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A FUNCTIONAL EQUATION WITH BIADDITIVE FUNCTIONS

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Abstract. Let S, H, X be groups. For two given biadditive functions $A\colon S^2 \to X$, $B\colon H^2 \to X$ and for two unknown mappings $T\colon S \to H$, $g\colon S \to S$ we will study the functional equation

$$B(T(x), T(y)) = A(x, g(y)), \quad x, y \in S,$$

which is a generalization of the orthogonality equation in Hilbert spaces.

1. Introduction

Let H, K be unitary spaces. It is easy to check that, if $f\colon H \to K$ satisfies the orthogonality equation

$$\langle f(x)|f(y)\rangle = \langle x|y\rangle, \tag{1.1}$$

then f is a linear isometry (see, e.g. [6, Lemma 2.1.1 and the following Remark]).

The above equation was generalized in normed spaces X, Y by considering a norm derivative $\rho'_+(x,y) := \|x\| \cdot \lim_{t\to 0^+} \frac{\|x+ty\|-\|x\|}{t}$ instead of inner

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product, i.e.

$$\rho'_+(f(x), f(y)) = \rho'_+(x, y), \quad x, y \in X, \tag{1.2}$$

with an unknown function $f\colon X \to Y$. Note that if the norm comes from an inner product $\langle\cdot,\cdot\rangle$, we obtain $\rho'_+(x, y) = \langle x|y\rangle$.

The second way of generalization of the orthogonality equation in Hilbert spaces H, K is to look for the solutions of

$$\langle f(x)|g(y)\rangle = \langle x|y\rangle, \quad x, y \in H, \tag{1.3}$$

where $f, g\colon H \to K$ are unknown functions. Solutions of (1.2) and (1.3), can be found in the papers [1], [4], [2], [8].

In [5] authors give a natural generalization of such functional equations in the case of commutative groups. They consider biadditive mappings instead of inner products.

Another generalization of (1.3) we can find in the paper [7] where the author studies the equation

$$\langle f(x)|g(y^*)\rangle = \langle x|y^*\rangle, \quad x \in E, y^* \in E^*,$$

where $f\colon E \to F$, $g\colon E^* \to F^*$, E, F are Banach spaces, E^*, F^* are spaces dual to E and F respectively, and $\langle a|\varphi\rangle := \varphi(a)$.

In [3] we can find a different approach. Instead of taking two different functions on the left side of (1.3), we change only the right side of (1.3), so we obtain

$$\langle f(x)|f(y)\rangle = \langle x|g(y)\rangle, \quad x, y \in X,$$

with two unknown functions $f\colon X \to Y$, $g\colon X \to X$.

In this paper we generalize the above equation – we consider biadditive mappings instead of inner products.

2. Preliminaries

We start by recalling definition of multi-additive functions. By $\mathrm{Perm}(n)$ we denote the set of all bijections of the set $\{1, \ldots, n\}$.

Definition 1. Let S be a semigroup, H be a group, $n \in \mathbb{N}$. The function $A\colon S^n \to H$ is called *n-additive* if

$$A(x_1, \dots, x_{i-1}, x_i + y, x_{i+1}, \dots, x_n) = A(x_1, \dots, x_n) + A(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n),$$

for all $y, x_1, \dots, x_n \in S$ and $i \in \{1, \dots, n\}$.

Moreover, A is called *symmetric* if

$$A(x_1, \dots, x_n) = A(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

for all $x_1, \dots, x_n \in S$ and $\sigma \in \mathrm{Perm}(n)$.

Now we introduce some theory of the adjoint operator on groups.

Definition 2. Let S, H, X be groups, $A\colon S^2 \to X$, $B\colon H^2 \to X$ be biadditive functions. Let further $T\colon S \to H$ and

$$D(T^*) := \{v \in H : \exists_{y \in S} \forall_{x \in S}\, B(T(x), v) = A(x, y)\}.$$

A function $T^*\colon D(T^*) \to S$ is called a *(B, A)-adjoint operator* (to T) if and only if

$$B(T(x), v) = A(x, T^*(v)),\ x \in S,\ v \in D(T^*).$$

Remark 1. Let S, X be groups, $A\colon S^2 \to X$ be a biadditive function. We observe that

$$\begin{aligned} A(x, v) + A(x, y) + A(u, v) + A(u, y) &= A(x, v + y) + A(u, v + y) \\ &= A(x + u, v + y) = A(x + u, v) + A(x + u, y) \\ &= A(x, v) + A(u, v) + A(x, y) + A(u, y), \quad x, y, u, v \in S, \end{aligned}$$

so

$$A(x, y) + A(u, v) = A(u, v) + A(x, y), \quad x, y, u, v \in S.$$

Hence the group generated by the image of A is a commutative subgroup of X (so we can assume that X is commutative).

Lemma 1 (see [5, Lemma 4]). *Let S, H, X be groups, $A\colon S^2 \to X$, $B\colon H^2 \to X$ be biadditive functions. Let further $T\colon S \to H$ and $T^*\colon D(T^*) \to S$ be a (B, A)-adjoint operator to T,*

$$S_{AR} := \{y \in S : \forall_{x\in S}\, A(x, y) = 0\}, \tag{2.1}$$

$$S_{ALT^*} := \{x \in S : \forall_{y\in \operatorname{im} T^*}\, A(x, y) = 0\}, \tag{2.2}$$

$$H_{BTR} := \{v \in H : \forall_{u\in \operatorname{im} T}\, B(u, v) = 0\}, \tag{2.3}$$

$$H_{BLD^*} := \{u \in H : \forall_{v\in D(T^*)}\, B(u, v) = 0\}. \tag{2.4}$$

Then

(a) *S_{AR}, S_{ALT^*} are normal subgroups of S, $D(T^*)$, H_{BTR}, H_{BLD^*} are normal subgroups of H. Moreover in the case when X is torsion-free, if H is divisible, then H_{BTR}, H_{BLD^*} are divisible, if S is divisible, then S_{AR}, S_{ALT^*} are divisible, if S, H are divisible, then $D(T^*)$ is divisible;*

(b) $\forall_{x,y\in S}\, T(x + y) - T(y) - T(x) \in H_{BLD^*}$*;*

(c) $\forall_{x,y\in S}\, x - y \in S_{ALT^*} \Leftrightarrow T(x) - T(y) \in H_{BLD^*}$*;*

(d) $\forall_{u,v\in D(T^*)}\, T^*(u + v) - T^*(v) - T^*(u) \in S_{AR}$*;*

(e) $\forall_{u,v\in D(T^*)}\, u - v \in H_{BTR} \Leftrightarrow T^*(u) - T^*(v) \in S_{AR}$*;*

(f) $H_{BTR} \subset D(T^*)$*;*

(g) *Let $\varkappa\colon S \to S/S_{AR}$ be a canonical homomorphism. Then the function $\widetilde{T^*}\colon D(T^*)/H_{BTR} \to \operatorname{im} T^*/S_{AR}$ given by*

$$\widetilde{T^*}(u + H_{BTR}) = T^*(u) + S_{AR}, \quad u \in D(T^*),$$

is well-defined and it is an isomorphism.

Proof. Proofs of (b)–(f) are in [5].

Let $v \in D(T^*)$, $x \in H$. From Remark 1 we have

$$\begin{aligned} B(T(x), x + v - x) &= B(T(x), x) + B(T(x), v) - B(T(x), x) \\ &= B(T(x), v) = B(x, T^*(v)), \end{aligned}$$

so $D(T^*)$ is a normal subgroup of G (the rest of proofs of (a) are in [5]).

Let $u, v \in D(T^*)$ be such that $u - v \in H_{BTR}$. Then from (e) we have $T^*(u) - T^*(v) \in S_{AR}$, so $\widetilde{T^*}$ is well-defined. From (d) we obtain that $\widetilde{T^*}$ is a homomorphism. From (e) we obtain the injectiveness. Obviously $\widetilde{T^*}$ is surjective. □

Remark 2. If in the previous lemma S, H, X are linear spaces over some field $\mathbb{K}$ and A, B are $\mathbb{K}$-linear, then S_{AR}, S_{ALT^*}, $D(T^*)$, H_{BTR}, H_{BLD^*} are linear spaces over $\mathbb{K}$.

Definition 3. Let S, H, X be groups, $A\colon S^2 \to X$, $B\colon H^2 \to X$ be biadditive functions, $T\colon S \to H$. A function T is called an (A,B)-*quasi isometry* if

$$B(T(x), T(y)) = A(x,y), \quad x, y \in S. \tag{2.5}$$

A function T is called an (A,B)-*isometry* if T is bijective and satisfies (2.5).

Lemma 2. *Let H, X be groups, $\widetilde{H}$ be a normal subgroup of H, $B\colon H^2 \to X$ be a biadditive function, the sets $\widetilde{H}_{BL}$, $\widetilde{H}_{BR}$ be given by formulas*

$$\widetilde{H}_{BL} = \{x \in \widetilde{H} : \forall_{y\in\widetilde{H}}\ B(x,y) = 0\}, \tag{2.6}$$

$$\widetilde{H}_{BR} = \{y \in \widetilde{H} : \forall_{x\in\widetilde{H}}\ B(x,y) = 0\}, \tag{2.7}$$

$$\widetilde{H}_{B0} := \widetilde{H}_{BL} \cap \widetilde{H}_{BR}. \tag{2.8}$$

Then $\widetilde{H}_{BL}$, $\widetilde{H}_{BR}$, and $\widetilde{H}_{B0}$ are normal subgroups of $\widetilde{H}$, the function $\widetilde{B}\colon (\widetilde{H}/\widetilde{H}_{B0})^2 \to X$ given by the formula

$$\widetilde{B}(x + \widetilde{H}_{B0}, y + \widetilde{H}_{B0}) := B(x,y), \quad x, y \in \widetilde{H}, \tag{2.9}$$

is well-defined and it is biadditive.

Proof. It is easy to observe that $\widetilde{H}_{BL}$, $\widetilde{H}_{BR}$, $\widetilde{H}_{B0}$ are normal subgroups of $\widetilde{H}$. Let $x_1, x_2, y_1, y_2 \in \widetilde{H}$ be such that $x_2 - x_1, y_2 - y_1 \in \widetilde{H}_{B0}$. Then

$$\begin{aligned} B(x_2, y_2) &= B(x_2 - x_1, y_2) + B(x_1, y_2) = B(x_1, y_2) \\ &= B(x_1, y_2 - y_1) + B(x_1, y_1) = B(x_1, y_1), \end{aligned}$$

so $\widetilde{B}$ is well-defined. It is easy to observe that $\widetilde{B}$ is biadditive. □

Of course each (A,B)-isometry is an (A,B)-quasi isometry. The following result shows that for any (A,B)-quasi isometry there exists some $(\widetilde{A}, \widetilde{B})$-isometry connected with it.

THEOREM 1. *Let S, H, X are groups, $A\colon S^2 \to X$, $B\colon H^2 \to X$ be biadditive functions, $T\colon S \to H$ be an (A,B)-quasi isometry, $\widetilde{H} = \langle \operatorname{im} T\rangle$. Let further $\widetilde{H}_{B0}$, S_{AR} be defined by* (2.8) *and* (2.1),

$$S_{AL} := \{x \in S : \forall_{y\in S}\ A(x,y) = 0\},$$
$$S_{A0} = S_{AL} \cap S_{AR}.$$

Then S_{AL}, S_{A0} are normal subgroups of S, the function $\widetilde{T}\colon S/S_{A0} \to \widetilde{H}/\widetilde{H}_{B0}$ given by the formula

$$\widetilde{T}(x + S_{A0}) := T(x) + \widetilde{H}_{B0},\ x \in S,$$

is well-defined and it is an $(\widetilde{A}, \widetilde{B})$-isometry, where $\widetilde{A}$, $\widetilde{B}$ are defined by (2.9) *(for $\widetilde{A}$ we take $B = A$, $\widetilde{H} = S$ in the previous lemma).*

PROOF. It is easy to observe that S_{AL}, S_{A0} are normal subgroups of S. Let $x, y \in S$. We observe that

$$A(x - y, z) = A(x, z) - A(y, z) = B(T(x), T(z)) - B(T(y), T(z))$$
$$= B(T(x) - T(y), T(z)),$$

so $x - y \in S_{AL} \Leftrightarrow T(x) - T(y) \in \widetilde{H}_{BL}$. In analogical way we can obtain that $x-y \in S_{AR} \Leftrightarrow T(x)-T(y) \in \widetilde{H}_{BR}$. Hence $x-y \in S_{A0} \Leftrightarrow T(x)-T(y) \in \widetilde{H}_{B0}$, so $\widetilde{T}$ is well-defined and it is injective.

Let $v \in \widetilde{H}$, then there exist $k_1, \ldots, k_n \in \mathbb{Z}$, $x_1, \ldots, x_n \in S$ such that $v = \sum_{i=1}^{n} k_i T(x_i)$. Since for $x, y, z \in S$ we have

$$0 = A(x + y - y - x, z) = A(x + y, z) - A(y, z) - A(x, z)$$
$$= B(T(x + y), T(z)) - B(T(y), T(z)) - B(T(x), T(z))$$
$$= B(T(x + y) - T(y) - T(x), T(z)),$$
$$0 = A(z, x + y - y - x) = A(z, x + y) - A(z, y) - A(z, x)$$
$$= B(T(z), T(x + y)) - B(T(z), T(y)) - B(T(z), T(x))$$
$$= B(T(z), T(x + y) - T(y) - T(x)),$$

then $T(x+y)-T(y)-T(x)\in \widetilde{H}_{B0}$. Hence we have

$$v+\widetilde{H}_{B0}=\sum_{i=1}^{n} k_iT(x_i)+\widetilde{H}_{B0}=T\Big(\sum_{i=1}^{n} k_ix_i\Big)+\widetilde{H}_{B0}=\widetilde{T}\Big(\sum_{i=1}^{n} k_ix_i+S_{A0}\Big),$$

which means that $\widetilde{T}$ is surjective. □

3. Main results

In this section we assume that S, H, X are groups, $A\colon S^2\to X$, $B\colon H^2\to X$ are biadditive functions, $\widetilde{H}$ is a normal subgroup of H, $\widetilde{H}_{BL}$, $\widetilde{H}_{BR}$, $\widetilde{H}_{B0}$, S_{AR} are given resp. by (2.6), (2.7), (2.8), (2.1).

Remark 3. Let $T\colon S\to H$, $g\colon S\to S$ satisfy the equation

$$B(T(x),T(y))=A(x,g(y)),\quad x,y\in S. \tag{3.1}$$

Then we can assume that H is generated by $\operatorname{im}T$.

Theorem 2. *Let $T\colon S\to H$, $g\colon S\to S$ satisfy equation (3.1), $\widetilde{H}:=\langle \operatorname{im}T\rangle$, $T^*\colon H\to S$ be a (B,A)-adjoint operator to T. Then $\widetilde{H}\subset D(T^*)$,*

$$T^*(T(y))-g(y)\in S_{AR},\quad y\in S.$$

Moreover, if $H=\widetilde{H}$ then $H_{BLD^*}=\widetilde{H}_{BL}$, $H_{BTR}=\widetilde{H}_{BR}$, where H_{BTR}, H_{BLD^*} are given respectively by (2.3), (2.4).

Proof. From (3.1) we obtain that $\operatorname{im}T\subset D(T^*)$, so $\widetilde{H}\subset D(T^*)$ and

$$\begin{aligned}A(x,T^*(T(y))-g(y))&=A(x,T^*(T(y)))-A(x,g(y))\\&=B(T(x),T(y))-B(T(x),T(y))=0,\quad x,y\in S.\end{aligned}$$

Hence $T^*(T(y))-g(y)\in S_{AR}$.

Assume that $H = \widetilde{H}$. We notice that $H_{BR} \subset H_{BTR}$ and since $H = D(T^*)$, we get $H_{BL} = H_{BLD^*}$. Let $v \in H_{BTR}$. For $u \in H$ there exist $x_1, \dots, x_n \in S$ and $k_1, \dots, k_n \in \mathbb{Z}$ such that $u = \sum_{i=1}^{n} k_i T(x_i)$. Then

$$B(u, v) = B\Big(\sum_{i=1}^{n} k_i T(x_i), v\Big) = \sum_{i=1}^{n} k_i B(T(x_i), v) = 0,$$

so $v \in H_{BR}$. □

Theorem 3. *Let $T\colon S \to H$, $g\colon S \to S$ satisfy equation (3.1), $\widetilde{H} := \langle \operatorname{im} T \rangle$, $T^*\colon H \to S$ be a (B, A)-adjoint operator to T, $\widetilde{H}_{BL} \subset \widetilde{H}_{BR}$. Then*

$$S_{ALT^*} \subset S_{ALg} := \{x \in S : \forall_{y \in S}\, A(x, g(y)) = 0\}, \tag{3.2}$$

$$g(x+y) - g(y) - g(x) \in S_{AR}, \quad x, y \in S, \tag{3.3}$$

$$g(S_{ALT^*}) \subset S_{AR},$$

and T is an (A_1, B)-quasi isometry, where S_{ALT^*} is given by (2.2) and $A_1\colon S^2 \to X$ is given by the formula

$$A_1(x, y) = A(x, g(y)), \quad x, y \in S. \tag{3.4}$$

Moreover, if $\widetilde{H} = H$, then $S_{ALT^*} = S_{ALg}$.

Proof. Using previous theorem we get $\langle \operatorname{im} g \rangle + S_{AR} \subset \langle \operatorname{im} T^* \rangle + S_{AR}$, so $S_{ALT^*} \subset S_{ALg}$ (when $\widetilde{H} = H$ we have $\langle \operatorname{im} g \rangle + S_{AR} = \langle \operatorname{im} T^* \rangle + S_{AR}$, so $S_{ALT^*} = S_{ALg}$).

Using Lemma 1 (b) we have also

$$\begin{aligned} A(z, g(x+y) - g(y) - g(x)) &= A(z, g(x+y)) - A(z, g(y)) - A(z, g(x)) \\ &= B(T(z), T(x+y)) - B(T(z), T(y)) - B(T(z), T(x)) \\ &= B(T(z), T(x+y) - T(y) - T(x)) = 0, \quad x, y, z \in S. \end{aligned}$$

Hence $g(x+y) - g(y) - g(x) \in S_{AR}$ for $x, y \in S$.

Let $x \in S$, $y \in S_{ALT^*}$. Then in view of Lemma 1 (c) we get $T(y) \in H_{BLD^*}$, so $T(y) \in \widetilde{H}_{BL} \subset \widetilde{H}_{BR}$. Hence we have

$$A(x, g(y)) = B(T(x), T(y)) = 0,$$

which means that $g(y) \in S_{AR}$.

For $x, y \in S$ we have

$$B(T(x), T(y)) = A(x, g(y)) = A_1(x, y),$$

which ends the proof. □

The following example shows that the assumption $\widetilde{H}_{BL} \subset \widetilde{H}_{BR}$ is important in the previous theorem.

EXAMPLE 1. Let $S = H = \mathbb{Q}^2$, $X = \mathbb{Q}$, $g = (g_1, g_2)\colon S \to S$ be an arbitrary function, $f\colon S \to H$ be a function given by the formula

$$f(x) = (x_1 + x_2, g_2(x)), \quad x = (x_1, x_2) \in \mathbb{Q}^2.$$

Let further $B\colon H^2 \to X$, $A\colon S^2 \to X$ be functions given by formulas

$$B((x_1, x_2), (y_1, y_2)) = x_1 y_2, \quad x_1, x_2, y_1, y_2 \in \mathbb{Q},$$

$$A((x_1, x_2), (y_1, y_2)) = (x_1 + x_2) y_2, \quad x_1, x_2, y_1, y_2 \in \mathbb{Q}.$$

It is easy to see that A, B are biadditive and $S_{AR} = \mathbb{Q} \times 0$.

We have also

$$A(x, g(y)) = (x_1 + x_2) g_2(y) = B((x_1 + x_2, g_2(x)), (y_1 + y_2, g_2(y)))$$

$$= B(f(x), f(y)), \quad x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{Q}^2.$$

Hence (3.1) holds. Of course g_2 can be nonadditive, so there exist $x, y \in S$ such that $g(x + y) - g(y) - g(x) \notin S_{AR}$.

It is a natural question whether given a function T there exists a function g such that (T, g) satisfies equation (3.1). The lemma below gives us an answer for this question.

LEMMA 3. *Let $T\colon S \to H$ be such that $\operatorname{im} T \subset D(T^*)$, where T^* is a (B, A)-adjoint operator to T. Then (T, g) satisfies equation (3.1), where $g = T^* \circ T\colon S \to S$.*

PROOF. We observe that

$$B(T(x), T(y)) = A(x, T^*(T(y))) = A(x, g(y)), \quad x, y \in S,$$

which ends the proof. □

The second natural question is whether given a function g there exists a function T such that (T, g) satisfies equation (3.1).

Lemma 4. *Let $g\colon S \to S$ satisfies conditions (3.3), $g(S_{ALg}) \subset S_{AR}$ and there exists a subgroup $\widetilde{H}$ of H such that there exists an (A_1, B)-quasi isometry $T\colon S \to \widetilde{H}$ where A_1, S_{ALg} are given resp. by (3.4), (3.2). Then (T, g) satisfies (3.1).*

Proof. Let $x, y \in S$. Then

$$B(T(x), T(y)) = A_1(x, y) = A(x, g(y)),$$

which ends the proof. □

In the previous lemma, in the case when S, H are unitary spaces and A, B are inner products, instead assuming the existence of an (A_1, B)-quasi isometry we can assume that g is positive and symmetric and there exists an isometry from S to some subspace of H (see [3, Theorem 8]). In general we do not have a similar result but we can write here the following

Theorem 4. *Let $g, h\colon S \to S$ be such that $\operatorname{im} h \subset D(h^*)$ and $g(x) - (h^* \circ h)(x) \in S_{AR}$ for $x \in S$, where $h^*\colon D(h^*) \to S$ is an (A, A)-adjoint operator to h. Assume that $\operatorname{im} h$ is (B, A)-isometric with some subset of H, i.e. there exists $I\colon S \to H$ such that*

$$B\big(I(h(x)), I(h(y))\big) = A(h(x), h(y)), \quad x, y \in S.$$

Then (T, g) satisfies (3.1) with $T := I \circ h$.

Proof. We observe that

$$\begin{aligned} B(T(x), T(y)) &= B\big(I(h(x)), I(h(y))\big) = A(h(x), h(y)) \\ &= A(x, (h^* \circ h)(y)) = A(x, g(y)), \quad x, y \in S. \end{aligned}$$ □

The following example shows that we cannot reverse the previous theorem.

Example 2. Let $S = H = X = \mathbb{Q}$, $A\colon S^2 \to X$, $B\colon H^2 \to X$, $g\colon S \to S$, $T\colon S \to H$ be maps given by formulas

$$A(x, y) = xy, \quad x, y \in S, \quad B(x, y) = -xy, \quad x, y \in S,$$

$$g(x) = -x, \quad x \in S, \quad T(x) = x, \quad x \in S.$$

Then it is easy to see that (T,g) satisfies (3.1). Suppose that there exists a map $h\colon S\to S$ such that $\operatorname{im} h\subset D(h^*)$, $g(x)-h^*\circ h(x)\in S_{AR}$, where $h^*\colon D(h^*)\to S$ is (A,A)-adjoint operator to h. Then

$$0\le h(x)^2=A(h(x),h(x))=A\big(x,(h^*\circ h)(x)\big)$$
$$=A(x,g(x))=A(x,-x)=-x^2,\quad x\in S,$$

which gives us a contradiction.

We can also say something about the family of functions T which satisfy equation (3.1) with the same g.

THEOREM 5. *Let $T_1,T\colon S\to H$, $g\colon S\to S$, (T,g) satisfies (3.1), $\widetilde{H}:=\langle\operatorname{im} T_1\rangle$. Then (T_1,g) satisfies (3.1) if and only if there exists a $(B|_{\langle\operatorname{im} T\rangle^2},B)$-quasi isometry $I\colon\langle\operatorname{im} T\rangle\to\langle\operatorname{im} T_1\rangle$ such that $T_1(x)-I(T(x))\in\widetilde{H}_{B0}$ for $x\in S$.*

PROOF. Assume that (T_1,g) satisfies (3.1). For $x\in\langle\operatorname{im} T\rangle$ let $\varphi(x)$ be an arbitrary element of the set $x+\widetilde{H}_{B0}$. We define $I\colon\langle\operatorname{im} T\rangle\to\widetilde{H}$ by the formula

$$I\Big(\sum_{i=1}^n k_iT(x_i)\Big):=\varphi\Big(\sum_{i=1}^n k_iT_1(x_i)+\widetilde{H}_{B0}\Big),\quad k_1,\dots,k_n\in\mathbb{Z},\ x_1,\dots,x_n\in S.$$

Let $k_1,\dots,k_n,r_1,\dots,r_n\in\mathbb{Z}$, $y,x_1,\dots,x_n\in S$ and $\sum_{i=1}^n k_iT(x_i)=\sum_{i=1}^n r_iT(x_i)$. Then

$$0=B\Big(\sum_{i=1}^n k_iT(x_i)-\sum_{i=1}^n r_iT(x_i),T(y)\Big)$$
$$=\sum_{i=1}^n k_iB\big(T(x_i),T(y)\big)-\sum_{i=1}^n r_iB\big(T(x_i),T(y)\big)$$
$$=\sum_{i=1}^n k_iA\big(x_i,g(y)\big)-\sum_{i=1}^n r_iA\big(x_i,g(y)\big)$$
$$=\sum_{i=1}^n k_iB\big(T_1(x_i),T_1(y)\big)-\sum_{i=1}^n r_iB\big(T_1(x_i),T_1(y)\big)$$
$$=B\Big(\sum_{i=1}^n k_iT_1(x_i)-\sum_{i=1}^n r_iT_1(x_i),T_1(y)\Big),$$

so $\sum\limits_{i=1}^{n} k_i T_1(x_i) - \sum\limits_{i=1}^{n} r_i T_1(x_i) \in \widetilde{H}_{BL}$. In analogical way we can prove that

$$\sum_{i=1}^{n} k_i T_1(x_i) - \sum_{i=1}^{n} r_i T_1(x_i) \in \widetilde{H}_{BR}.$$

Hence $\sum\limits_{i=1}^{n} k_i T_1(x_i) - \sum\limits_{i=1}^{n} r_i T_1(x_i) \in \widetilde{H}_0$, so I is well-defined. We have also

$$\begin{aligned}
B\Big(\sum_{i=1}^{n} k_i T(x_i), \sum_{j=1}^{n} r_j T(x_j)\Big) &= \sum_{i=1}^{n} k_i \sum_{j=1}^{n} r_j B\big(T(x_i), T(x_j)\big) \\
&= \sum_{i=1}^{n} k_i \sum_{j=1}^{n} r_j B\big(T_1(x_i), T_1(x_j)\big) \\
&= B\Big(\sum_{i=1}^{n} k_i T_1(x_i), \sum_{j=1}^{n} r_j T_1(x_j)\Big) \\
&= B\bigg(\varphi\Big(\sum_{i=1}^{n} k_i T_1(x_i)\Big), \varphi\Big(\sum_{j=1}^{n} r_j T_1(x_j)\Big)\bigg),
\end{aligned}$$

so I is a $(B|_{\langle \operatorname{im} T\rangle^2}, B)$-quasi isometry. We have also

$$I(T(x)) + \widetilde{H}_{B0} = \varphi(T_1(x) + \widetilde{H}_{B0}) + \widetilde{H}_{B0} = T_1(x) + \widetilde{H}_{B0}, \quad x \in S,$$

so $T_1(x) - I(T(x)) \in \widetilde{H}_{B0}$ for $x \in S$.

Let $I\colon \langle \operatorname{im} T\rangle \to \widetilde{H}$ be a $(B|_{\langle \operatorname{im} T\rangle^2}, B)$-quasi isometry such that $T_1(x) - I(T(x)) \in \widetilde{H}_{B0}$ for $x \in S$. Then

$$\begin{aligned}
B(T_1(x), T_1(y)) &= B\big(I\left(T(x)\right), I(T(y))\big) \\
&= B(T(x), T(y)) = A(x, g(y)), \quad x, y \in S,
\end{aligned}$$

which ends the proof. □

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