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THE HYBRID NUMBERS OF PADOVAN AND SOME IDENTITIES

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Abstract. In this article, we will define Padovan's hybrid numbers, based on the new noncommutative numbering system studied by Özdemir ([7]). Such a system that is a set involving complex, hyperbolic and dual numbers. In addition, Padovan's hybrid numbers are created by combining this set, satisfying the relation $ih = -hi = \varepsilon + i$. Given this, some properties and identities are shown for these numbers, such as Binet's formula, generating matrix, characteristic equation, norm, and generating function. In addition, these numbers are extended to the integer field and some identities are made.

1. Introduction

A new numeric set was presented by Özdemir in [7] where he studied three numerical systems together, be they: the complex, hyperbolic and dual-resistant numbers combined with each other. Thus, he called this new number as a hybrid number, denoted by $\mathbb{K}$. From this, the hybrid numbers, some

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theorems, properties and their matrix form will be defined. An extension is also made to the field of the whole numbers, verifying its evolutionary process.

DEFINITION 1.1. The set of all hybrid numbers is defined as:

$$\mathbb{K} = \{z = a + bi + c\varepsilon + dh : a, b, c, d \in \mathbb{R},$$
$$i^2 = -1, \varepsilon^2 = 0, h^2 = 1, ih = -hi = \varepsilon + 1\}$$

Also, you can perform some properties and operations with the hybrid numbers, be they: Taking two hybrid numbers $z_1 = a_1 + b_1 i + c_1\varepsilon + d_1 h$ and $z_2 = a_2 + b_2 i + c_2\varepsilon + d_2 h$ and $s \in \mathbb{R}$ get:

- Equality $z_1 = z_2$, if and only if, $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$ and $d_1 = d_2$;
- Sum $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)\varepsilon + (d_1 + d_2)h$;
- Subtraction $z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)i + (c_1 - c_2)\varepsilon + (d_1 - d_2)h$;
- Multiplication by scalar $s \cdot z = s \cdot a + s \cdot bi + s \cdot c\varepsilon + s \cdot dh$

The hybrid product is obtained by distributing the terms to the right, preserving the order of multiplication of the units and then writing the values of the following substituting of each product of units by the equalities $i^2 = -1$, $\varepsilon^2 = 0$, $h^2 = 1$, $ih = -hi = \varepsilon + 1$. Using these equalities, we can find the product of any two hybrid units. And we can still present the multiplication table of a hybrid number, as shown in Table 1. The multiplication operation on

Table 1. Multiplication table for $\mathbb{K}$

$\cdot$	1	i	ε	h
1	1	i	ε	h
i	i	-1	$1-h$	$\varepsilon + i$
ε	ε	$1+h$	0	$-\varepsilon$
h	h	$-\varepsilon - i$	ε	1

the hybrid numbers is not commutative, but it has the Associativity property and the set $\mathbb{K}$ of the hybrid numbers forms a non-commutative ring relative to the addition and multiplication.

And yet, we have the conjugate of a hybrid number $z = a + bi + c\varepsilon + dh$, denoted by $\overline{z}$, given by

$$\overline{z} = a - bi - c\varepsilon - dh$$

and the real number

$$C(z) = z\overline{z} = \overline{z}z = a^2 + (b-c)^2 - c^2 - d^2 = a^2 + b^2 - 2bc - d^2$$

is called the hybrid number character, where the root of this real number will be the standard of the hybrid number z, so we have that: $\|z\| = \sqrt{C(z)}$.

Motivated by the work of [2], [3], [9], [11], [10], [5] in which authors deal with the process of hybridization of linear and recurrent sequences, in this article will be defined the hybrid numbers of Padovan. Inherent investigations into this process of these new numbers will be performed, finding their recurrence, characteristic equation, norm, matrix form, generating function and formula of Binet.

2. Padovan sequence

In this section, we will begin the studies on the sequence of Padovan, presenting this name in honor of Italian architect Richard Padovan, born in the city of Padua [8]. Regarded as a raw sequence of the Fibonacci sequence, the Padovan sequence is a sequence of the linear and recurrent type of integers, being third order. Described as: $1, 1, 1, 2, 2, 3, 4, 5, 7, \ldots$, these numbers display the recurrence formula defined below.

Definition 2.1. The recurrence ratio of the sequence with the initial terms $P_0 = P_1 = P_2 = 1$ and P_n the n-th term, is given by:

$$P_n = P_{n-2} + P_{n-3}, n \geqslant 3$$

From this recurrence formula, we can perform algebraic operations in order to obtain the characteristic polynomial of this sequence. Thus, using or reasoning performed by Koshy [6] the behavior of the convergence of the quotients is conjectured as being of the type $\frac{P_n}{P_{n-1}} > 0$, so:

$$P_n = P_{n-2} + P_{n-3},$$

$$\frac{P_n}{P_{n-2}} = 1 + \frac{P_{n-3}}{P_{n-2}},$$

$$\frac{P_n}{P_{n-2}} \frac{P_{n-1}}{P_{n-1}} = 1 + \frac{1}{\frac{P_{n-2}}{P_{n-3}}},$$

$$\lim_{n\to\infty} x_{n-1} \lim_{n\to\infty} x_n = 1 + \frac{1}{\lim_{n\to\infty} x_{n-2}},$$

$$x^2 = 1 + \frac{1}{x}.$$

The characteristic polynomial of the Padovan sequence is defined as:

$$x^3 - x - 1 = 0.$$

The above equation has three roots, two of which are complex and conjugate and one real solution given by ([1]): $\psi = \sqrt[3]{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{27}}} + \sqrt[3]{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{27}}} \approx 1,324717957244746025960908 5...$

Richard Padovan found a convergence relationship between the numbers in the Padovan sequence, presenting the plastic number, which is the relationship between P_{n+1} with P_n. Let P_n be the Padovan sequence and ψ the plastic constant defined in [4]:

$$\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \psi \approx 1,32.$$

This number found from the convergence of the Padovan sequence numbers is known as the plastic number or radiant number. It is also worth mentioning that this sequence has a form of algebraic representation, as shown in the work of Vieira and Alves [12].

Motivated by the work [7] and based on the importance of this sequence, we will apply the definition of hybrid numbers to Padovan numbers and in the next sections we will introduce Padovan hybrid numbers and will examine some properties of them.

3. Main results

In this section we will define Padovan's hybrid numbers and will present some results obtained from this definition.

Definition 3.1. Padovan's hybrid number, denoted HP_n is defined as:

$$HP_n = P_n + P_{n+1}i + P_{n+2}\varepsilon + P_{n+3}h.$$

Theorem 3.2. *With the following initial conditions: $HP_0 = 1+i+\varepsilon+2h$, $HP_1 = 1+i+2\varepsilon+2h$ and $HP_2 = 1+2i+2\varepsilon+3h$, the hybrid number sequence HP_n satisfies the recurrence relationship*

$$HP_n = HP_{n-2} + HP_{n-3}. \tag{3.1}$$

Proof. It is enough to observe that:

$$\begin{aligned} HP_{n-2}+HP_{n-3} &= (P_{n-2}+P_{n-1}i+P_n\varepsilon+P_{n+1}h) \\ &\quad +(P_{n-3}+P_{n-2}i+P_{n-1}\varepsilon+P_nh) \\ &= (P_{n-2}+P_{n-3})+(P_{n-1}+P_{n-2})i \\ &\quad +(P_n+P_{n-1})\varepsilon\ +(P_{n+1}+P_n)h \\ &= P_n+P_{n+1}i+P_{n+2}\varepsilon+P_{n+3}h \\ &= HP_n. \end{aligned}$$

□

According to the recurrence relationship, $HP_n = HP_{n-2} - HP_{n-3}$, we can now present its characteristic equation. Rewrite this recurrence equation in the following form:

$$\frac{HP_n}{HP_{n-2}} = 1 + \frac{HP_{n-3}}{HP_{n-2}},$$

$$\frac{HP_{n-1}}{HP_{n-1}} \cdot \frac{HP_n}{HP_{n-2}} = 1 + \frac{1}{\dfrac{HP_{n-2}}{HP_{n-3}}}.$$

Denoting $x_n = \frac{HP_n}{HP_{n-1}}$, we have: $x_{n-1} = \frac{HP_{n-1}}{HP_{n-2}}$ and $x_{n-2} = \frac{HP_{n-2}}{HP_{n-3}}$. We get the equation $x_{n-1}x_n = 1+\frac{1}{x_{n-2}}$. Assuming that the proposed limit exists and is equal to x, passing to the limit in this last equation gives:

$$\lim_{n\to\infty} x_{n-1} \lim_{n\to\infty} x_n = 1 + \frac{1}{\lim_{n\to\infty} x_{n-2}},$$

$$x^2 = 1 + \frac{1}{x},$$

$$x^3 - x - 1 = 0,$$

which is a third degree equation having three roots equal to the roots of the characteristic equation to the Padovan sequence, given in the previous section.

Based on what has been presented above we have that the character of Padovan's hybrid number is given as $C(HP_n) = P_n^2+(P_{n+1}-P_{n+2})^2-P_{n+2}^2-P_{n+3}^2$ and we can still present the norm of a Padovan hybrid number.

Theorem 3.3. *The standard of a Padovan hybrid number is given as:*

$$\|HP_n\|^2 = |-2P_{n+1}P_{n+2} - 2P_{n+1}P_n|.$$

PROOF. We have

$$\begin{aligned}\|HP_n\| &= \sqrt{|C(HP_n)|},\\ \|HP_n\|^2 &= |P_n^2 + (P_{n+1} - P_{n+2})^2 - P_{n+2}^2 - P_{n+3}^2|\\ &= |P_n^2 + P_{n+1}^2 - 2P_{n+1}P_{n+2} + P_{n+2}^2 - P_{n+2}^2 - (P_{n+1} + P_n)^2|\\ &= |P_n^2 + P_{n+1}^2 - 2P_{n+1}P_{n+2} - P_{n+1}^2 - 2P_{n+1}P_n - P_n^2|\\ &= |-2P_{n+1}P_{n+2} - 2P_{n+1}P_n|.\end{aligned}$$

□

In addition, Padovan's hybrid number can be represented in matrix form.

THEOREM 3.4. *An array of Padovan's hybrid number, with* $n \in \mathbb{N}$, *is defined as:*

$$\varphi_{HP_n} = \begin{bmatrix} P_n + P_{n+2} & 2P_{n+1} + P_n - P_{n+2} \\ P_{n+2} + P_n & P_n - P_{n+2} \end{bmatrix}.$$

PROOF. In [7] can be found a matrix form of a hybrid number, denoted by $\varphi_{a+bi+c\varepsilon+dh}$, given as:

$$\varphi_{a+bi+c\varepsilon+dh} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + c\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} + d\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, to present the matrix form of hybrid Padovan numbers, we have:

$$\begin{aligned}\varphi_{HP_n} &= P_n\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + P_{n+1}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + P_{n+2}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} + P_{n+3}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\\ &= \begin{bmatrix} P_n + P_{n+2} & P_{n+1} - P_{n+2} + P_{n+3} \\ P_{n+2} - P_{n+1} + P_{n+3} & P_n - P_{n+2} \end{bmatrix}\\ &= \begin{bmatrix} P_n + P_{n+2} & P_{n+1} - P_{n+2} + P_{n+1} + P_n \\ P_{n+2} - P_{n+1} + P_{n+1} + P_n & P_n - P_{n+2} \end{bmatrix}\\ &= \begin{bmatrix} P_n + P_{n+2} & 2P_{n+1} + P_n - P_{n+2} \\ P_{n+2} + P_n & P_n - P_{n+2} \end{bmatrix}.\end{aligned}$$

□

And yet, we have the following property that relates the determinant of a matrix to its norm.

REMARK 3.5. If φ_{HP_n} corresponds to the hybrid matrix of the Padovan hybrid number, HP_n, then $\|HP_n\|^2 = \det(\varphi_{HP_n})$, since we have

$$\begin{aligned}
\det(\varphi_{HP_n}) = \det \begin{bmatrix} P_n + P_{n+2} & 2P_{n+1} + P_n - P_{n+2} \\ P_{n+2} + P_n & P_n - P_{n+2} \end{bmatrix} \\
&= |P_n^2 - P_{n+2}^2 - (2P_{n+2}P_{n+1} + 2P_{n+1}P_n + P_nP_{n+2} \\
&\quad + P_n^2 - P_{n+2}^2 - P_nP_{n+2})| \\
&= |P_n^2 - P_{n+2}^2 - 2P_{n+1}P_{n+2} - 2P_{n+1}P_n - P_n^2 + P_{n+2}^2| \\
&= | - 2P_{n+1}P_{n+2} - 2P_{n+1}P_n| = \|HP_n\|^2.
\end{aligned}$$

Next, we will give the HP_n generating function, its Binet formula, and some identities related to this sequence.

THEOREM 3.6. *The generating function of the generalized hybrid number of Padovan, denoted by $G_{HP_n}(x)$, is:*

$$G_{HP_n}(x) = \frac{HP_0 + HP_1x + (HP_2 - HP_0)x^2}{1 - x^2 - x^3}.$$

PROOF. To define the Padovan hybrid number generating function, denoted by $G_{HP_n}(x)$, let's write a series in which each term in the sequence corresponds to the coefficient:

$$G_{HP_n}(x) = \sum_{n=0}^{\infty} HP_n x^n.$$

Making algebraic manipulations due to recurrence relationship we can write this sequence as:

$$\begin{aligned}
G_{HP_n}(x) &= HP_0 + HP_1x + HP_2x^2 + HP_3x^3 + HP_4x^4 + \dots, \\
x^2G_{HP_n}(x) &= HP_0x^2 + HP_1x^3 + HP_2x^4 + HP_3x^5 + HP_4x^6 + \dots, \\
x^3G_{HP_n}(x) &= HP_0x^3 + HP_1x^4 + HP_2x^5 + HP_3x^6 + HP_4x^7 + \dots, \\
(1 - x^2 - x^3)G_{HP_n}(x) &= HP_0 + HP_1x + (HP_2 - HP_0)x^2 \\
&\quad + (HP_3 - HP_1 - HP_0)x^3 + \dots, \\
G_{HP_n}(x) &= \frac{HP_0 + HP_1x + (HP_2 - HP_0)x^2}{1 - x^2 - x^3}.
\end{aligned}$$

□

Thus, we have the generating function for Padovan hybrid sequence.

Now we will explore the existence of an explicit formula for calculating the n-th term of the sequence, without relying on recurrence using Binet's formula, where it is necessary to use the roots of the characteristic equation of this sequence.

THEOREM 3.7. *For $n \geq 0$ we have the following Binet formula for Padovan hybrid number:*

$$HP_n = A(x_1)^n + B(x_2)^n + C(x_3)^n,$$

where x_1, x_2 and x_3 are the roots of the characteristic equation of the Padovan hybrid sequence and A, B and C the coefficients equal to:

$$A = \frac{HP_2 - HP_1x_2 - HP_1x_3 + HP_0x_2x_3}{(x_1 - x_2)(x_1 - x_3)},$$

$$B = \frac{HP_2 - HP_1x_1 - HP_1x_3 + HP_0x_1x_3}{(x_2 - x_1)(x_1 - x_3)},$$

$$C = \frac{HP_2 - HP_1x_1 - HP_1x_2 + HP_0x_1x_2}{(x_3 - x_1)(x_3 - x_2)}.$$

PROOF. We have that Binet's formula can be represented as follows:

$$HP_n = A(x_1)^n + B(x_2)^n + C(x_3)^n.$$

For $n = 0$, we have: $A + B + C = HP_0$, for $n = 1$, we have $Ax_1 + Bx_2 + Cx_3 = HP_1$ and for $n = 2$, we have $Ax_1^2 + Bx_2^2 + Cx_3^2 = HP_2$.

We can build a system of linear equations as follows:

$$\begin{cases} A + B + C = HP_0, \\ Ax_1 + Bx_2 + Cx_3 = HP_1, \\ Ax_1^2 + Bx_2^2 + Cx_3^2 = HP_2. \end{cases}$$

Solving the linear system, we obtain the following formulas on A, B and C:

$$A = \frac{HP_2 - HP_1x_2 - HP_1x_3 + HP_0x_2x_3}{x_1^2 - x_1x_2 - x_1x_3 + x_2x_3}$$

$$= \frac{HP_2 - HP_1x_2 - HP_1x_3 + HP_0x_2x_3}{(x_1 - x_2)(x_1 - x_3)},$$

$$B = \frac{HP_2 - HP_1x_1 - HP_1x_3 + HP_0x_1x_3}{x_2^2 - x_1x_2 - x_1x_3 + x_2x_3}$$
$$= \frac{HP_2 - HP_1x_1 - HP_1x_3 + HP_0x_1x_3}{(x_2 - x_1)(x_1 - x_3)},$$
$$C = \frac{HP_2 - HP_1x_1 - HP_1x_2 + HP_0x_1x_2}{x_3^2 - x_1x_2 - x_1x_3 + x_2x_3}$$
$$= \frac{HP_2 - HP_1x_1 - HP_1x_2 + HP_0x_1x_2}{(x_3 - x_1)(x_3 - x_2)}.$$ □

Definition 3.8. Similar to the definition of Padovan hybrid sequence, recurrence for the negative terms of this sequence can be given; with $n \geqslant 1$ and $n \in \mathbb{N}$, we put:

$$HP_{-n} = HP_{-n+3} - HP_{-n+1}.$$

Thus, it is possible to calculate non-positive integer terms of the Padovan hybrid sequence, e.g., $HP_{-1} = i, HP_{-2} = 1 + h$ and $HP_{-3} = \varepsilon - h$.

4. Some identities of Padovan's hybrid numbers

Here are some identities regarding Padovan's hybrid numbers.

From the recurrence relationship on Padovan hybrid numbers given in (3.1), it can be concluded that: $HP_{n+1} = HP_{n-1} + HP_{n-2}$ and $HP_{n-2} = HP_{n+1} - HP_{n-1}$ allowing to write Padovan's hybrid numbers as follows:

$$HP_1 = HP_4 - HP_2,$$
$$HP_2 = HP_5 - HP_3,$$
$$HP_3 = HP_6 - HP_4,$$
$$HP_4 = HP_7 - HP_5,$$
$$\vdots$$
$$HP_{2n-1} = HP_{2n+2} - HP_{2n},$$
$$HP_{2n} = HP_{2n+3} - HP_{2n+1},$$
$$HP_{2n+1} = HP_{2n+4} - HP_{2n+2}.$$

From now on some identities about this hybrid model of this sequence will be discussed.

Identity 4.1. *The following summation describes the sum of Padovan hybrid numbers, up to the $2n$, odd index order:*

$$\sum_{i=1}^{n} HP_{2i-1} = HP_{2n+2} - HP_2.$$

Proof. Starting from the Padovan hybrid numbers described through the notational definition and considering only the odd index terms, we have that:

$$\begin{aligned}\sum_{i=1}^{n} HP_{2i-1} &= HP_1 + HP_3 + HP_5 + \ldots + HP_{2n-1}\\ &= (HP_4 - HP_2) + (HP_6 - HP_4) + (HP_8 - HP_6)\\ &\quad + \ldots + (HP_{2n+2} - HP_{2n})\\ &= HP_{2n+2} - HP_2.\end{aligned}$$

□

Identity 4.2. *The sum of Padovan's hybrid numbers, up to the $2n$, even index order can be described by:*

$$\sum_{i=1}^{n} HP_{2i} = HP_{2n+3} - HP_3.$$

Proof. Now, considering only the even index terms, we have:

$$\begin{aligned}\sum_{i=1}^{n} HP_{2i} &= HP_2 + HP_4 + HP_6 + \ldots + HP_{2n}\\ &= (HP_5 - HP_3) + (HP_7 - HP_5) + (HP_9 - HP_7)\\ &\quad + \ldots + (HP_{2n+3} - HP_{2n+1})\\ &= HP_{2n+3} - HP_3.\end{aligned}$$

□

Identity 4.3. *The sum of the first Padovan hybrid numbers, up to n, of index greater than zero can be obtained by developing:*

$$\sum_{i=1}^{n} HP_i = HP_{n+5} - HP_2 - HP_3.$$

Proof. We have:

$$\begin{aligned}\sum_{i=1}^{n} HP_i &= HP_1 + HP_2 + HP_3 + HP_4 + \ldots + HP_n \\ &= (HP_4 - HP_2) + (HP_5 - HP_3) + (HP_6 - HP_4) \\ &\quad + (HP_7 - HP_5) + \ldots + (HP_{n+3} - HP_{n+1}) \\ &= HP_{n+2} + HP_{n+3} - HP_2 - HP_3. \qquad \square\end{aligned}$$

Identity 4.4. *The sum of the three consecutive terms of Padovan hybrid numbers results in another Padovan hybrid term, which is:*

$$HP_n + HP_{n+1} + HP_{n+2} = HP_{n+5}.$$

Proof. Applying the recurrence of Padovan's hybrid numbers, one has:

$$\begin{aligned}HP_n + HP_{n+1} + HP_{n+2} &= HP_{n+3} + HP_{n+2} \\ &= HP_{n+5}. \qquad \square\end{aligned}$$

5. Conclusion

In this work it was possible to introduce and study a new type of numbers Padovan hybrid numbers, being a combination of the complex, hyperbolic and Padovan dual numbers. Via relation defined by Özdemir [7] which establishes some relations fundamental to the development of this work, such as $ih = -hi = \varepsilon + i$ between the units i, ε, h of these three number systems.

Some properties involving these sequences have been presented, including the Binet formula, generating matrix, characteristic equation, norm, generating function, extension to the integer field and some identities.

For future work, it is important to investigate more properties around these numbers, as well as their application in the area of nature and physics.

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References

[1] J. Aarts, R. Fokkink, and G. Kruijtzer, *Morphic numbers*, Nieuw Arch. Wiskd. (5) **2** (2001), no. 1, 56–58.

[2] P. Catarino, *On k-Pell hybrid numbers*, J. Discrete Math. Sci. Cryptogr. **22** (2019), no. 1, 83–89.

[3] G. Cerda-Morales, *Investigation of generalized hybrid Fibonacci numbers and their properties*, arXiv preprint (2018). Avaliable at arXiv:1806.02231.

[4] R. de C. Ferreira, *Números Mórficos*, Dissertação (Mestrado Profissional em Matemática), Universidade Federal da Paraíba, João Pessoa, 2015.

[5] C. Kızılateş, *A new generalization of Fibonacci hybrid and Lucas hybrid numbers*, Chaos Solitons Fractals **130** (2020), 109449, 5pp.

[6] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley–Interscience, New York, 2001.

[7] M. Özdemir, *Introducion to hybrid numbers*, Adv. Appl. Clifford Algebr. **28** (2018), no. 1, Paper No. 11, 32 pp.

[8] I. Stewart, *Tales of a neglected number*, Sci. Amer. **274** (1996), no. 6, 102–103.

[9] A. Szynal-Liana, *The Horadam hybrid numbers*, Discuss. Math. Gen. Algebra Appl. **38** (2018), no. 1, 91–98.

[10] A. Szynal-Liana and I. Włoch, *On Pell and Pell-Lucas hybrid numbers*, Comment. Math. **58** (2018), no. 1–2, 11–17.

[11] A. Szynal-Liana and I. Włoch, *On Jacobsthal and Jacobsthal-Lucas hybrid numbers*, Ann. Math. Sil. **33** (2019), no. 1, 276–283.

[12] R.P.M. Vieira and F.R.V. Alves, *Propriedades das extensões da Sequência de Padovan*, C.Q.D. Revista Eletrônica Paulista de Matemática, Bauru **15** (2019), 24–40.

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