valued measures

In this section we consider extensions of transfunctions to vector measures. We start with three technical lemmas that will be used in the proof of the main result.

Lemma 3.1. *Let $\mu_1, \dots, \mu_n$ be finite positive measures on (X, Σ). Then there exists a measure μ on (X, Σ) such that for every $\varepsilon > 0$ there is a finite partition $\pi \subset \Sigma$ of X such that for $i = 1, \dots, n$ we have*

$$\mu_i = \sum_{S \in \pi} \alpha_{i,S} \mu_S + \kappa_i$$

where $\alpha_{i,S} \geq 0$ and $\kappa_1, \dots, \kappa_n$ are positive measures on (X, Σ) such that

$$\kappa_1(X) + \cdots + \kappa_n(X) < \varepsilon. \tag{3.1}$$

In particular, we may define μ to be $\sum_{i=1}^n \mu_i$.

Proof. Let $\varepsilon > 0$. Notice that the measures $\mu_1, \dots, \mu_n$ are absolutely continuous with respect to μ as defined above. Consider the Radon-Nikodym derivatives f_i of μ_i, that is, $\mu_i(B) = \int_B f_i d\mu$ for all $B \in \Sigma$. Since each f_i is non-negative and integrable with respect to μ, there are simple functions $\sum_{A \in \pi_i} \alpha_{i,A} A(x)$, with respect to finite partitions π_i of X, such that $\alpha_{i,A} \geq 0$, $\sum_{A \in \pi_i} \alpha_{i,A} A(x) \leq f_i$, and

$$\int_X \left(f_i - \sum_{A \in \pi_i} \alpha_{i,A} A(x) \right) d\mu < \frac{\varepsilon}{n}.$$

Now define the common refinement of the partitions π_i to be π, and define the measures κ_i by the equation

$$\kappa_i(B) = \int_B \left(f_i - \sum_{A \in \pi_i} \alpha_{i,A} A(x) \right) d\mu$$

for all $B \in \Sigma$. Notice that each simple function with respect to π_i is also a simple function with respect to π, that is,

$$\sum_{A \in \pi_i} \alpha_{i,A} A(x) = \sum_{S \in \pi} \alpha_{i,S} S(x)$$

where $\alpha_{i,A} = \alpha_{i,S}$ if $S \subseteq A$. Consequently, for every $B \in \Sigma$, we have

$$\begin{aligned} \mu_i(B) &= \int_B \sum_{S \in \pi} \alpha_{i,S} S(x) d\mu + \int_B \left(f_i - \sum_{S \in \pi} \alpha_{i,S} S(x) \right) d\mu \\ &= \sum_{S \in \pi} \alpha_{i,S} \mu_S(B) + \kappa_i(B), \end{aligned}$$

and the κ_i's were constructed to satisfy (3.1). $\square$

Let $\Phi\colon \mathcal{M}(X,\Sigma_X,\mathbb{R}^+) \to \mathcal{M}(Y,\Sigma_Y,\mathbb{R}^+)$ be a transfunction. We say that Φ is bounded if $\|\Phi(\mu)\| \le C\|\mu\|$ for some $C>0$ and all $\mu \in \mathcal{M}$ and we define

$$\|\Phi\| = \inf\{C : \|\Phi(\mu)\| \le C\|\mu\| \text{ for all } \mu \in \mathcal{M}(X,\Sigma_X,\mathbb{R}^+)\},$$

so that we have $\|\Phi(\mu)\| \le \|\Phi\|\|\mu\|$ for all $\mu \in \mathcal{M}(X,\Sigma_X,\mathbb{R}^+)$. We say that Φ is strongly additive if $\Phi(\mu_1+\mu_2) = \Phi(\mu_1)+\Phi(\mu_2)$ for all $\mu_1,\mu_2 \in \mathcal{M}$, and we say that Φ is homogeneous if $\Phi(\alpha\mu) = \alpha\Phi(\mu)$ for any $\alpha > 0$.

Lemma 3.2. *Let $\Phi\colon \mathcal{M}(X,\Sigma_X,\mathbb{R}^+) \to \mathcal{M}(Y,\Sigma_Y,\mathbb{R}^+)$ be a bounded, strongly additive, and homogeneous transfunction and let $\mathbb{E}$ be a Banach space. Then*

$$\|v_1\Phi\mu_1 + \dots + v_n\Phi\mu_n\| \le \|\Phi\|\|v_1\mu_1 + \dots + v_n\mu_n\| \tag{3.2}$$

for all $v_1,\dots,v_n \in \mathbb{E}$, $\mu_1,\dots,\mu_n \in \mathcal{M}(X,\Sigma_X,\mathbb{R}^+)$, and $n \in \mathbb{N}$.

Proof. Let $v_1,\dots,v_n \in \mathbb{E}$, $\mu_1,\dots,\mu_n \in \mathcal{M}(X,\Sigma_X,\mathbb{R}^+)$, and let $\mu = \sum_{k=1}^n \mu_k$. By Lemma 3.1, for any $\varepsilon > 0$ there are disjoint sets $S_1,\dots,S_N \in \Sigma$ such that for $i = 1,\dots,n$ we have

$$\mu_i = \sum_{j=1}^N \alpha_{i,j}\mu_{S_j} + \kappa_i, \tag{3.3}$$

where $\alpha_{k,j} \ge 0$ and $\kappa_1,\dots,\kappa_n$ are measures such that

$$\|v_1\kappa_1 + \dots + v_n\kappa_n\| < \frac{\varepsilon}{2\|\Phi\|} \quad \text{and} \quad \|v_1\Phi(\kappa_1) + \dots + v_n\Phi(\kappa_n)\| < \frac{\varepsilon}{2}. \tag{3.4}$$

Using that Φ is strongly additive and homogeneous,

$$\begin{aligned}
\|v_1\Phi(\mu_1) + \dots + v_n\Phi(\mu_n)\| &= \left\|\sum_{i=1}^n v_i\Phi\left(\sum_{j=1}^N \alpha_{i,j}\mu_{S_j} + \kappa_i\right)\right\| \\
&= \left\|\sum_{i=1}^n v_i\left(\sum_{j=1}^N \alpha_{i,j}\Phi(\mu_{S_j}) + \Phi(\kappa_i)\right)\right\| \\
&= \left\|\sum_{j=1}^N\left(\sum_{i=1}^n \alpha_{i,j}v_i\right)\Phi(\mu_{S_j}) + \sum_{i=1}^n v_i\Phi(\kappa_i)\right\| \\
&\le \left\|\sum_{j=1}^N\left(\sum_{i=1}^n \alpha_{i,j}v_i\right)\Phi(\mu_{S_j})\right\| + \left\|\sum_{i=1}^n v_i\Phi(\kappa_i)\right\|.
\end{aligned}$$

Then the second bound of (3.4) implies

$$\begin{aligned}
\|v_1\Phi(\mu_1)+\cdots+v_n\Phi(\mu_n)\| &\le \left\|\sum_{j=1}^{N}\Bigl(\sum_{i=1}^{n}\alpha_{i,j}v_i\Bigr)\Phi(\mu_{S_j})\right\|+\frac{\varepsilon}{2}\\
&\le \sum_{j=1}^{N}\left\|\Bigl(\sum_{i=1}^{n}\alpha_{i,j}v_i\Bigr)\Phi(\mu_{S_j})\right\|+\frac{\varepsilon}{2}\\
&\le \sum_{j=1}^{N}\left\|\sum_{i=1}^{n}\alpha_{i,j}v_i\right\|\left\|\Phi(\mu_{S_j})\right\|+\frac{\varepsilon}{2}\\
&\le \|\Phi\|\sum_{j=1}^{N}\left\|\sum_{i=1}^{n}\alpha_{i,j}v_i\right\|\left\|\mu_{S_j}\right\|+\frac{\varepsilon}{2}.
\end{aligned}$$

For the right-hand side of (3.2), we apply (3.3) to obtain

$$\begin{aligned}
\|\Phi\|\,\|v_1\mu_1+\cdots+v_n\mu_n\| &= \|\Phi\|\left\|\sum_{i=1}^{n}v_i\sum_{j=1}^{N}\alpha_{i,j}\mu_{S_j}+\sum_{i=1}^{n}v_i\kappa_i\right\|\\
&\ge \|\Phi\|\left(\left\|\sum_{i=1}^{n}v_i\sum_{j=1}^{N}\alpha_{i,j}\mu_{S_j}\right\|-\left\|\sum_{i=1}^{n}v_i\kappa_i\right\|\right),
\end{aligned}$$

and then using (3.4),

$$\begin{aligned}
\|\Phi\|\,\|v_1\mu_1+\cdots+v_n\mu_n\| &\ge \|\Phi\|\left(\left\|\sum_{i=1}^{n}v_i\sum_{j=1}^{N}\alpha_{i,j}\mu_{S_j}\right\|-\frac{\varepsilon}{2\|\Phi\|}\right)\\
&= \|\Phi\|\left\|\sum_{j=1}^{N}\sum_{i=1}^{n}\alpha_{i,j}v_i\mu_{S_j}\right\|-\frac{\varepsilon}{2}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\|\Phi\|\,\|v_1\mu_1+\cdots+v_n\mu_n\| &\ge \|\Phi\|\sum_{j=1}^{N}\left\|\sum_{i=1}^{n}\alpha_{i,j}v_i\mu_{S_j}\right\|-\frac{\varepsilon}{2}\\
&= \|\Phi\|\sum_{j=1}^{N}\left\|\sum_{i=1}^{n}\alpha_{i,j}v_i\right\|\left\|\mu_{S_j}\right\|-\frac{\varepsilon}{2},
\end{aligned}$$

and we get

$$\|v_1\Phi(\mu_1) + \cdots + v_n\Phi(\mu_n)\| \leq \|\Phi\| \, \|v_1\mu_1 + \cdots + v_n\mu_n\| + \varepsilon.$$

Since ε is an arbitrary positive number, the desired inequality follows. □

Corollary 3.3. *Let $\Phi\colon \mathcal{M}(X, \Sigma_X, \mathbb{R}^+) \to \mathcal{M}(Y, \Sigma_Y, \mathbb{R}^+)$ be a bounded, strongly additive, and homogeneous transfunction and let $\mathbb{E}$ be a Banach space. If*

$$\sum_{n=1}^{\infty} v_n\mu_n = 0,$$

for some $v_n \in \mathbb{E}$ and $\mu_n \in \mathcal{M}(X, \Sigma_X, \mathbb{R}^+)$, then

$$\sum_{n=1}^{\infty} v_n\Phi\mu_n = 0.$$

Proof. Since

$$\left\| \sum_{j=1}^{n} v_n\Phi\mu_n \right\| \leq \|\Phi\| \left\| \sum_{j=1}^{n} v_n\mu_n \right\|,$$

we have

$$\left\| \sum_{j=1}^{\infty} v_n\Phi\mu_n \right\| \leq \|\Phi\| \left\| \sum_{j=1}^{\infty} v_n\mu_n \right\| = 0. \qquad \square$$

In the next theorem we show that bounded, strongly additive, and homogeneous transfunctions can be extended to vector measures of a special type, namely measures that can be defined as sums of series of positive measures multiplied by elements of a Banach space $\mathbb{E}$. We will denote this space of measures by $\mathcal{M}_s(X, \Sigma_X, \mathbb{E})$, that is,

$$\mathcal{M}_s(X, \Sigma_X, \mathbb{E}) = \left\{ \sum_{n=1}^{\infty} v_n\mu_n : \mu_n \in \mathcal{M}(X, \Sigma_X, \mathbb{R}^+), v_n \in \mathbb{E}, \sum_{n=1}^{\infty} \|v_n\| \|\mu_n\| < \infty \right\}.$$

Theorem 3.4. *Let $\Phi\colon \mathcal{M}(X, \Sigma_X, \mathbb{R}^+) \to \mathcal{M}(Y, \Sigma_Y, \mathbb{R}^+)$ be a bounded, strongly additive, and homogeneous transfunction and let $\mathbb{E}$ be a Banach space. Then there is a unique bounded linear transfunction $\tilde{\Phi}\colon \mathcal{M}_s(X, \Sigma_X, \mathbb{E}) \to \mathcal{M}(Y, \Sigma_Y, \mathbb{E})$ satisfying $\tilde{\Phi}(v\mu) = v\Phi\mu$ for every $\mu \in \mathcal{M}(X, \Sigma_X, \mathbb{R}^+)$ and every $v \in \mathbb{E}$.*

PROOF. Let $\Phi\colon \mathcal{M}(X,\Sigma_X,\mathbb{R}^+) \to \mathcal{M}(Y,\Sigma_Y,\mathbb{R}^+)$ be a bounded, strongly additive, and homogeneous transfunction and let $\mathbb{E}$ be a Banach space.

If $\mu = \sum_{n=1}^{\infty} v_n\mu_n$, for some $v_n \in \mathbb{E}$ and $\mu_n \in \mathcal{M}(X,\Sigma_X,\mathbb{R}^+)$ such that $\sum_{n=1}^{\infty} \|v_n\|\|\mu_n\| < \infty$, then we define

$$\tilde{\Phi}(\mu) = \sum_{n=1}^{\infty} v_n \Phi\mu_n.$$

Since

$$\sum_{n=1}^{\infty} \|v_n\Phi\mu_n\| \le \sum_{n=1}^{\infty} \|v_n\|\|\Phi\mu_n\| \le \|\Phi\| \sum_{n=1}^{\infty} \|v_n\|\|\mu_n\| < \infty,$$

the series converges. Moreover, if $\sum_{n=1}^{\infty} v_n\mu_n = \sum_{n=1}^{\infty} w_n\kappa_n$, for some $v_n, w_n \in \mathbb{E}$ and $\mu_n, \kappa_n \in \mathcal{M}(X,\Sigma_X,\mathbb{R}^+)$ such that

$$\sum_{n=1}^{\infty} \|v_n\|\|\mu_n\| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \|w_n\|\|\kappa_n\| < \infty,$$

then

$$\sum_{n=1}^{\infty} (v_n\mu_n - w_n\kappa_n) = 0.$$

By Corollary 3.3,

$$\sum_{n=1}^{\infty} (v_n\Phi\mu_n - w_n\Phi\kappa_n) = 0,$$

so $\sum_{n=1}^{\infty} v_n\Phi\mu_n = \sum_{n=1}^{\infty} w_n\Phi\kappa_n$. This shows that the extension is well-defined.

Clearly, $\tilde{\Phi}$ is a linear transfunction from $\mathcal{M}_s(X,\Sigma_X,\mathbb{E})$ to $\mathcal{M}(Y,\Sigma_Y,\mathbb{E})$. Since, by Lemma 3.2, for every $n \in \mathbb{N}$ we have

$$\|v_1\Phi\mu_1 + \cdots + v_n\Phi\mu_n\| \le \|\Phi\|\|v_1\mu_1 + \cdots + v_n\mu_n\|,$$

we have $\|\tilde{\Phi}(\mu)\| \le \|\Phi\|\|\mu\|$, so $\tilde{\Phi}$ is bounded.

Now let $\Psi\colon \mathcal{M}_s(X,\Sigma_X,\mathbb{E}) \to \mathcal{M}(Y,\Sigma_Y,\mathbb{E})$ be a bounded linear transfunction satisfying $\Psi(v\mu) = v\Psi\mu$ for every $\mu \in \mathcal{M}(X,\Sigma_X,\mathbb{R}^+)$ and every $v \in \mathbb{E}$. If $\mu = \sum_{n=1}^{\infty} v_n\mu_n$, for some $v_n \in \mathbb{E}$ and $\mu_n \in \mathcal{M}(X,\Sigma_X,\mathbb{R}^+)$ such that $\displaystyle\sum_{n=1}^{\infty} \|v_n\|\|\mu_n\| < \infty$, then for every $n \in \mathbb{N}$ we have

$$\Psi(v_1\mu_1 + \cdots + v_n\mu_n) = \tilde{\Phi}(v_1\mu_1 + \cdots + v_n\mu_n)$$

and consequently $\tilde{\Phi} = \Psi$ by continuity. □

From the above theorem and Theorem 2.2 we obtain the following result.

Corollary 3.5. *Let $\mathbb{E}$ be a Banach space with the Radon-Nikodym property. For every bounded, strongly additive, and homogeneous $\Phi\colon \mathcal{M}(X,\Sigma_X,\mathbb{R}^+) \to \mathcal{M}(Y,\Sigma_Y,\mathbb{R}^+)$ there is a unique bounded linear transfunction*

$$\tilde{\Phi}\colon \mathcal{M}(X,\Sigma_X,\mathbb{E}) \to \mathcal{M}(Y,\Sigma_Y,\mathbb{E})$$

satisfying $\tilde{\Phi}(v\mu) = v\Phi\mu$ for every $\mu \in \mathcal{M}(X,\Sigma_X,\mathbb{R}^+)$ and every $v \in \mathbb{E}$.

Corollary 3.6. *Every bounded positive operator*

$$\tilde{\Phi}\colon \mathcal{M}(X,\Sigma_X,\mathbb{R}) \to \mathcal{M}(Y,\Sigma_Y,\mathbb{R})$$

is the unique extension of a bounded, strongly additive, and homogeneous transfunction $\Phi\colon \mathcal{M}(X,\Sigma_X,\mathbb{R}^+) \to \mathcal{M}(Y,\Sigma_Y,\mathbb{R}^+)$.

The previous results raise questions about the structure of $\mathcal{M}(X,\Sigma_X,\mathbb{E})$. For example, is $\mathcal{M}_s(X,\Sigma_X,\mathbb{E})$ complemented in $\mathcal{M}(X,\Sigma_X,\mathbb{E})$? If so, we could define an extension of transfunctions without the Radon-Nikodym assumption on $\mathbb{E}$.

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