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NOTES ON A GENERAL SEQUENCE

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Abstract. Let $\{r_n\}_{n\in\mathbb{N}}$ be a strictly increasing sequence of nonnegative real numbers satisfying the asymptotic formula $r_n \sim \alpha\beta^n$, where α, β are real numbers with $\alpha > 0$ and $\beta > 1$. In this note we prove some limits that connect this sequence to the number e. We also establish some asymptotic formulae and limits for the counting function of this sequence. All of the results are applied to some well-known sequences in mathematics.

1. Introduction

Let $\{r_n\}_{n\in\mathbb{N}}$ be a strictly increasing sequence of nonnegative real numbers satisfying the asymptotic formula

$$r_n \sim \alpha\beta^n, \quad \alpha > 0, \beta > 1, \tag{1.1}$$

i.e., $\lim_{n\to\infty} \frac{r_n}{\alpha\beta^n} = 1$.

In this paper, we are interested in finding some general results for this sequence. Afterwards, we show that all of the results are applied to some well-known sequences in mathematics.

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2. Main results

In this section we aim to present our main results on sequences satisfying asymptotic formula (1.1). We have the following theorems.

THEOREM 2.1. *Let $r_1 > 1$. The following limit holds:*

$$\lim_{n\to\infty} \frac{\sqrt[n]{\log r_1 \log r_2 ... \log r_n}}{\log r_n} = \frac{1}{e}.$$

PROOF. By (1.1), we have

$$\log\log r_n = \log n + \log\log\beta + o(1). \tag{2.1}$$

On the other hand, from the Stirling's approximation for $n!$ (i.e. $n! \sim (\frac{n}{e})^n\sqrt{2\pi n}$) [1], we obtain

$$\sum_{i=1}^{n} \log i = n\log n - n + o(n). \tag{2.2}$$

Hence, (2.1) and (2.2) give

$$\begin{aligned}
&\log\left(\frac{\sqrt[n]{\log r_1 \log r_2 ... \log r_n}}{\log r_n}\right)\\
&\quad = \frac{1}{n}\big(\log\log r_1 + \log\log r_2 + \cdots + \log\log r_n\big) - \log\log r_n\\
&\quad = \frac{1}{n}\left(\sum_{i=1}^{n}\log i + n\log\log\beta + o(n)\right) - \big(\log n + \log\log\beta + o(1)\big)\\
&\quad = -1 + o(1).
\end{aligned}$$

This completes the proof. □

THEOREM 2.2. *Let $r_1 > 1$. If k is an arbitrary but fixed positive integer, then*

$$\lim_{n\to\infty} \frac{\Big((\log r_1)^{1^k}(\log r_2)^{2^k}\ldots(\log r_n)^{n^k}\Big)^{\frac{k+1}{n^{k+1}}}}{\log r_n} = \frac{1}{e^{k+1}}.$$

PROOF. By (1.1), we have

$$\log\log r_i = \log i + \log\log\beta + f(i) \quad (i \geq 1), \tag{2.3}$$

where

$$f(i) \to 0. \tag{2.4}$$

Now, we have (see (2.3))

$$\begin{aligned} \log\left((\log r_1)^{1^k}(\log r_2)^{2^k}\dots(\log r_n)^{n^k}\right) &= \sum_{i=1}^{n} i^k \log\log r_i \\ &= \sum_{i=1}^{n}\left(i^k\log i + i^k\log\log\beta + f(i)i^k\right) \\ &= \sum_{i=1}^{n} i^k \log i + \sum_{i=1}^{n} i^k\log\log\beta + \sum_{i=1}^{n} f(i)i^k. \end{aligned} \tag{2.5}$$

We know that the function $x^k \log x$ (with $k \geq 1$) is strictly increasing and nonnegative on the interval $[1,\infty)$. Hence, as an immediate consequence of the definition of integral as area below the curve $x^k \log x$, as well as using integration by parts, we find that

$$\begin{aligned} \sum_{i=1}^{n} i^k \log i &= \int_1^n x^k \log x \; dx + O(n^k \log n) \\ &= \frac{n^{k+1}}{k+1}\log n - \frac{n^{k+1}}{(k+1)^2} + o(n^{k+1}). \end{aligned} \tag{2.6}$$

A similar argument shows that

$$\sum_{i=1}^{n} i^k = \int_1^n x^k \; dx + O(n^k) = \frac{n^{k+1}}{k+1} + o(n^{k+1}). \tag{2.7}$$

Now, given $\epsilon > 0$, there exists n_0 such that if $n \geq n_0$ we have $|f(i)| < \epsilon$ (see (2.4)). Hence,

$$\begin{aligned} \left|\sum_{i=1}^{n} f(i)i^k\right| &\leq \sum_{i=1}^{n} |f(i)|i^k \leq \sum_{i=1}^{n_0-1} |f(i)|i^k + \epsilon\sum_{i=n_0}^{n} i^k \\ &\leq \sum_{i=1}^{n_0-1} |f(i)|i^k + \epsilon\sum_{i=1}^{n} i^k. \end{aligned} \tag{2.8}$$

Therefore (see (2.7) and (2.8)) from a certain value of n, we have

$$\left|\frac{\sum_{i=1}^{n} f(i)i^k}{n^{k+1}}\right| \leq \frac{\sum_{i=1}^{n_0-1} |f(i)|i^k}{n^{k+1}} + \epsilon\left(\frac{\sum_{i=1}^{n} i^k}{n^{k+1}}\right) \leq \epsilon,$$

where ϵ is arbitrarily small. That is,

$$\sum_{i=1}^{n} f(i)i^k = o(n^{k+1}). \tag{2.9}$$

Equalities (2.5), (2.6), (2.7), and (2.9) give

$$\begin{aligned} &\log\left((\log r_1)^{1^k}(\log r_2)^{2^k}\dots(\log r_n)^{n^k}\right) \\ &\qquad = \frac{n^{k+1}}{k+1}\log n + \frac{n^{k+1}}{k+1}\log\log\beta - \frac{n^{k+1}}{(k+1)^2} + o(n^{k+1}). \end{aligned} \tag{2.10}$$

Hence, (2.3) and (2.10) give

$$\begin{aligned} &\log\left(\frac{\left((\log r_1)^{1^k}(\log r_2)^{2^k}\dots(\log r_n)^{n^k}\right)^{\frac{k+1}{n^{k+1}}}}{\log r_n}\right) \\ &\quad = \frac{k+1}{n^{k+1}}\log\left((\log r_1)^{1^k}(\log r_2)^{2^k}\dots(\log r_n)^{n^k}\right) - \log\log r_n \\ &\quad = -\frac{1}{k+1} + o(1). \end{aligned}$$

This completes the proof. □

Theorem 2.3. *The following limit holds:*

$$\lim_{n\to\infty}\left(\frac{\log r_{n+1}}{\log r_n}\right)^n = e.$$

Proof. Condition (1.1) gives

$$\log r_n = n\log\beta + \log\alpha + o(1).$$

Therefore if we put, for sake of simplicity, $c = \frac{\log\alpha}{\log\beta}$, then

$$\left(\frac{\log r_{n+1}}{\log r_n}\right)^n = \left(\frac{(n+1)\log\beta + \log\alpha + o(1)}{n\log\beta + \log\alpha + o(1)}\right)^n$$

$$= \left(1+\frac{1}{n}\right)^n \frac{\left(1+\frac{c+o(1)}{n+1}\right)^n}{\left(1+\frac{c+o(1)}{n}\right)^n} \to e\frac{e^c}{e^c} = e:$$

in fact, it is well-known that if the sequence $a_n \to \infty$, then

$$\lim_{n\to\infty}\left(1+\frac{1}{a_n}\right)^{a_n} = e,$$

hence

$$\left(1+\frac{c+o(1)}{n+1}\right)^n = \left(\left(1+\frac{1}{\frac{n+1}{c+o(1)}}\right)^{\frac{n+1}{c+o(1)}}\right)^{\frac{n}{n+1}(c+o(1))} \to e^c,$$

analogously the other limit. This completes the proof. $\square$

Now, let $A(x)$ be the number of r_n not exceeding x, that is $A(x)$ is the counting function of the sequence $\{r_n\}_{n\in\mathbb{N}}$. In following we establish some asymptotic formulae for the function $A(x)$, and we also obtain some general results for the sequence $\{A(n)\}_{n\in\mathbb{N}}$.

THEOREM 2.4. *We have*

$$A(x) \sim \frac{1}{\log\beta}\log x. \tag{2.11}$$

More precisely, we have

$$A(x) = \frac{1}{\log\beta}\log x + O(1). \tag{2.12}$$

PROOF. By (1.1), we have $\log r_n \sim n\log\beta$, that is, $\log r_n \sim A(r_n)\log\beta$, thus, $A(r_n) \sim \frac{1}{\log\beta}\log r_n$. Clearly $\frac{\log r_{n+1}}{\log r_n} \to 1$ and if $x \in [r_n, r_{n+1})$, then $A(x) = A(r_n)$. Therefore

$$1 \leftarrow \frac{\log r_n}{\log r_{n+1}}\frac{A(r_n)}{\frac{1}{\log\beta}\log r_n} = \frac{A(r_n)}{\frac{1}{\log\beta}\log r_{n+1}} \le \frac{A(x)}{\frac{1}{\log\beta}\log x} \le \frac{A(r_n)}{\frac{1}{\log\beta}\log r_n} \to 1.$$

Property (2.11) is proved.

We have $\log r_n = \log \alpha + n \log \beta + o(1)$ and consequently

$$A(r_n) = \frac{\log r_n - \log \alpha - o(1)}{\log \beta}.$$

If $x \in [r_n, r_{n+1})$, then $A(x) = A(r_n)$ and the last equation can be written in the form

$$A(x) = \frac{\log x - (\log x - \log r_n) - \log \alpha - o(1)}{\log \beta} = \frac{\log x}{\log \beta} + O(1),$$

since $0 \le \log x - \log r_n \le \log r_{n+1} - \log r_n = \log \beta + o(1)$. This proves equality (2.12). The theorem is proved. $\square$

THEOREM 2.5. *If k is an arbitrary but fixed positive integer, then*

$$\big(A(1)^k + A(2)^k + \cdots + A(n)^k\big) \sim \frac{1}{\log^k \beta} n \log^k n, \tag{2.13}$$

$$\frac{A(1)^k + A(2)^k + \cdots + A(n)^k}{n} \sim A(n)^k. \tag{2.14}$$

PROOF. First, let us recall the well-known proposition (see [7, page 332]) that states for two series of positive terms $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$, if $\sum_{i=1}^{\infty} b_i$ diverges and $a_i \sim b_i$, then $\sum_{i=1}^{n} a_i \sim \sum_{i=1}^{n} b_i$ and consequently also $\sum_{i=1}^{\infty} a_i$ diverges. Now, using this fact and by use of (2.11), we have

$$\big(A(1)^k + A(2)^k + \cdots + A(n)^k\big) \sim \sum_{i=1}^{n} \frac{\log^k i}{\log^k \beta}. \tag{2.15}$$

On the other hand, we know that $\frac{\log^k x}{\log^k \beta}$ is increasing and positive in the interval $(1, \infty)$. Hence, as an immediate consequence of the definition of integral as area below the curve $\frac{\log^k x}{\log^k \beta}$, we find that

$$\sum_{i=1}^{n} \frac{\log^k i}{\log^k \beta} = \int_1^n \frac{\log^k x}{\log^k \beta}\, dx + O\left(\frac{\log^k n}{\log^k \beta}\right) \sim \frac{n \log^k n}{\log^k \beta}, \tag{2.16}$$

since (L'Hospital's rule) $\lim_{x\to\infty} \frac{\int_1^x \log^k t\ dt}{x \log^k x} = 1$ (see also [3]). Hence, (2.15) and (2.16) give equality (2.13). Property (2.14) is an immediate consequence of (2.11) and (2.13). The theorem is proved. $\square$

Meanwhile, as an immediate consequence of previous theorem, we obtain the following corollary.

COROLLARY 2.6. *If k is an arbitrary but fixed positive integer, then*

$$\big(A(1)^k + A(2)^k + \cdots + A(n)^k\big)^{\frac{1}{\log n}} \to e, \tag{2.17}$$

$$\big(A(1)^k + A(2)^k + \cdots + A(n)^k\big)^{\frac{1}{A(n)}} \to \beta. \tag{2.18}$$

PROOF. Equality (2.17) is an immediate consequence of (2.13). Equality (2.18) is an immediate consequence of (2.11) and (2.13). The corollary is proved. □

3. Some sequences with property (1.1)

Some well-known sequences follow condition (1.1). For example, the Fibonacci sequence $\{F_n\}_{n\geq 0}$ is one of them, which is defined as follows:

$$F_{n+1} = F_n + F_{n-1},$$

with $F_0 = 0$ and $F_1 = 1$. The Fibonacci numbers are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \dots .$$

Now, we shall show that the Fibonacci sequence satisfies condition (1.1). By the well-known Binet's formula, we have

$$F_n = \frac{\varphi^n - (-\frac{1}{\varphi})^n}{\sqrt{5}}, \tag{3.1}$$

where $\varphi = \frac{1+\sqrt{5}}{2}(\approx 1.6180339...)$ is the *golden ratio*.

Hence, the Binet's formula can also be written as

$$F_n = \frac{1}{\sqrt{5}}\left(\varphi^n + \frac{(-1)^{n+1}}{\varphi^n}\right) = \frac{1}{\sqrt{5}}\varphi^n\left(1 + \frac{(-1)^{n+1}}{\varphi^{2n}}\right),$$

which implies that

$$F_n \sim \frac{1}{\sqrt{5}}\varphi^n.$$

Therefore, the Fibonacci sequence holds under condition (1.1), and then all of the results proved in this paper can be applied to the Fibonacci sequence. Note that Theorem 2.1 and Theorem 2.2 hold for the sequence $\{F_n\}_{n\geq 3}$ of Fibonacci numbers, and therefore

$$\lim_{n\to\infty} \frac{\sqrt[n]{\log F_3 \log F_4 \ldots \log F_n}}{\log F_n} = \frac{1}{e},$$

$$\lim_{n\to\infty} \frac{\left((\log F_3)^{3^k} (\log F_4)^{4^k} \ldots (\log F_n)^{n^k}\right)^{\frac{k+1}{n^{k+1}}}}{\log F_n} = \frac{1}{e^{k+1}}.$$

By Theorem 2.3, we have also the following limit which connects the number e with Fibonacci numbers:

$$\lim_{n\to\infty} \left(\frac{\log F_{n+1}}{\log F_n}\right)^n = e.$$

Now, recall the golden ratio φ. It is well-known that the ratio of two consecutive Fibonacci numbers tends to the golden number, i.e.,

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi. \tag{3.2}$$

In fact, the limit in (3.2) is an immediate consequence of Binet's formula (3.1). The golden number φ, can be expressed exactly by the following infinite series of continued fractions and that of continued square roots (see, for example, [8]):

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}},$$

and

$$\varphi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}.$$

Here, Corollary 2.6 gives a new expansion of the golden number φ. Clearly, our new expansion is based on the counting function of the Fibonacci numbers.

More exactly, if $\wp(x)$ denotes the counting function of the Fibonacci numbers, i.e., the number of F_n not exceeding x, then (2.18) gives

$$\lim_{n\to\infty} \sqrt[\wp(n)]{\wp(1)+\wp(2)+\cdots+\wp(n)} = \varphi. \tag{3.3}$$

As can be seen, the new expansion in (3.3) shows a new strong relationship of Fibonacci numbers with the golden ratio.

Further examples of the sequences that apply to condition (1.1) are the Lucas, Jacobsthal, Pell, Pell-Lucas, and Jacobsthal-Lucas sequences that are defined as follows (for $n \geq 0$), respectively:

$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2,\ L_1 = 1,$$

$$J_{n+2} = J_{n+1} + 2J_n, \quad J_0 = 0,\ J_1 = 1,$$

$$P_{n+2} = 2P_{n+1} + P_n, \quad P_0 = 0,\ P_1 = 1,$$

$$Q_{n+2} = 2Q_{n+1} + Q_n, \quad Q_0 = 2,\ Q_1 = 2,$$

$$j_{n+2} = j_{n+1} + 2j_n, \quad j_0 = 2,\ j_1 = 1.$$

The Lucas, Jacobsthal, Pell, Pell-Lucas, and Jacobsthal-Lucas numbers are

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \ldots,$$

$$0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, \ \ldots,$$

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, \ldots,$$

$$1, 1, 3, 7, 17, 41, 99, 239, 577, 1393, 3363, \ \ldots,$$

and

$$2, 1, 5, 7, 17, 31, 65, 127, 257, 511, 1025, \ldots,$$

respectively. For more information see, [2],[4], [6], and [9].

Similarly, the following well-known formulae exist for the Lucas, Jacobsthal, Pell, Pell-Lucas, and Jacobsthal-Lucas numbers (see [5]), respectively:

$$L_n = \frac{(1+\sqrt{5})^n + (1-\sqrt{5})^n}{2^n},$$

$$J_n = \frac{2^n - (-1)^n}{3},$$

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}},$$

$$Q_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n,$$

$$j_n = 2^n + (-1)^n.$$

Consequently $L_n \sim (\frac{1+\sqrt{5}}{2})^n$, $J_n \sim \frac{1}{3}2^n$, $P_n \sim \frac{1}{2\sqrt{2}}(1+\sqrt{2})^n$, $Q_n \sim \frac{1}{2}(1+\sqrt{2})^n$, and $j_n \sim 2^n$. Hence, the Lucas, Jacobsthal, Pell, Pell-Lucas, and Jacobsthal-Lucas numbers satisfy condition (1.1).

We invite the interested reader to research further sequences.

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References

[1] M. Abramowitz and I.A. Stegun (eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, New York, 1972.
[2] G. Bilgici and T.D. Şentürk, *Some addition formulas for Fibonacci, Pell and Jacobsthal numbers*, Ann. Math. Sil. **33** (2019), 55–65.
[3] R. Jakimczuk, *Functions of slow increase and integer sequences*, J. Integer Seq. **13** (2010), Article 10.1.1, 14 pp.
[4] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley-Interscience, New York, 2001.
[5] T. Koshy, *Sums of Fibonacci–Pell–Jacobsthal products*, Internat. J. Math. Ed. Sci. Tech. **44** (2013), 559–568.
[6] B. Pritsker, *The Equations World*, Dover Publications, New York, 2019.
[7] J. Rey Pastor, P. Pi Calleja and C.A. Trejo, *Análisis Matemático*, Vol. 1, Editorial Kapelusz, Buenos Aires, 1969.
[8] A.P. Stakhov, *The golden section in the measurement theory*, Comput. Math. Appl. **17** (1989), 613–638.
[9] A. Szynal-Liana and I. Włoch, *On Jacobsthal and Jacobsthal-Lucas hybrid numbers*, Ann. Math. Sil. **33** (2019), 276–283.

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