

NUMERICAL COMPARISON OF FNVIM AND FNHPM FOR SOLVING A CERTAIN TYPE OF NONLINEAR CAPUTO TIME-FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. This work presents a numerical comparison between two efficient methods namely the fractional natural variational iteration method (FNVIM) and the fractional natural homotopy perturbation method (FNHPM) to solve a certain type of nonlinear Caputo time-fractional partial differential equations in particular, nonlinear Caputo time-fractional wave-like equations with variable coefficients. These two methods provided an accurate and efficient tool for solving this type of equations. To show the efficiency and capability of the proposed methods we have solved some numerical examples. The results show that there is an excellent agreement between the series solutions obtained by these two methods. However, the FNVIM has an advantage over FNHPM because it takes less time to solve this type of nonlinear problems without using He's polynomials. In addition, the FNVIM enables us to overcome the difficulties arising in identifying the general Lagrange multiplier and it may be considered as an added advantage of this technique compared to the FNHPM.

1. Introduction

In mathematics, the fractional calculus is a branch of the analysis, which studies the generalization of the derivation and integration of integer

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order n (ordinary) to the non-integer order (fractional). It has turned out that many phenomena in engineering, physics and other sciences can be described very successfully by models using mathematical tools from fractional calculus. Recently, The nonlinear fractional partial differential equations appeared in many branches of physics, engineering and applied mathematics including fluid mechanics, viscoelasticity, aerodynamics, electrodynamics, rheology, mathematical biology and so on (see [6],[7],[11],[13],[16],[17]). Hence, it is important to solve nonlinear fractional partial differential equations. In general, there exists no method that yields an exact solution for nonlinear fractional partial differential equations due to the computational complexities of nonlinear parts involving them. Therefore, several different and powerful methods for solving fractional partial differential equations have been proposed in order to obtain the approximate solutions. The most commonly used ones are: the adomian decomposition method (ADM) [4], variational iteration method (VIM) [15] homotopy analysis method (HAM) [12], homotopy perturbation method (HPM) [5], fractional reduced differential transform method [10], and fractional residual power series method (FRPSM) [9].

The main objective of this paper is to introduce a numerical comparison of two powerful methods, the fractional natural variational iteration method (FNVIM) and the fractional natural homotopy perturbation method (FNHPM) for solving certain type of nonlinear Caputo time-fractional partial differential equations in particular, nonlinear Caputo time-fractional wave-like equation with variable coefficients of the form ([8],[9])

$$(1.1) \quad D_t^\alpha v = \sum_{i,j=1}^n F_{1ij}(X, t, v) \frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(v_{x_i}, v_{x_j}) \\ + \sum_{i=1}^n G_{1i}(X, t, v) \frac{\partial^p}{\partial x_i^p} G_{2i}(v_{x_i}) + H(X, t, v) + S(X, t),$$

with the initial conditions

$$(1.2) \quad v(X, 0) = a_0(X), \quad v_t(X, 0) = a_1(X),$$

where D_t^α is the Caputo fractional derivative operator of order α , $1 < \alpha \leq 2$ and v is a function of $(X, t) \in \mathbb{R}^n \times \mathbb{R}^+$, F_{1ij}, G_{1i} $i, j \in \{1, 2, \dots, n\}$ are nonlinear functions of X, t and v , F_{2ij}, G_{2i} $i, j \in \{1, 2, \dots, n\}$, are nonlinear functions of derivatives of v with respect to x_i and x_j $i, j \in \{1, 2, \dots, n\}$, respectively. Also H, S are nonlinear functions and k, m, p are integers.

2. Definitions and properties

We present some definitions and important properties of the fractional calculus theory and natural transform that will be widely used in this paper.

DEFINITION 2.1 ([11]). Let $f \in L^1(0, T), T > 0$. The *Riemann-Liouville fractional integral of order $\alpha \geq 0$* is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

DEFINITION 2.2 ([11]). Let $f^{(n)} \in L^1(0, T), T > 0$. The *Liouville-Caputo fractional derivative of order $\alpha \geq 0$* is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - \tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau,$$

where $n - 1 < \alpha \leq n, n = [\alpha] + 1$ with $[\alpha]$ being the integer part of α .

DEFINITION 2.3 ([11]). The *Mittag-Leffler function* is defined as follows

$$(2.1) \quad E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha + 1)}, \quad \alpha \in \mathbb{C}, \operatorname{Re}(\alpha) > 0.$$

For $\alpha = 1, E_\alpha(z)$ reduces to e^z . A further generalization of (2.1) is given in the form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0.$$

DEFINITION 2.4 ([1]). The *natural transform* is defined over the set of functions

$$A = \left\{ f(t) / \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty), j \in \mathbb{Z}^+ \right\},$$

by the following integral

$$\mathcal{N}^+[f(t)] = R^+(s, u) = \frac{1}{u} \int_0^{+\infty} e^{-\frac{st}{u}} f(t) dt, \quad s, u \in (0, \infty).$$

THEOREM 2.5 ([8]). *Let $n \in \mathbb{N}^*$ and $\alpha > 0$ be such that $n - 1 < \alpha \leq n$ and $R^+(s, u)$ be the natural transform of the function $f(t)$, then the natural transform denoted by $R_\alpha^+(s, u)$ of the Caputo fractional derivative of the function $f(t)$ of order α , is given by*

$$\mathcal{N}^+[D^\alpha f(t)] = R_\alpha^+(s, u) = \frac{s^\alpha}{u^\alpha} R^+(s, u) - \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{u^{\alpha-k}} [D_t^k f(t)]_{t=0}.$$

3. FNVIM for nonlinear Caputo time-fractional wave-like equations

THEOREM 3.1. *Consider the nonlinear Caputo time-fractional wave-like equations (1.1) with initial conditions (1.2). Then, by the FNVIM the exact solution of the equations (1.1) and (1.2) is given as a limit of the successive approximations $v_n(X, t)$, $n = 0, 1, 2, \dots$, in other words*

$$v(X, t) = \lim_{n \rightarrow \infty} v_n(X, t).$$

PROOF. To prove the above theorem, firstly we define the nonlinear operators

$$\begin{aligned} Nv &= \sum_{i,j=1}^n F_{1ij}(X, t, v) \frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(v_{x_i}, v_{x_j}), \\ Mv &= \sum_{i=1}^n G_{1i}(X, t, v) \frac{\partial^p}{\partial x_i^p} G_{2i}(v_{x_i}), \\ Kv &= H(X, t, v). \end{aligned}$$

Then, the equation (1.1) is written in the form

$$(3.1) \quad D_t^\alpha v(X, t) = Nv(X, t) + Mv(X, t) + Kv(X, t) + S(X, t).$$

The methodology consists of applying the natural transform first on both sides of (3.1) and using the Theorem 2.5, we have

$$(3.2) \quad \mathcal{N}^+ [v(X, t)] = \frac{u^\alpha}{s^\alpha} \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{u^{\alpha-k}} [D^k v(X, t)]_{t=0} + \frac{u^\alpha}{s^\alpha} \mathcal{N}^+ [Nv(X, t) + Mv(X, t) + Kv(X, t) + S(X, t)].$$

Operating the inverse natural transform on both sides of (3.2), we get

$$(3.3) \quad v(X, t) = L(X, t) + \mathcal{N}^{-1} \left(\frac{u^\alpha}{s^\alpha} \mathcal{N}^+ [Nv(X, t) + Mv(X, t) + Kv(X, t)] \right),$$

where $L(X, t)$ is a term arising from the source term and the prescribed initial conditions. After that, let us take the first partial derivative with respect to t of the equation (3.3), to obtain

$$\frac{\partial}{\partial t} v(X, t) = \frac{\partial}{\partial t} L(X, t) + \frac{\partial}{\partial t} \mathcal{N}^{-1} \left(\frac{u^\alpha}{s^\alpha} \mathcal{N}^+ [Nv(X, t) + Mv(X, t) + Kv(X, t)] \right).$$

According to the variational iteration method ([3]), we can construct a correct functional as follows

$$(3.4) \quad v_{n+1}(X, t) = v_n(X, t) + \int_0^t \lambda(\tau) \left[\frac{\partial v_n}{\partial \tau} - \frac{\partial}{\partial \tau} \mathcal{N}^{-1} \left(\frac{u^\alpha}{s^\alpha} \mathcal{N}^+ [N\tilde{v}_n + M\tilde{v}_n + K\tilde{v}_n] \right) - \frac{\partial L}{\partial \tau} \right] d\tau.$$

where $\lambda(\tau)$ is a general Lagrange multiplier which can be identified optimally via the variational theory and integration by parts. The subscript n denotes the n^{th} -order approximation, \tilde{v}_n is considered as a restricted variation (i.e. $\delta\tilde{v}_n = 0$). Making the above correction functional stationary, and noting that $\delta\tilde{v}_n = 0$,

$$\delta v_{n+1}(X, t) = \delta v_n(X, t) + \delta \int_0^t \lambda(\tau) \left[\frac{\partial v_n}{\partial \tau} - \frac{\partial L}{\partial \tau} \right] d\tau,$$

we obtain the following stationary conditions

$$\begin{aligned} 1 + \lambda(\tau)|_{\tau=t} &= 0, \\ \lambda'(\tau)|_{\tau=t} &= 0. \end{aligned}$$

Therefore, the Lagrange multiplier can be easily identified as

$$(3.5) \quad \lambda(\tau) = -1.$$

Substituting equation (3.5) into the correction functional equation (3.4), we get the iterative formula for $n = 0, 1, 2, \dots$, as follows

$$\begin{aligned} v_{n+1}(X, t) &= v_n(X, t) \\ &\quad - \int_0^t \left[\frac{\partial v_n}{\partial \tau} - \frac{\partial}{\partial \tau} \mathcal{N}^{-1} \left(\frac{u^\alpha}{s^\alpha} \mathcal{N}^+ [Nv_n + Mv_n + Kv_n] \right) - \frac{\partial L}{\partial \tau} \right] d\tau. \end{aligned}$$

Or

$$v_{n+1}(X, t) = L(X, t) + \mathcal{N}^{-1} \left(\frac{u^\alpha}{s^\alpha} \mathcal{N}^+ [Nv_n(X, t) + Mv_n(X, t) + Kv_n(X, t)] \right).$$

Finally, the exact solution of the equations (1.1) and (1.2) is given as a limit of the successive approximations $v_n(X, t)$, $n = 0, 1, 2, \dots$, in other words

$$v(X, t) = \lim_{n \rightarrow \infty} v_n(X, t). \quad \square$$

4. FNHPM for nonlinear Caputo time-fractional wave-like equations

THEOREM 4.1. *Consider the following nonlinear Caputo time-fractional wave-like equations (1.1) with the initial conditions (1.2). Then, by FNHPM the solution of the equations (1.1) and (1.2) is given in the form of infinite series which converges rapidly to the exact solution as follows*

$$v(X, t) = \sum_{n=0}^{\infty} v_n(X, t).$$

PROOF. Similarly like in the proof of Theorem 3.1, we have

$$(4.1) \quad v(X, t) = L(X, t) + \mathcal{N}^{-1} \left(\frac{u^\alpha}{s^\alpha} \mathcal{N}^+ [Nv(X, t) + Mv(X, t) + Kv(X, t)] \right).$$

Now, applying the homotopy perturbation method ([2]), we can assume that the solution can be expressed as a power series in p as given below

$$(4.2) \quad v(X, t) = \sum_{n=0}^{\infty} p^n v_n(X, t),$$

where the homotopy parameter p is considered as a small parameter $p \in [0, 1]$.

The nonlinear terms can be decomposed as

$$(4.3) \quad \begin{aligned} Nv(X, t) &= \sum_{n=0}^{\infty} p^n H_n(v), & Mv(X, t) &= \sum_{n=0}^{\infty} p^n K_n(v), \\ Kv(X, t) &= \sum_{n=0}^{\infty} p^n J_n(v), \end{aligned}$$

where $H_n(v)$, $K_n(v)$ and $J_n(v)$ are He's polynomials ([14]), and it can be calculated by the formulas given below

$$(4.4) \quad \begin{aligned} H_n(v_0, v_1, \dots, v_n) &= \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N \left(\sum_{i=0}^{\infty} p^i v_i \right) \right]_{p=0}, \\ K_n(v_0, v_1, \dots, v_n) &= \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[M \left(\sum_{i=0}^{\infty} p^i v_i \right) \right]_{p=0}, \\ J_n(v_0, v_1, \dots, v_n) &= \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[K \left(\sum_{i=0}^{\infty} p^i v_i \right) \right]_{p=0}, \end{aligned}$$

Substituting the equalities (4.2) and (4.3) into (4.1), we get

$$(4.5) \quad \begin{aligned} \sum_{n=0}^{\infty} p^n v_n(X, t) &= L(X, t) + p \left[\mathcal{N}^{-1} \left(\frac{u^\alpha}{s^\alpha} \mathcal{N}^+ \left[\sum_{n=0}^{\infty} p^n H_n(v) \right. \right. \right. \\ &\quad \left. \left. \left. + \sum_{n=0}^{\infty} p^n K_n(v) + \sum_{n=0}^{\infty} p^n J_n(v) \right] \right) \right]. \end{aligned}$$

Using the coefficient of the like powers of p in (4.5), the following approximations are obtained

$$\begin{aligned} p^0 : \quad & v_0(X, t) = L(X, t), \\ p^1 : \quad & v_1(X, t) = \mathcal{N}^{-1} \left(\frac{u^\alpha}{s^\alpha} \mathcal{N}^+ [H_0(v) + K_0(v) + J_0(v)] \right), \\ p^2 : \quad & v_2(X, t) = \mathcal{N}^{-1} \left(\frac{u^\alpha}{s^\alpha} \mathcal{N}^+ [H_1(v) + K_1(v) + J_1(v)] \right), \\ p^3 : \quad & v_3(X, t) = \mathcal{N}^{-1} \left(\frac{u^\alpha}{s^\alpha} \mathcal{N}^+ [H_2(v) + K_2(v) + J_2(v)] \right), \\ & \vdots \end{aligned}$$

Finally, the solution of the equations (1.1) and (1.2) is given in the form of infinite series as follows

$$v(X, t) = \sum_{n=0}^{\infty} v_n(X, t). \quad \square$$

5. Numerical applications

In order to evaluate the advantages and the accuracy of the FNVIM and FNHPM for the resolution of nonlinear Caputo time-fractional wave-like equations with variable coefficients, we will consider the following three numerical examples. All the results are calculated using Matlab (version 7.9.0.529 (R2009b)).

EXAMPLE 5.1. Consider the 2-dimensional nonlinear Caputo time-fractional wave-like equation with variable coefficients

$$(5.1) \quad D_t^\alpha v = \frac{\partial^2}{\partial x \partial y} (v_{xx} v_{yy}) - \frac{\partial^2}{\partial x \partial y} (xy v_x v_y) - v,$$

with the initial conditions

$$(5.2) \quad v(x, y, 0) = e^{xy}, \quad v_t(x, y, 0) = e^{xy},$$

where D_t^α is the Caputo fractional derivative operator of order α , $1 < \alpha \leq 2$ and v is a function of $(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}^+$.

Application of the FNVIM. Following the description of the FNVIM presented in Section 3, we obtain the iteration formula as follows

$$v_{n+1} = e^{xy} + te^{xy} + \mathcal{N}^{-1} \left(\frac{u^\alpha}{s^\alpha} \mathcal{N}^+ \left[\frac{\partial^2}{\partial x \partial y} (v_{nxx} v_{nyy}) - \frac{\partial^2}{\partial x \partial y} (xy v_{nx} v_{ny}) - v_n \right] \right),$$

and

$$v_0 = v_0(x, y, t) = (1 + t) e^{xy},$$

$$v_1 = v_1(x, y, t) = \left(1 + t - \frac{t^\alpha}{\Gamma(\alpha + 1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} \right) e^{xy},$$

$$v_2 = v_2(x, y, t) = \left(1 + t - \frac{t^\alpha}{\Gamma(\alpha + 1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} \right) e^{xy},$$

⋮

Then, the general term in successive approximation is given by

$$v_n(x, y, t) = \sum_{k=0}^n \left(\frac{(-1)^k t^{k\alpha}}{\Gamma(k\alpha + 1)} + \frac{(-1)^k t^{k\alpha+1}}{\Gamma(k\alpha + 2)} \right) e^{xy}.$$

Therefore, the exact solution of the equations (5.1) and (5.2) using Mittag-Leffler functions, is

$$(5.3) \quad v(x, y, t) = \lim_{n \rightarrow \infty} v_n(x, y, t) = \sum_{k=0}^{\infty} \left(\frac{(-1)^k t^{k\alpha}}{\Gamma(k\alpha + 1)} + \frac{(-1)^k t^{k\alpha+1}}{\Gamma(k\alpha + 2)} \right) e^{xy} = (E_\alpha(-t^\alpha) + tE_{\alpha,2}(-t^\alpha)) e^{xy}.$$

Application of the FNHPM. Following the description of the FNHPM presented in Section 4, gives

$$(5.4) \quad \sum_{n=0}^{\infty} p^n v_n(x, y, t) = (1 + t) e^{xy} + p \left[\mathcal{N}^{-1} \left(\frac{u^\alpha}{s^\alpha} \mathcal{N}^+ \left[\sum_{n=0}^{\infty} p^n H_n(v) - xy \sum_{n=0}^{\infty} p^n K_n(v) - \sum_{n=0}^{\infty} p^n v_n \right] \right) \right],$$

where $H_n(v)$ and $K_n(v)$ are He's polynomials that represents the nonlinear terms, $\frac{\partial^2}{\partial x \partial y}(v_{xx}v_{yy})$ and $\frac{\partial^2}{\partial x \partial y}(v_x v_y)$ respectively.

Using (4.4), the first few components of He's polynomials, are given by

$$\begin{aligned} H_0(v) &= \frac{\partial^2}{\partial x \partial y} \left((v_0)_{xx} (v_0)_{yy} \right), \\ H_1(v) &= \frac{\partial^2}{\partial x \partial y} \left((v_1)_{xx} (v_0)_{yy} + (v_0)_{xx} (v_1)_{yy} \right), \\ H_2(v) &= \frac{\partial^2}{\partial x \partial y} \left((v_2)_{xx} (v_0)_{yy} + (v_1)_{xx} (v_1)_{yy} + (v_0)_{xx} (v_2)_{yy} \right), \\ &\vdots \end{aligned}$$

and

$$\begin{aligned} K_0(v) &= \frac{\partial^2}{\partial x \partial y} \left((v_0)_x (v_0)_y \right), \\ K_1(v) &= \frac{\partial^2}{\partial x \partial y} \left((v_1)_x (v_0)_y + (v_0)_x (v_1)_y \right), \\ K_2(v) &= \frac{\partial^2}{\partial x \partial y} \left((v_2)_x (v_0)_y + (v_1)_x (v_1)_y + (v_0)_x (v_2)_y \right), \\ &\vdots \end{aligned}$$

Equating the coefficients of corresponding power of p on both sides in (5.4), we get

$$\begin{aligned} p^0 : \quad v_0(x, y, t) &= (1 + t)e^{xy}, \\ p^1 : \quad v_1(x, y, t) &= - \left(\frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} \right) e^{xy}, \\ p^2 : \quad v_2(x, y, t) &= \left(\frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} \right) e^{xy}, \\ &\vdots \end{aligned}$$

So, the solution of the equations (5.1) and (5.2) using Mittag-Leffler functions can be expressed as

$$v(x, y, t) = \left(1 + t - \frac{t^\alpha}{\Gamma(\alpha + 1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} + \dots \right) e^{xy}$$

$$(5.5) \quad = (E_\alpha(-t^\alpha) + tE_{\alpha,2}(-t^\alpha)) e^{xy}.$$

Taking $\alpha = 2$ in equalities (5.3) and (5.5), the solution will be as follows

$$v(x, y, t) = (E_2(-t^2) + tE_{2,2}(-t^2)) e^{xy} = (\cos t + \sin t)e^{xy},$$

which is exactly the same solution obtained by FNDM ([8]) and FRPSM ([9]).

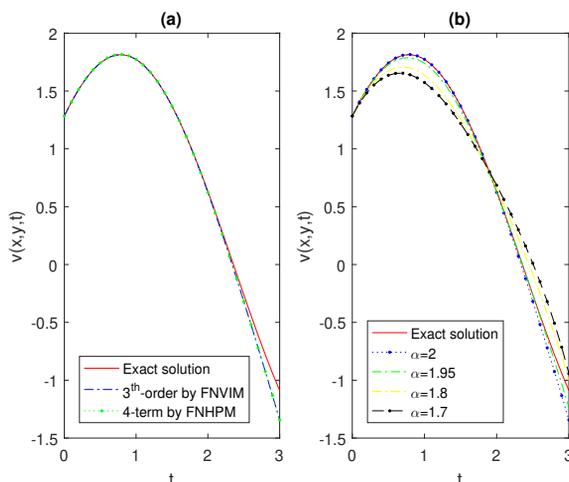


Figure 1. (a) The comparison of the approximate solutions by FNVIM, FNHPM and exact solution when $\alpha = 2$ and $x = y = 0.5$, (b) The behavior of the exact solution and approximate solutions by FNVIM and FNHPM for different values of α when $x = y = 0.5$.

Table 1. The absolute errors for differences between the exact solution and approximate solutions by the FNVIM and FNHPM for Example 5.1 when $\alpha = 2$.

	$ v_{exact} - v_{FNVIM} $	$ v_{exact} - v_{FNHPM} $	$ v_{exact} - v_{FNVIM} $	$ v_{exact} - v_{FNHPM} $
$t/x, y$	0.5	0.5	0.7	0.7
0.1	3.2196×10^{-13}	3.2196×10^{-13}	4.0929×10^{-13}	4.0929×10^{-13}
0.5	1.3095×10^{-7}	1.3095×10^{-7}	1.6647×10^{-7}	1.6647×10^{-7}
1	3.5001×10^{-5}	3.5001×10^{-5}	4.4495×10^{-5}	4.4495×10^{-5}
1.5	9.2940×10^{-4}	9.2940×10^{-4}	1.1815×10^{-3}	1.1815×10^{-3}
2	9.5484×10^{-3}	9.5484×10^{-3}	1.2138×10^{-2}	1.2138×10^{-2}

EXAMPLE 5.2. Consider the following nonlinear Caputo time-fractional wave-like equation with variable coefficients

$$(5.6) \quad D_t^\alpha v = v^2 \frac{\partial^2}{\partial x^2} (v_x v_{xx} v_{xxx}) + v_x^2 \frac{\partial^2}{\partial x^2} (v_{xx}^3) - 18v^5 + v,$$

with the initial conditions

$$(5.7) \quad v(x, 0) = e^x, \quad v_t(x, 0) = e^x,$$

where D_t^α is the Caputo fractional derivative operator of order α , $1 < \alpha \leq 2$ and v is a function of $(x, t) \in]0, 1[\times \mathbb{R}^+$.

Application of the FNHPM. Following the description of the FNVIM presented in Section 3, we obtain the iteration formula as follows

$$v_{n+1} = e^x + t e^x + \mathcal{N}^{-1} \left(\frac{u^\alpha}{s^\alpha} \mathcal{N}^+ \left[v_n^2 \frac{\partial^2}{\partial x^2} (v_{nx} v_{nxx} v_{nxxx}) + v_{nx}^2 \frac{\partial^2}{\partial x^2} (v_{nxx}^3) - 18v_n^5 + v_n \right] \right),$$

and

$$\begin{aligned} v_0 &= v_0(x, t) = (1 + t) e^x, \\ v_1 &= v_1(x, t) = \left(1 + t + \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} \right) e^x, \\ v_2 &= v_2(x, t) = \left(1 + t + \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} \right) e^x, \\ &\vdots \end{aligned}$$

Then, the general term in successive approximation is given by

$$v_n(x, t) = \sum_{k=0}^n \left(\frac{t^{k\alpha}}{\Gamma(k\alpha + 1)} + \frac{t^{k\alpha+1}}{\Gamma(k\alpha + 2)} \right) e^x.$$

Therefore, the exact solution of the equations (5.6) and (5.7) using Mittag-Leffler functions, is

$$(5.8) \quad \begin{aligned} v(x, t) &= \lim_{n \rightarrow \infty} v_n(x, t) = \sum_{k=0}^{\infty} \left(\frac{t^{k\alpha}}{\Gamma(k\alpha + 1)} + \frac{t^{k\alpha+1}}{\Gamma(k\alpha + 2)} \right) e^x \\ &= (E_\alpha(t^\alpha) + tE_{\alpha,2}(t^\alpha)) e^x. \end{aligned}$$

Application of the FNHPM. Following the description of the FNHPM presented in Section 4, gives

$$(5.9) \quad \sum_{n=0}^{\infty} p^n v_n(x, t) = (1 + t)e^x + p \left[\mathcal{N}^{-1} \left(\frac{u^\alpha}{s^\alpha} \mathcal{N}^+ \left[\sum_{n=0}^{\infty} p^n H_n(v) + \sum_{n=0}^{\infty} p^n K_n(v) - 18 \sum_{n=0}^{\infty} p^n J_n(v) + \sum_{n=0}^{\infty} p^n v_n \right] \right) \right],$$

where $H_n(v)$, $K_n(v)$, and $J_n(v)$ are He's Polynomials which represent the non-linear terms, $v^2 \frac{\partial^2}{\partial x^2} (v_x v_{xx} v_{xxx})$, $v_x^2 \frac{\partial^2}{\partial x^2} (v_{xx}^3)$ and v^5 respectively.

Using (4.4), the first few components of He's polynomials, are given by

$$\begin{aligned} H_0(v) &= v_0^2 \frac{\partial^2}{\partial x^2} ((v_0)_x (v_0)_{xx} (v_0)_{xxx}), \\ H_1(v) &= 2v_0 v_1 \frac{\partial^2}{\partial x^2} ((v_0)_x (v_0)_{xx} (v_0)_{xxx}) + v_0^2 \frac{\partial^2}{\partial x^2} ((v_1)_x (v_0)_{xx} (v_0)_{xxx} \\ &\quad + (v_0)_x (v_1)_{xx} (v_0)_{xxx} + (v_0)_x (v_0)_{xx} (v_1)_{xxx}), \\ &\vdots \\ K_0(v) &= (v_0)_x^2 \frac{\partial^2}{\partial x^2} ((v_0)_{xx}^3), \\ K_1(v) &= 2(v_0)_x (v_1)_x \frac{\partial^2}{\partial x^2} ((v_0)_{xx}^3) + (v_0)_x^2 \frac{\partial^2}{\partial x^2} (3(v_0)_{xx}^2 (v_1)_{xx}), \\ &\vdots \end{aligned}$$

and

$$J_0(v) = v_0^5, \quad J_1(v) = 5v_0^4 v_1, \quad \dots$$

Equating the coefficients of corresponding power of p on both sides in (5.9), we obtain

$$\begin{aligned} p^0 : \quad v_0(x, t) &= (1 + t)e^x, \\ p^1 : \quad v_1(x, t) &= \left(\frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} \right) e^x, \\ p^2 : \quad v_2(x, t) &= \left(\frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} \right) e^x, \\ &\vdots \end{aligned}$$

So, the solution of the equations (5.6) and (5.7) using Mittag-Leffler functions can be expressed as

$$\begin{aligned}
 v(x, t) &= \left(1 + t + \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} + \dots \right) e^x \\
 (5.10) \quad &= (E_\alpha(t^\alpha) + tE_{\alpha,2}(t^\alpha)) e^x,
 \end{aligned}$$

Taking $\alpha = 2$ in equalities (5.8) and (5.10), the solution will be as follows

$$v(x, t) = (E_2(t^2) + tE_{2,2}(t^2)) e^x = e^{x+t},$$

which is exactly the same solution obtained by FNDM ([8]) and FRPSM ([9]).

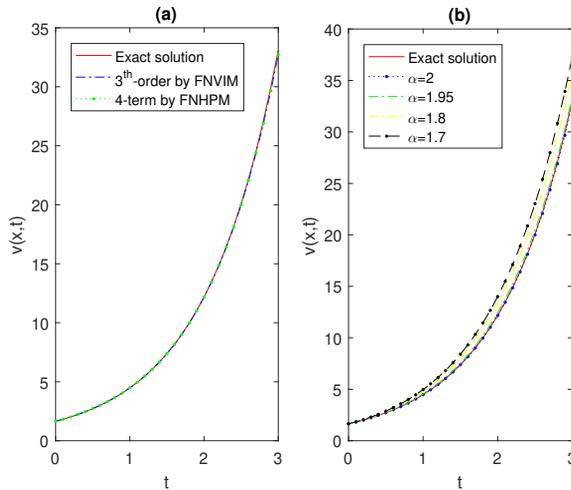


Figure 2. (a) The comparison of the approximate solutions by FNVIM, FNHPM and exact solution when $\alpha = 2$ and $x = 0.5$, (b) The behavior of the exact solution and approximate solutions by FNVIM and FNHPM for different values of α when $x = 0.5$.

Table 2. The absolute errors for differences between the exact solution and approximate solutions by the FNVIM and FNHPM for Example 5.2 when $\alpha = 2$.

	$ v_{exact} - v_{FNVIM} $	$ v_{exact} - v_{FNHPM} $	$ v_{exact} - v_{FNVIM} $	$ v_{exact} - v_{FNHPM} $
t/x	0.5	0.5	0.7	0.7
0.1	4.1350×10^{-13}	4.1350×10^{-13}	5.0505×10^{-13}	5.0505×10^{-13}
0.5	1.6907×10^{-7}	1.6907×10^{-7}	2.0650×10^{-7}	2.0650×10^{-7}
1	4.5934×10^{-5}	4.5934×10^{-5}	5.6104×10^{-5}	5.6104×10^{-5}
1.5	1.2529×10^{-3}	1.2529×10^{-3}	1.5303×10^{-3}	1.5303×10^{-3}
2	1.3361×10^{-2}	1.3361×10^{-2}	1.6319×10^{-2}	1.6319×10^{-2}

EXAMPLE 5.3. Consider the following one dimensional nonlinear Caputo time-fractional wave-like equation with variable coefficients

$$(5.11) \quad D_t^\alpha v = x^2 \frac{\partial}{\partial x} (v_x v_{xx}) - x^2 (v_{xx})^2 - v,$$

with the initial conditions

$$(5.12) \quad v(x, 0) = 0, \quad v_t(x, 0) = x^2,$$

where D_t^α is the Caputo fractional derivative operator of order α , $1 < \alpha \leq 2$ and v is a function of $(x, t) \in]0, 1[\times \mathbb{R}^+$.

Application of the FNVIM. Following the description of the FNVIM presented in Section 3, we obtain the iteration formula as follows

$$v_{n+1} = tx^2 + \mathcal{N}^{-1} \left(\frac{u^\alpha}{s^\alpha} \mathcal{N}^+ \left[x^2 \frac{\partial}{\partial x} (v_{nx} v_{nxx}) - x^2 (v_{nxx})^2 - v_n \right] \right),$$

and

$$v_0 = v_0(x, t) = tx^2,$$

$$v_1 = v_1(x, t) = \left(t - \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} \right) x^2,$$

$$v_2 = v_2(x, t) = \left(t - \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} \right) x^2,$$

⋮

Then, the general term in successive approximation is given by

$$v_n(x, t) = x^2 \left(\sum_{k=0}^n \frac{(-1)^k t^{k\alpha+1}}{\Gamma(k\alpha + 2)} \right).$$

Therefore, the exact solution of the equations (5.11) and (5.12) using Mittag-Leffler functions, is

$$(5.13) \quad \begin{aligned} v(x, t) &= \lim_{n \rightarrow \infty} v_n(x, t) = x^2 \left(\sum_{i=0}^{\infty} \frac{(-1)^i t^{i\alpha+1}}{\Gamma(i\alpha + 2)} \right) \\ &= x^2 (tE_{\alpha,2}(-t^\alpha)), \end{aligned}$$

Application of the FNHPM. Following the description of the FNHPM presented in Section 4, gives

$$(5.14) \quad \sum_{n=0}^{\infty} p^n v_n(x, t) = tx^2 + p \left[\mathcal{N}^{-1} \left(\frac{u^\alpha}{s^\alpha} \mathcal{N}^+ \left[x^2 \sum_{n=0}^{\infty} p^n H_n(v) - x^2 \sum_{n=0}^{\infty} p^n K_n(v) - \sum_{n=0}^{\infty} p^n v_n \right] \right) \right],$$

where $H_n(v)$, and $K_n(v)$, are He's Polynomials which represent the nonlinear terms, $\frac{\partial}{\partial x}(v_x v_{xx})$, $(v_{xx})^2$, respectively.

Using (4.4), the first few components of He's polynomials, are given by

$$\begin{aligned} H_0(v) &= \frac{\partial}{\partial x} ((v_0)_x (v_0)_{xx}), \\ H_1(v) &= \frac{\partial}{\partial x} ((v_0)_x (v_1)_{xx} + (v_1)_x (v_0)_{xx}), \\ H_2(v) &= \frac{\partial}{\partial x} ((v_0)_x (v_2)_{xx} + (v_1)_x (v_1)_{xx} + (v_2)_x (v_0)_{xx}), \\ &\vdots \end{aligned}$$

and

$$\begin{aligned} K_0(v) &= (v_0)_{xx}^2, \\ K_1(v) &= 2(v_0)_{xx} (v_1)_{xx}, \\ K_2(v) &= (v_1)_{xx}^2 + 2(v_0)_{xx} (v_2)_{xx}, \\ &\vdots \end{aligned}$$

Equating the coefficients of corresponding power of p on both sides in (5.14), we get

$$\begin{aligned} p^0 : \quad v_0(x, t) &= tx^2, \\ p^1 : \quad v_1(x, t) &= -\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} x^2, \\ p^2 : \quad v_2(x, t) &= \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} x^2, \\ &\vdots \end{aligned}$$

So, the solution of the equations (5.11) and (5.12) using Mittag-Leffler function can be expressed as

$$(5.15) \quad v(x, t) = x^2 \left(t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} \dots \right) = x^2 (tE_{\alpha,2}(-t^\alpha)),$$

Taking $\alpha = 2$ in equalities (5.13) and (5.15), the solution will be as follows

$$v(x, t) = x^2 (tE_{2,2}(-t^2)) = x^2 \sin t,$$

which is exactly the same solution obtained by FNDM ([8]) and FRPSM ([9]).

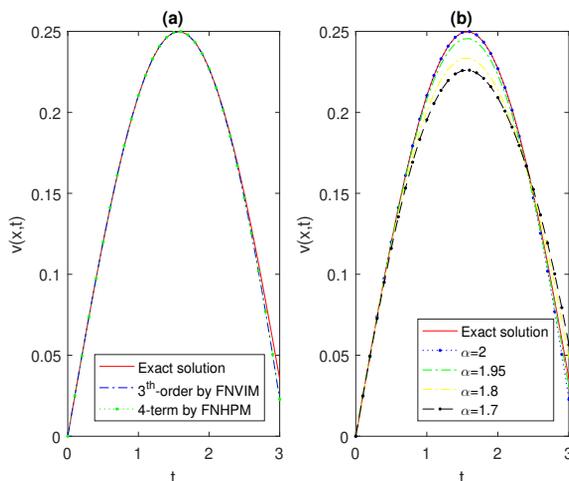


Figure 3. (a) The comparison of the approximate solutions by FNVIM, FNHPM and exact solution when $\alpha = 2$ and $x = 0.5$, (b) The behavior of the exact solution and approximate solutions by FNVIM and FNHPM for different values of α when $x = 0.5$.

Table 3. The absolute errors for differences between the exact solution and approximate solutions by the FNVIM and FNHPM for Example 5.3 when $\alpha = 2$.

t/x	$ v_{exact} - v_{FNVIM} $	$ v_{exact} - v_{FNHPM} $	$ v_{exact} - v_{FNVIM} $	$ v_{exact} - v_{FNHPM} $
	0.5	0.5	0.7	0.7
0.1	6.8887×10^{-16}	6.8887×10^{-16}	1.3502×10^{-15}	1.3502×10^{-15}
0.5	1.3425×10^{-9}	1.3425×10^{-9}	2.6313×10^{-9}	2.6313×10^{-9}
1	6.8271×10^{-7}	6.8271×10^{-7}	1.3381×10^{-6}	1.3381×10^{-6}
1.5	2.5951×10^{-5}	2.5951×10^{-5}	5.0864×10^{-5}	5.0864×10^{-5}
2	3.4023×10^{-4}	3.4023×10^{-4}	6.6685×10^{-4}	6.6685×10^{-4}

6. Numerical results and discussion

In Figures 1,2 and 3 (a): represents the comparison of the 3th order approximate solutions obtained by FNVIM and the 4-term approximate solution obtained by FNHPM and the exact solution at $\alpha = 2$, when $x = y = 0.5$ for Example 5.1 and $x = 0.5$ for Examples 5.2 and 5.3. The numerical results show that the FNVIM and FNHPM are highly accurate. (b): represents the behavior of the exact solutions and the 3th order approximate solution by FNVIM and the 4-term approximate solution by FNHPM at $\alpha = 1.7, 1.8, 1.95, 2$. These figures affirm that when the order of the fractional derivative α tends to 2, the approximate solutions obtained by FNVIM and FNHPM tends continuously to the exact solutions. In Tables 1, 2 and 3, we compute the absolute errors for differences between the exact solutions and the 3th order approximate solution by FNVIM and the 4-term approximate solution by FNHPM at $\alpha = 2$. The absolute errors obtained by the FNVIM are the same results obtained by FNHPM.

7. Conclusion

In this work, we compared the fractional natural variational iteration method (FNVIM) and the fractional natural homotopy perturbation method (FNHPM) as applied to nonlinear Caputo time-fractional wave-like equations with variable coefficients. For illustration purposes, we consider three different numerical examples. The results show that FNVIM has advantages over FNHPM, it is that it takes less time to solve this type of nonlinear problems without using He's polynomials and enables us to overcome the difficulties arising in identifying the general Lagrange multipliers. However, there is the high agreement of the numerical results obtained between the FNVIM and the FNHPM. Therefore, it may be concluded that both methods are powerful and efficient techniques for finding exact as well as approximate solutions for wide classes of nonlinear fractional partial differential equations.

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