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# AN EXTENSION OF THE ABEL–LIOUVILLE IDENTITY

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**Abstract.** In this note, we present an extension of the celebrated Abel–Liouville identity in terms of noncommutative complete Bell polynomials for generalized Wronskians. We also characterize the range equivalence of $n$-dimensional vector-valued functions in the subclass of $n$-times differentiable functions with a nonvanishing Wronskian.

## 1. Introduction

Throughout this paper let $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of real and the sets of positive and nonnegative integers, respectively, and let $I$ stand for a nonempty open real interval.

For an $n$-dimensional vector-valued $(n-1)$-times continuously differentiable function $f\colon I \to \mathbb{R}^n$, its *Wronskian* $W_f\colon I \to \mathbb{R}$ is defined by

$$W_f := \left| f^{(n-1)} \quad \dots \quad f' \quad f \right|.$$

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Here we usually interpret the elements of $\mathbb{R}^n$ as column vectors. In the sequel, the standard inner product on $\mathbb{R}^n$ will be denoted by $\langle\cdot,\cdot\rangle$.

Consider now the $n$th-order homogeneous linear differential equation

$$y^{(n)} = a_1 y^{(n-1)} + \cdots + a_n y, \tag{1}$$

where $a_1, \ldots, a_n \colon I \to \mathbb{R}$ are continuous functions. By the classical *Abel–Liouville identity* (cf. [4]), if $f \colon I \to \mathbb{R}^n$ is a fundamental system of solutions of (1), then $W_f$ does not vanish on $I$ and

$$W_f' = a_1 W_f.$$

For a sufficiently smooth function $f \colon I \to \mathbb{R}^n$ and $k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$, we introduce now the *generalized Wronskian* $W_f^k \colon I \to \mathbb{R}$ by

$$W_f^k := \begin{vmatrix} f^{(k_1)} & \ldots & f^{(k_n)} \end{vmatrix}.$$

One can easily see that, with this notation, we have

$$W_f = W_f^{(n-1,n-2,\ldots,0)} \quad \text{and} \quad W_f' = W_f^{(n,n-2,\ldots,0)}.$$

Therefore, the Abel–Liouville identity can be rewritten as

$$W_f^{(n,n-2,\ldots,0)} = a_1 W_f^{(n-1,n-2,\ldots,0)}. \tag{2}$$

One of the main goals of this short paper is to establish a formula for $W_f^k$ in terms of the coefficients of differential equation (1). Another goal is to introduce the range equivalence for $n$-dimensional vector-valued functions and to characterize this equivalence relation in the subclass of $n$-times differentiable functions with a nonvanishing Wronskian.

## 2. Main results

For the description of our main result, we recall the notion of *noncommutative complete Bell polynomials*, which was introduced by Schimming and Rida ([3]). Let $\mathbb{R}^{n\times n}$ denote the ring of $n \times n$ matrices with real entries and let $\mathbb{I}_n$ denote the $n \times n$ unit matrix. Now define $B_m\colon (\mathbb{R}^{n\times n})^m \to \mathbb{R}^{n\times n}$ by the following recursive formula

$$B_0 := \mathbb{I}_n, \quad B_{m+1}(X_1,\dots,X_{m+1}) := \sum_{j=0}^{m} \binom{m}{j} B_j(X_1,\dots,X_j)X_{m+1-j}.$$

The notion of *complete Bell polynomials* in the commutative setting (i.e., when $n = 1$) was introduced by Bell ([1], [2]). One can easily compute the first few Bell polynomials as follows:

$$\begin{aligned}
B_1(X_1) &= X_1,\\
B_2(X_1,X_2) &= X_1^2 + X_2,\\
B_3(X_1,X_2,X_3) &= X_1^3 + 2X_1X_2 + X_2X_1 + X_3,\\
B_4(X_1,X_2,X_3,X_4) &= X_1^4 + 3X_1^2X_2 + 2X_1X_2X_1 + 3X_1X_3 + 3X_2^2\\
&\quad + X_2X_1^2 + X_3X_1 + X_4,\\
B_5(X_1,X_2,X_3,X_4,X_5) &= X_1^5 + 4X_1^3X_2 + 3X_1^2X_2X_1 + 6X_1^2X_3 + 8X_1X_2^2\\
&\quad + 2X_1X_2X_1^2 + 3X_1X_3X_1 + 4X_1X_4 + 3X_2^2X_1\\
&\quad + X_2X_1^3 + 4X_2X_1X_2 + 6X_2X_3 + X_3X_1^2\\
&\quad + 4X_3X_2 + X_4X_1 + X_5.
\end{aligned}$$

The statement of the next basic lemma was proved in the paper [3].

Lemma 1. *For every $j \in \mathbb{N}_0$, and $j$-times differentiable matrix-valued function $X\colon I \to \mathbb{R}^{n\times n}$,*

$$B_{j+1}\big(X,\dots,X^{(j)}\big) = XB_j\big(X,\dots,X^{(j-1)}\big) + \Big(B_j\big(X,\dots,X^{(j-1)}\big)\Big)'.$$

Lemma 2. *Let $n, m \in \mathbb{N}$, let $X\colon I \to \mathbb{R}^{n\times n}$ be an $(m-1)$-times continuously differentiable function and $Y\colon I \to \mathbb{R}^{n\times n}$ be a differentiable function such that*

$$Y' = YX \tag{3}$$

*holds on $I$. Then $Y$ is $m$-times continuously differentiable and*

$$Y^{(j)} = YB_j\big(X,\dots,X^{(j-1)}\big) \quad (j \in \{0,\dots,m\}). \tag{4}$$

Proof. If $m=1$, then $X$ is continuous, hence the continuity of $Y$ and (3) imply that $Y$ is continuously differentiable. If $m>1$, then using (3), a simple inductive argument shows that $Y$ is $m$-times continuously differentiable.

The equality (4) is trivial if $j=0$, because $B_0=\mathbb{I}_n$. For $j=1$, the equality (4) is equivalent to (3). Now assume that (4) has been verified for some $j$ with $1\le j<m$. Then, using (3) and Lemma 1, we get

$$\begin{aligned}
Y^{(j+1)} &= \big(Y^{(j)}\big)' = \Big(YB_j\big(X,\dots,X^{(j-1)}\big)\Big)' \\
&= Y'B_j\big(X,\dots,X^{(j-1)}\big) + Y\Big(B_j\big(X,\dots,X^{(j-1)}\big)\Big)' \\
&= Y\Big[XB_j\big(X,\dots,X^{(j-1)}\big) + \Big(B_j\big(X,\dots,X^{(j-1)}\big)\Big)'\Big] \\
&= YB_{j+1}\big(X,\dots,X^{(j)}\big).
\end{aligned}$$

This proves the assertion for $j+1$. □

In what follows, let $e_1,\dots,e_n$ denote the elements of the standard basis in $\mathbb{R}^n$.

Corollary 3. *Let $n,m\in\mathbb{N}$, let $a=(a_1,\dots,a_n)\colon I\to\mathbb{R}^n$ be an $(m-1)$-times continuously differentiable function and let $f\colon I\to\mathbb{R}^n$ be a fundamental system of solutions of the differential equation* (1)*. Let the matrix-valued functions $X_a\colon I\to\mathbb{R}^{n\times n}$ and $Y_f\colon I\to\mathbb{R}^{n\times n}$ be defined by*

$$X_a := \begin{pmatrix} a & e_1 & \dots & e_{n-1}\end{pmatrix} \quad \textit{and} \quad Y_f := \begin{pmatrix} f^{(n-1)} & \dots & f' & f\end{pmatrix}. \tag{5}$$

*Then $Y_f$ is $m$-times continuously differentiable and*

$$Y_f^{(j)} = Y_fB_j\big(X_a,\dots,X_a^{(j-1)}\big) \quad (j\in\{0,\dots,m\}).$$

Proof. The function $f$ satisfies the differential equation (1), therefore $f^{(n)}=Y_f\cdot a$. On the other hand, $f^{(n-i)}=Y_f\cdot e_i$ holds for $i\in\{1,\dots,n-1\}$. These equalities imply that

$$\begin{aligned}
Y_f' &= \begin{pmatrix} f^{(n)} & f^{(n-1)} & \dots & f'\end{pmatrix} \\
&= \begin{pmatrix} Y_f\cdot a & Y_f\cdot e_1 & \dots & Y_f\cdot e_{n-1}\end{pmatrix} = Y_fX_a.
\end{aligned}$$

Therefore, equation (3) holds with $Y:=Y_f$ and $X:=X_a$, consequently, the statement is a consequence of Lemma 2. □

Using the above corollary, we can easily establish a formula for the computation of the generalized Wronskian $W_f^k$.

THEOREM 4. *Let $n, m \in \mathbb{N}$, let $a = (a_1, \dots, a_n)\colon I \to \mathbb{R}^n$ be an $(m-1)$-times continuously differentiable function and let $f\colon I \to \mathbb{R}^n$ be a fundamental system of solutions of the differential equation* (1)*. Let the matrix-valued functions $X_a\colon I \to \mathbb{R}^{n\times n}$ be defined by* (5)*. Then, for $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n$ with $\max(k_1, \dots, k_n) \le m + n - 1$,*

$$(6)\quad W_f^k = W_f \Big| B_{\ell_1}\big(X_a, \dots, X_a^{(\ell_1-1)}\big)e_{n+\ell_1-k_1} \quad \dots \quad B_{\ell_n}\big(X_a, \dots, X_a^{(\ell_n-1)}\big)e_{n+\ell_n-k_n}\Big|,$$

*where, for $i \in \{1, \dots, n\}$, $\ell_i := (k_i - n + 1)^+ := \max(k_i - n + 1, 0)$.*

PROOF. Define the matrix valued function $Y_f\colon I \to \mathbb{R}^{n\times n}$ by (5) and observe that, in view of Corollary 3, for all $\ell \in \{0, \dots, m + n - 1\}$ and $i \in \{(\ell - n + 1)^+, \dots, \min(\ell, m)\}$, we have that

$$f^{(\ell)} = Y_f^{(i)} e_{n+i-\ell} = Y_f B_i\big(X_a, \dots, X_a^{(i-1)}\big)e_{n+i-\ell}.$$

By taking the smallest possible value for $i$ in the above formula, we get

$$f^{(\ell)} = Y_f B_{(\ell-n+1)^+}\big(X_a, \dots, X_a^{((\ell-n+1)^+-1)}\big)e_{n+(\ell-n+1)^+-\ell}.$$

Applying this equality for $\ell \in \{k_1, \dots, k_n\}$, we obtain

$$\big(f^{(k_1)} \quad \dots \quad f^{(k_n)}\big) = Y_f\Big(B_{\ell_1}\big(X_a, \dots, X_a^{(\ell_1-1)}\big)e_{n+\ell_1-k_1} \quad \dots \quad B_{\ell_n}\big(X_a, \dots, X_a^{(\ell_n-1)}\big)e_{n+\ell_n-k_n}\Big).$$

Now taking the determinant side by side and using the product rule for determinants, the equality (6) follows. $\square$

In the subsequent corollary, we consider the case when $\ell_i = 0$ for $i \in \{2, \dots, n\}$. In this particular setting, the determinant on the left hand side of (6) can easily be computed.

COROLLARY 5. *Let $n, m \in \mathbb{N}$, let $a = (a_1, \dots, a_n)\colon I \to \mathbb{R}^n$ be an $(m-1)$-times continuously differentiable function and let $f\colon I \to \mathbb{R}^n$ be a fundamental system of solutions of the differential equation* (1)*. Let the matrix-valued* *functions* $X_a\colon I \to \mathbb{R}^{n\times n}$ *be defined by* (5) *and let* $d \in \{0,...,m-1\}$ *and* $j \in \{0,\dots,n-1\}$*. Then*

$$\text{(7)}\quad W_f^{(n+d,n-1,\dots,j+1,j-1,\dots,0)} = (-1)^{n-j-1} W_f \langle B_{d+1}\big(X_a,\dots,X_a^{(d)}\big)e_1, e_{n-j}\rangle.$$

*If* $d=0$ *and* $j=n-1$*, then this equality reduces to the Abel–Liouville identity* (2)*. More generally, for* $d=0,1,2$*, we get the following formulas:*

$$\begin{aligned}
W_f^{(n,n-1,\dots,j+1,j-1,\dots,0)} &= (-1)^{n-j-1} W_f a_{n-j},\\
W_f^{(n+1,n-1,\dots,j+1,j-1,\dots,0)} &= (-1)^{n-j-1} W_f (a_1 a_{n-j} + a_{n-j+1} + a'_{n-j}),\\
W_f^{(n+2,n-1,\dots,j+1,j-1,\dots,0)} &= (-1)^{n-j-1} W_f (a_1^2 a_{n-j} + a_1 a_{n-j+1} + a_2 a_{n-j}\\
&\quad + a_{n-j+2} + a_1 a'_{n-j} + 2a'_1 a_{n-j} + 2a'_{n-j+1} + a''_{n-j}). \qquad (8)
\end{aligned}$$

*(Here we define* $a_{n+1} := a_{n+2} := 0$*.)*

Proof. We apply the previous theorem for $k := (n+d, n-1,\dots, j+1, j-1,\dots,0)$, where $d \in \{0,...,m-1\}$ and $j \in \{0,\dots,n-1\}$. Then we get that $\ell_1 = d+1$, and $\ell_i = 0$ for $i \in \{2,\dots,n\}$. Therefore,

$$\begin{aligned}
W_f^{(n+d,n-1,\dots,j+1,j-1,\dots,0)} &= W_f \Big| B_{d+1}\big(X_a,\dots,X_a^{(d)}\big)e_1 \quad \mathbb{I}_n e_1\\
&\qquad \dots \quad \mathbb{I}_n e_{n-j-1} \quad \mathbb{I}_n e_{n-j+1} \quad \dots \quad \mathbb{I}_n e_n \Big|\\
&= (-1)^{n-j-1} W_f \langle B_{d+1}\big(X_a,\dots,X_a^{(d)}\big)e_1, e_{n-j}\rangle.
\end{aligned}$$

Thus, equality (7) has been shown. In the case $d=0$, we have that

$$\langle B_1(X_a)e_1, e_{n-j}\rangle = \langle X_a e_1, e_{n-j}\rangle = a_{n-j}$$

because the $(n-j)$th entry of $X_a$ equals $a_{n-j}$. This implies the first equality in (8) for $j \in \{0,\dots,n-1\}$. In particular, for $j=n-1$, this equality is equivalent to the Abel–Liouville identity (2).

In the case $d=1$, a simple computation gives that

$$\langle B_2(X_a, X'_a)e_1, e_{n-j}\rangle = \langle (X_a^2 + X'_a)e_1, e_{n-j}\rangle = a_1 a_{n-j} + a_{n-j+1} + a'_{n-j},$$

which yields the second equality in (8) for $j \in \{0,\dots,n-1\}$.

In the case $d = 2$, a somewhat more difficult computation gives that

$$\begin{aligned}\langle B_3(X_a, X_a', X_a'')e_1, e_{n-j}\rangle &= \langle (X_a^3 + 2X_aX_a' + X_a'X_a + X_a'')e_1, e_{n-j}\rangle \\ &= a_1^2 a_{n-j} + a_1 a_{n-j+1} + a_2 a_{n-j} + a_{n-j+2} \\ &\quad + a_1 a_{n-j}' + 2a_1' a_{n-j} + 2a_{n-j+1}' + a_{n-j}'',\end{aligned}$$

which then yields the third equality in (8). □

For the sake of convenience and brevity, we introduce the following notation: for an $n$-times continuously differentiable function $f: I \to \mathbb{R}^n$ such that $W_f$ is nonvanishing and $j \in \{0, \dots, n-1\}$, the function $\Phi_f^{[j]}: I \to \mathbb{R}$ is defined by

$$\Phi_f^{[j]} := (-1)^{n-j-1} \frac{W_f^{(n,\dots,j+1,j-1,\dots,0)}}{W_f}.$$

For instance, if $f$ is $n$-times continuously differentiable function whose components form a fundamental system of solutions for (1), then the Abel–Liouville identity (2) can be rewritten as

$$\Phi_f^{[n-1]} = a_1.$$

More generally, the first equality in (8) gives that

$$\Phi_f^{[j]} = a_{n-j} \quad (j \in \{0, \dots, n-1\})$$

or, equivalently,

$$a_j = \Phi_f^{[n-j]} \quad (j \in \{1, \dots, n\}). \tag{9}$$

Lemma 6. *Let $f: I \to \mathbb{R}^n$ be an $n$-times continuously differentiable function such that $W_f$ is nonvanishing. Then the components of $f$ form a fundamental system of solutions of the $n$th-order homogeneous linear differential equation*

$$y^{(n)} = \sum_{j=0}^{n-1} \Phi_f^{[j]} y^{(j)}. \tag{10}$$

Proof. This equation is equivalent to the following identity

$$\begin{aligned}&\left|f^{(n-1)} \quad \dots \quad f^{(0)}\right| y^{(n)}\\ &\qquad = \sum_{j=0}^{n-1} (-1)^{n-j-1} \left|f^{(n)} \quad \dots \quad f^{(j+1)} \quad f^{(j-1)} \quad \dots \quad f^{(0)}\right| y^{(j)}.\end{aligned}$$

We can now rearrange this equation to obtain

$$\begin{vmatrix} y^{(n)} & y^{(n-1)} & \dots & y \\ f_1^{(n)} & f_1^{(n-1)} & \dots & f_1 \\ \vdots & \vdots & \ddots & \vdots \\ f_n^{(n)} & f_n^{(n-1)} & \dots & f_n \end{vmatrix} = 0.$$

It is easily seen that if $y \in \{f_1, \dots, f_n\}$, then the determinant vanishes. Therefore, $f_1, \dots, f_n$ are solutions of (10). Due to the condition that $W_f$ is nonvanishing, the components of $f$ are linearly independent, therefore they form a fundamental solution system for (10). □

Corollary 7. *Let $n, m \in \mathbb{N}$ with $m \geq n$ and let $f \colon I \to \mathbb{R}^n$ be an $m$-times continuously differentiable function such that $W_f$ is nonvanishing. Define $a = (a_1, \dots, a_n) \colon I \to \mathbb{R}^n$ by (9) and $X_a \colon I \to \mathbb{R}^{n \times n}$ by (5). Then the equality (6) holds for $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n$, if $k_i \leq m$ and $\ell_i := (k_i - n + 1)^+$ for $i \in \{1, \dots, n\}$.*

Proof. It follows from the definition of $a$, that it is $(m-n)$-times continuously differentiable. On the other hand, by Lemma 6, we have that $f$ satisfies the $n$-th order homogeneous linear differential equation (1). Thus, the statement is a consequence of Theorem 4. □

We say that two continuous functions $f, g \colon I \to \mathbb{R}^n$ are *range equivalent*, denoted by $f \sim g$, if there exists a nonsingular $n \times n$-matrix $A$ such that

$$f = Ag. \tag{11}$$

Theorem 8. *Let $f, g \colon I \to \mathbb{R}^n$ be $n$-times continuously differentiable functions such that $W_f$ and $W_g$ are nonvanishing. Then $f \sim g$ holds if and only if*

$$\Phi_f^{[j]} = \Phi_g^{[j]} \quad (j \in \{0, \dots, n-1\}). \tag{12}$$

Proof. If $f \sim g$, then there exists a nonsingular $n \times n$-matrix $A$ such that $f = Ag$. The product rule for determinants shows that $W_f^k = |A| W_g^k$ for every $k \in \mathbb{N}_0^n$. Using this identity and the definition of $\Phi_f^{[j]}$ and $\Phi_g^{[j]}$, we obtain the equalities in (12).

On the other hand, if the identities (12) are valid on $I$, then the $n$th-order homogeneous linear differential equation (10) is equivalent to the following one

$$y^{(n)} = \sum_{j=0}^{n-1} \Phi_g^{[j]} y^{(j)}.$$

Therefore, the ($n$-dimensional) solution spaces of these differential equations are identical, which in view of Lemma 6 yields that the components of $f$ are linear combinations of the components of $g$. Thus identity (11) holds for some nonsingular $n \times n$-matrix $A$. $\square$

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