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A GENERAL FIXED POINT THEOREM FOR TWO PAIRS OF ABSORBING MAPPINGS IN G_P-METRIC SPACES

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Abstract. A general fixed point theorem for two pairs of absorbing mappings satisfying a new type of implicit relation ([37]), without weak compatibility in G_p-metric spaces is proved. As applications, new results for mappings satisfying contractive conditions of integral type and for ϕ-contractive mappings are obtained.

1. Introduction

Let (X, d) be a metric space and S, T be two self mappings of X. In [19], Jungck defined S and T to be compatible if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t,$$

for some $t \in X$.

This concept has been frequently used to prove existence theorems in fixed point theory.

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Let f, g be self mappings of a nonempty set X. A point $x \in X$ is a coincidence point of f and g if $fx = gx$. The set of all coincidence points of f and g is denoted by $\mathcal{C}(f, g)$.

The study of common fixed points for noncompatible mappings is also interesting, the work in this regard being initiated by Pant in [30]–[32].

Aamri and El-Moutawakil ([1]) introduced a generalization of noncompatible mappings.

Definition 1.1 ([1]). Let S and T be self mappings of a metric space (X, d). We say that S and T satisfy $(E.A)$*-property* if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t,$$

for some $t \in X$.

Remark 1.2. It is clear that two self mappings S and T of a metric space (X, d) are noncompatible if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$, but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is nonzero or does not exist. Therefore, two noncompatible self mappings of a metric space (X, d) satisfy $(E.A)$-property.

In 2005, Liu et al. ([23]) defined the notion of common $(E.A)$-property.

Definition 1.3 ([23]). Two pairs (A, S) and (B, T) of self mappings on a metric space (X, d) are said to satisfy *common* $(E.A)$*-property* if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t,$$

for some $t \in X$.

There exists a vast literature concerning the study of fixed points for mappings satisfying common $(E.A)$-property.

In 2011, Sintunavarat and Kumam ([48]) introduced the concept of common limit range property.

Definition 1.4 ([48]). A pair (A, S) of self mappings on a metric space (X, d) is said to satisfy *common limit range property with respect to* S, denoted $CLR_{(S)}$-property, if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t,$$

for some $t \in S(X)$.

Thus, we can infer that a pair (A, S) satisfying $(E.A)$-property, along with the closedness of the subspace $S(X)$, always has $CLR_{(S)}$-property.

Recently, Imdad et al. ([17]) extended the notion of common limit range property for two pairs of mappings in metric spaces.

Definition 1.5 ([17]). Two pairs (A, S) and (B, T) of self mappings of a metric space (X, d) are said to satisfy *common limit range property with respect to S and T*, denoted $CLR_{(S,T)}$- property, if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t,$$

for some $t \in S(X) \cap T(X)$.

Some results for pairs of mappings satisfying $CLR_{(S)}$- and $CLR_{(S,T)}$-property are obtained in [15], [16], [18] and in other papers.

Quite recently, the present author introduced in [37] a new type of common limit range property.

Definition 1.6 ([37]). *Let A, S and T be self mappings of a metric space* (X, d). The pair (A, S) is said to satisfy *common limit range property with respect to T*, denoted $CLR_{(A,S)T}$-property, if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t,$$

for some $t \in S(X) \cap T(X)$.

Remark 1.7 ([37]). Let A, B, S and T be self mappings of a metric space (X, d). If (A, S) and (B, T) satisfy $CLR_{(S,T)}$-property, then A, S and T satisfy $CLR_{(A,S)T}$-property.

Definition 1.8 ([22]). An *altering distance* is a function $\psi\colon [0,\infty) \to [0,\infty)$ such that

(ψ_1) ψ is increasing and continuous,

(ψ_2) $\psi(t) = 0$ if and only if $t = 0$.

Fixed point theorems involving altering distances have been studied in [38], [44], [45] and in other papers.

The notion of almost altering distance is introduced in [41].

Definition 1.9 ([41]). A function $\psi\colon [0,\infty) \to [0,\infty)$ is *an almost altering distance* if

(ψ_1) ψ is continuous,
(ψ_2) $\psi(t) = 0$ if and only if $t = 0$.

2. Preliminaries

In [11], [12] Dhage introduced a new class of generalized metric spaces named D-metric spaces. Mustafa and Sims ([28], [29]) proved that most of the claims concerning the fundamental topological structures on D-metric spaces are incorrect and introduced an appropriate notion of generalized metric space, named G-metric spaces. In fact, Mustafa, Sims and other authors studied many fixed point results for self mappings in G-metric spaces under certain conditions in [27]–[29], [47] and in other papers.

Definition 2.1 ([29]). Let X be a nonempty set and $G\colon X^3 \to \mathbb{R}_+$ be a function satisfying the following properties:

(G_1) $G(x,y,z) = 0$ if $x = y = z$,
(G_2) $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$,
(G_3) $G(x,y,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$,
(G_4) $G(x,y,z) = G(y,z,x) = \ldots$ (symmetry in all three variables),
(G_5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (triangle inequality).

The function G is called a *G-metric on X* and (X,G) is called a *G-metric space.*

Remark 2.2. Let (X,G) be a G-metric space. If $y = z$, then $G(x,y,y)$ is a quasi-metric on X ([36, Lemma 2.1]). Hence, (X,Q), where $Q(x,y) = G(x,y,y)$ is a quasi-metric and since every metric space is a particular case of quasi-metric space it follows that the notion of G-metric space is a generalization of a metric space.

In 1994, Matthews ([25]) introduced the notion of partial metric space as a part of study of denotional semantics of dataflows networks and proved the Banach contraction principle in such spaces.

Quite recently, in [4], [9], [10], [20], [21] and in other papers, some fixed point theorems under various contractive conditions in partial metric spaces have been proved.

Definition 2.3 ([25]). Let X be a nonempty set. A function $p\colon X^2 \to \mathbb{R}_+$ is said to be a *partial metric on* X if for all $x, y, z \in X$:
(P_1) $p(x,x) = p(x,y) = p(y,y)$ if and only if $x = y$,
(P_2) $p(x,x) \leq p(x,y)$,
(P_3) $p(x,y) = p(y,x)$,
(P_4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.
The pair (X,p) is called a *partial metric space*.

Remark 2.4. Obviously, every metric space is a partial metric space.

Quite recently, Ahmadi Zand and Dehghan Nezhad ([2]) introduced a generalization and unification of G-metric spaces and partial metric spaces, named G_p -metric spaces. Some fixed point results in G_p-metric spaces are obtained in [5]–[7], [33] and in other papers.

Definition 2.5 ([2, 33]). Let X be a nonempty set. A function $G_p\colon X^3 \to \mathbb{R}_+$ is called a *G_p-metric on X* if the following conditions are satisfied:
(G_{p1}) $x = y = z$ if $G_p(x,y,z) = G_p(x,x,x) = G_p(y,y,y) = G_p(z,z,z)$,
(G_{p2}) $0 \leq G_p(x,x,x) \leq G_p(x,x,y) \leq G_p(x,y,z)$ for all $x, y, z \in X$ with $y \neq z$,
(G_{p3}) $G_p(x,y,z) = G_p(y,z,x) = \ldots$ (symmetry in all three variables),
(G_{p4}) $G_p(x,y,z) \leq G_p(x,a,a) + G_p(a,y,z)$ for all $x, y, z, a \in X$.
The pair (X, G_p) is called a *G_p-metric space*.

Lemma 2.6 ([5]). *Let (X, G_p) be a G_p-metric space. Then:*
1) *if $G_p(x,y,z) = 0$, then $x = y = z$,*
2) *if $x \neq y$, then $G_p(x,y,y) > 0$.*

Definition 2.7 ([5]). Let (X, G_p) be a G_p-metric space and $\{x_n\}$ be a sequence of points in X. A point $x \in X$ is said to be the *limit of the sequence* $\{x_n\}$, denoted by $x_n \to x$, if $\lim_{m,n\to\infty} G_p(x, x_n, x_m) = G_p(x,x,x)$. Then the sequence $\{x_n\}$ is called *G_p-convergent to x*.

Lemma 2.8 ([5]). *Let (X, G_p) be a G_p-metric space. Then, for any $\{x_n\}$ in X and $x \in X$, the following conditions are equivalent:*
a) *$\{x_n\}$ is G_p-convergent to x,*
b) *$G_p(x_n, x_n, x) \to G_p(x,x,x)$ as $n \to \infty$,*
c) *$G_p(x_n, x, x) \to G_p(x,x,x)$ as $n \to \infty$.*

Lemma 2.9 ([5]). *If $x_n \to x$ in a G_p-metric space (X, G_p) and $G_p(x,x,x) = 0$, then for every $y \in X$*

a) $\lim_{n\to\infty} G_p(x_n, y, y) = G_p(x, y, y)$,
b) $\lim_{n\to\infty} G_p(x_n, x_n, y) = G_p(x, x, y)$.

DEFINITION 2.10 ([46]). Let A, S and T be self mappings of a G_p-metric space (X, G_p). The pair (A, S) satisfy (A, S) *common limit range property with respect to* T, if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$$

for some $z \in S(X) \cap T(X)$ with $G_p(z, z, z) = 0$.

The notion of absorbing mappings is introduced in [13, 14, 26] and in other papers.

We introduce the notion of absorbing mapping in G_p-metric spaces.

DEFINITION 2.11. Let A and S be self mappings of a G_p-metric space (X, G_p). Then

1) A is called S *absorbing* if there exists $R \geq 0$ such that

$$G_p(Sx, SAx, SAx) \leq RG_p(Sx, Ax, Ax), \ \forall x \in X.$$

Similarly, S is A absorbing.

2) A is called *pointwise* S *absorbing* if for given $x \in X$, there exists $R \geq 0$ such that

$$G_p(Sx, SAx, SAx) \leq RG_p(Sx, Ax, Ax).$$

Similarly, S is pointwise A absorbing.

3. Implicit relations

Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit relation in [34, 35] and in other papers.

The study of fixed points for a pair of mappings satisfying an implicit relation in G-metric spaces is initiated in [39] and [40].

The study of fixed points for a pair of mappings with common limit range property satisfying implicit relations is initiated in [15].

The study of fixed points for pairs of mappings with common limit range property in G-metric spaces is initiated in [41].

Recently, fixed point results for mappings satisfying an implicit relation in partial metric spaces are obtained in [49].

Fixed point theorems for mappings satisfying implicit relations in G_p-metric spaces are obtained in [42, 43].

In 2008, Ali and Imdad ([3]) introduced a new type of implicit relations.

Let $\mathcal{F}$ be the family of lower semi-continuous functions $F\colon \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(F_1) $F(t,0,t,0,0,t) > 0$, $\forall t > 0$,
(F_2) $F(t,0,0,t,t,0) > 0$, $\forall t > 0$,
(F_3) $F(t,t,0,0,t,t) > 0$, $\forall t > 0$.

Example 3.1. $F(t_1,\ldots,t_6) = t_1 - k\max\{t_2,t_3,\ldots,t_6\}$, where $k \in [0,1)$.

Example 3.2. $F(t_1,\ldots,t_6) = t_1 - k\max\left\{t_2,t_3,t_4,\dfrac{t_5+t_6}{2}\right\}$, where $k \in [0,1)$.

Example 3.3. $F(t_1,\ldots,t_6) = t_1 - k\max\left\{t_2,\dfrac{t_3+t_4}{2},\dfrac{t_5+t_6}{2}\right\}$, where $k \in [0,1)$.

Example 3.4. $F(t_1,\ldots,t_6) = t_1 - at_2 - b\max\{t_3,t_4\} - c\max\{t_5,t_6\}$, where $a,b,c \geq 0$ and $a+b+c<1$.

Example 3.5. $F(t_1,\ldots,t_6) = t_1 - \alpha\max\{t_2,t_3,t_4\} - (1-\alpha)(at_5+bt_6)$, where $\alpha \in (0,1)$, $a,b \geq 0$ and $a+b<1$.

Example 3.6. $F(t_1,\ldots,t_6) = t_1 - at_2 - b(t_3+t_4) - c\min\{t_5,t_6\}$, where $a,b,c \geq 0$ and $a+b+c<1$.

Example 3.7. $F(t_1,\ldots,t_6) = t_1 - at_2 - \dfrac{b(t_5+t_6)}{1+t_3+t_4}$, where $a,b \geq 0$ and $a+2b<1$.

Example 3.8. $F(t_1,\ldots,t_6) = t_1 - \max\{ct_2,ct_3,ct_4,at_5+bt_6\}$, where $c \in (0,1)$, $a,b \geq 0$ and $a+b<1$.

Other examples are in [15].

The purpose of this paper is to prove a general fixed point theorem for two pairs of absorbing mappings satisfying a new type of common limit range property in G_p-metric spaces. As applications we obtain new results for mappings satisfying contractive conditions of integral type and for φ-contractive mappings.

4. Main results

THEOREM 4.1. *Let A, B, S and T be self mappings of a G_p-metric space (X, G_p) such that*

$$F\big(\psi\left(G_p(Ax,By,By)\right),\psi\left(G_p(Sx,Ty,Ty)\right),\psi\left(G_p(Ax,Sx,Sx)\right),\\ \psi\left(G_p(Ty,By,By)\right),\psi\left(G_p(Sx,By,By)\right),\psi\left(G_p(Ax,Ty,Ty)\right)\big)\leq 0 \tag{4.1}$$

for all $x, y \in X$, where $F \in \mathcal{F}$ and ψ is an almost altering distance.

If (A, S) and T satisfy $CLR_{(A,S)T}$-property, then $\mathcal{C}(A, S) \neq \emptyset \neq \mathcal{C}(B, T)$.

Moreover, if A is pointwise S absorbing and B is pointwise T absorbing, then A, B, S and T have a unique common fixed point z with $G_p(z, z, z) = 0$.

PROOF. The proof that $Bu = Tu = z = Av = Sv$ for some $u, v \in X$ and $z \in S(X) \cap T(X)$ with $G_p(z, z, z) = 0$ is similar to the first part of the proof of [46, Theorem 4.1]. Fix now the points u, v, z satisfying these properties.

If A is pointwise S absorbing, there exists $R_1 \geq 0$ such that

$$G_p\left(Sv, SAv, SAv\right) \leq R_1 G_p\left(Sv, Av, Av\right) = R_1 G_p\left(z, z, z\right) = 0.$$

Hence, by Lemma 2.6 (1), $z = Sv = SAv = Sz$ and z is a fixed point of S.

By (4.1) for $x = z$ and $y = u$ we obtain

$$F\big(\psi\left(G_p(Az,Bu,Bu)\right),\psi\left(G_p(Sz,Tu,Tu)\right),\psi\left(G_p(Az,Sz,Sz)\right),\\ \psi\left(G_p(Tu,Bu,Bu)\right),\psi\left(G_p(Sz,Bu,Bu)\right),\psi\left(G_p(Az,Tu,Tu)\right)\big)\leq 0,$$

$$F\big(\psi\left(G_p(Az,z,z)\right),0,\psi\left(G_p(Az,z,z)\right),0,0,\psi\left(G_p(Az,z,z)\right)\big)\leq 0,$$

a contradiction of (F_1) if $\psi\left(G_p(Az, z, z)\right) > 0$. Hence, $G_p(Az, z, z) = 0$ which implies by Lemma 2.6 (1) that $z = Az = Sz$. Therefore, z is a common fixed point of A and S with $G(z, z, z) = 0$.

If B is pointwise T absorbing, there exists $R_2 \geq 0$ such that

$$G_p\left(Tu, TBu, TBu\right) \leq R_2 G_p\left(Tu, Bu, Bu\right) = R_2 G_p\left(z, z, z\right) = 0.$$

Hence, $z = Tu = TBu = Tz$ and z is a fixed point of T.

By (4.1) for $x = v$ and $y = z$ we obtain

$$F\big(\psi\left(G_p(Av, Bz, Bz)\right), \psi\left(G_p(Sv, Tz, Tz)\right), \psi\left(G_p(Av, Sv, Sv)\right),$$
$$\psi\left(G_p(Tz, Bz, Bz)\right), \psi\left(G_p(Sv, Bz, Bz)\right), \psi\left(G_p(Av, Tz, Tz)\right)\big) \leq 0,$$
$$F\big(\psi\left(G_p(z, Bz, Bz)\right), 0, 0, \psi\left(G_p(z, Bz, Bz)\right), \psi\left(G_p(z, Bz, Bz)\right), 0\big) \leq 0,$$

a contradiction of (F_2) if $\psi\left(G_p(z, Bz, Bz)\right) > 0$. Hence, $G_p(z, Bz, Bz) = 0$ which implies $z = Bz = Tz$ and z is a common fixed point of B and T with $G(z, z, z) = 0$.

Hence, z is a common fixed point of A, B, S and T with $G_p\left(z, z, z\right) = 0$.

Suppose that there exists another common fixed point z_1 for A, B, S and T with $G_p\left(z_1, z_1, z_1\right) = 0$. Then, by (4.1) we obtain

$$F\big(\psi\left(G_p(Az, Bz_1, Bz_1)\right), \psi\left(G_p(Sz, Tz_1, Tz_1)\right), \psi\left(G_p(Az, Sz, Sz)\right),$$
$$\psi\left(G_p(Tz_1, Bz_1, Bz_1)\right), \psi\left(G_p(Sz, Bz_1, Bz_1)\right), \psi\left(G_p(Az, Tz_1, Tz_1)\right)\big) \leq 0,$$
$$F\big(\psi\left(G_p(z, z_1, z_1)\right), \psi\left(G_p(z, z_1, z_1)\right), 0, 0,$$
$$\psi\left(G_p(z, z_1, z_1)\right), \psi\left(G_p(z, z_1, z_1)\right)\big) \leq 0,$$

a contradiction of (F_3) if $\psi\left(G_p(z, z_1, z_1)\right) > 0$. Hence, $G_p(z, z_1, z_1) = 0$ which implies by Lemma 2.6 (1) that $z = z_1$. Hence, z is the unique common fixed point of A, B, S and T with $G_p\left(z, z, z\right) = 0$. □

REMARK 4.2. In [46, Theorem 4.1], the fact that z is the unique point of coincidence of (A, S) and (B, T) must be completed with the additional assumption, namely that $G_p(Sx, Sx, Sx) = 0$ for $x \in \mathcal{C}\left(A, S\right)$ and $G_p(Ty, Ty, Ty) = 0$ for $y \in \mathcal{C}\left(B, T\right)$.

A similar remark refers to [46, Theorems 4.2, 5.2–5.5].

If $\psi\left(t\right) = t$, by Theorem 4.1 we obtain

THEOREM 4.3. *Let A, B, S and T be self mappings of a G_p-metric space (X, G_p) such that*

$$F\big(G_p(Ax, By, By), G_p(Sx, Ty, Ty), G_p(Ax, Sx, Sx),$$
$$G_p(Ty, By, By), G_p(Sx, By, By), G_p(Ax, Ty, Ty)\big) \leq 0$$

for all $x, y \in X$ and some $F \in \mathcal{F}$.

If (A, S) and T satisfy $CLR_{(A,S)T}$-property, then $\mathcal{C}\left(A, S\right) \neq \emptyset \neq \mathcal{C}\left(B, T\right)$.

Moreover, if A is pointwise S absorbing and B is pointwise T absorbing, then A, B, S and T have a unique common fixed point z with $G_p(z,z,z)=0$.

THEOREM 4.4. *Let A, B, S and T be self mappings of a G_p-metric space (X, G_p) such that*

$$G_p(Ax,By,By) \le k \max\big\{G_p(Sx,Ty,Ty), G_p(Ax,Sx,Sx), G_p(Ty,By,By), G_p(Sx,By,By), G_p(Ax,Ty,Ty)\big\},$$

for all $x, y \in X$, where $k \in [0,1)$.

If (A,S) and T satisfy $CLR_{(A,S)T}$-property, then $\mathcal{C}(A,S) \neq \emptyset \neq \mathcal{C}(B,T)$.

Moreover, if A is pointwise S absorbing and B is pointwise T absorbing, then A, B, S and T have a unique common fixed point $z=0$ with $G_p(z,z,z)=0$.

PROOF. It follows by Theorem 4.3 and Example 3.1. □

EXAMPLE 4.5 ([46]). Let $X=[0,1]$ with $G_p(x,y,z)=\max\{x,y,z\}$. Then (X,G_p) is a G_p-metric space. Let $Ax=0$, $Sx=\dfrac{x}{x+1}$, $Bx=\dfrac{x}{3}$, $Tx=x$. Then $S(X)=\left[0,\frac{1}{2}\right]$, $T(X)=[0,1]$, $S(X)\cap T(X)=\left[0,\frac{1}{2}\right]$. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} x_n = 0$. Then,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z = 0 \in S(X)\cap T(X)$$

and $G_p(z,z,z)=0$. Hence, (A,S) and T satisfy $CLR_{(A,S)T}$-property. Note that

$$G_p(Sx,SAx,SAx)=\frac{x}{x+1}, \quad G_p(Sx,Ax,Ax)=\frac{x}{x+1}.$$

Hence,

$$G_p(Sx,SAx,SAx) \le R_1 G_p(Sx,Ax,Ax) \quad \text{for } R_1 \ge 1.$$

Thus, A is pointwise S absorbing. We have also

$$G_p(Tx,TBx,TBx)=x, \quad G_p(Tx,Bx,Bx)=x.$$

Hence,

$$G_p(Tx,TBx,TBx) \le R_2 G_p(Tx,Bx,Bx) \quad \text{for } R_2 \ge 1.$$

Thus, B is T pointwise absorbing.

On the other hand,

$$G_p\left(Ax, By, By\right) = \frac{y}{3}, \quad G\left(Ty, By, By\right) = y.$$

Hence,

$$G_p\left(Ax, By, By\right) \leq kG_p\left(Ty, By, By\right)$$

for $k \in \left[\frac{1}{3}, 1\right)$, which implies

$$G_p\left(Ax, By, By\right) \leq k \max\left\{G_p\left(Sx, Ty, Ty\right), G_p\left(Ax, Sx, Sx\right), G_p\left(Ty, By, By\right), G_p\left(Sx, By, By\right), G_p\left(Ax, Ty, Ty\right)\right\}$$

for $k \in \left[\dfrac{1}{3}, 1\right)$. By Theorem 4.4, A, B, S and T have a unique common fixed point $z = 0$ with $G_p\left(z, z, z\right) = 0$.

5. Applications

5.1. Fixed points for mappings satisfying contractive conditions of integral type in G_p-metric spaces

In [8], Branciari established the following theorem, which opened the way to the study of fixed points for mappings satisfying contractive conditions of integral type.

Theorem 5.1 ([8]). *Let (X, d) be a complete metric space, $c \in (0, 1)$ and $f \colon X \to X$ such that for all $x, y \in X$,*

$$\int_0^{d(fx,fy)} h(t)dt \leq c \int_0^{d(x,y)} h(t)dt,$$

whenever $h \colon [0, \infty) \to [0, \infty)$ is a Lebesgue measurable mapping which is summable (i.e. with finite integral) on each compact subset of $[0, \infty)$, such that $\int_0^{\varepsilon} h(t)dt > 0$ for all $\varepsilon > 0$.

Then f has a unique fixed point z such that for all $x \in X$, $z = \lim_{n\to\infty} f^n x$.

Some fixed point theorems for mappings satisfying contractive conditions of integral type are obtained in [38].

Lemma 5.2. *Let* $h\colon [0,\infty) \to [0,\infty)$ *be as in Theorem 5.1. Then* $\psi(t) = \int_0^t h(x)dx$ *is an almost altering distance.*

Proof. It follows by [38, Lemma 2.5]. □

Theorem 5.3. *Let* A, B, S *and* T *be self mappings of a* G_p*-metric space* (X, G_p) *such that*

$$(5.1)\quad F\Bigg(\int_0^{G_p(Ax,By,By)} h(t)\,dt, \int_0^{G_p(Sx,Ty,Ty)} h(t)\,dt, \int_0^{G_p(Ax,Sx,Sx)} h(t)\,dt, \int_0^{G_p(Ty,By,By)} h(t)\,dt, \int_0^{G_p(Sx,By,By)} h(t)\,dt, \int_0^{G_p(Ax,Ty,Ty)} h(t)\,dt\Bigg) \le 0$$

for all $x, y \in X$*, where* $F \in \mathcal{F}$ *and* $h(t)$ *is as in Theorem 5.1. If* (A, S) *and* T *satisfy* $CLR_{(A,S)T}$*-property, then* $\mathcal{C}(A,S) \neq \emptyset \neq \mathcal{C}(B,T)$.

Moreover, if A *is pointwise* S *absorbing and* B *is pointwise* T *absorbing, then* A, B, S *and* T *have a unique common fixed point* z *with* $G_p(z,z,z) = 0$.

Proof. By Lemma 5.2, $\psi(t) = \int_0^t h(x)dx$ is an almost altering distance. Then

$$\int_0^{G_p(Ax,By,By)} h(t)\,dt = \psi(G_p(Ax,By,By)),$$

$$\int_0^{G_p(Sx,Ty,Ty)} h(t)\,dt = \psi(G_p(Sx,Ty,Ty)),$$

$$\int_0^{G_p(Ax,Sx,Sx)} h(t)\,dt = \psi(G_p(Ax,Sx,Sx)),$$

$$\int_0^{G_p(Ty,By,By)} h(t)\,dt = \psi(G_p(Ty,By,By)),$$

$$\int_0^{G_p(Sx,By,By)} h(t)\,dt = \psi(G_p(Sx,By,By)),$$

$$\int_0^{G_p(Ax,Ty,Ty)} h(t)\,dt = \psi(G_p(Ax,Ty,Ty)).$$

By (5.1) we obtain

$$F\big(\psi\left(G_p\left(Ax,By,By\right)\right),\psi\left(G_p\left(Sx,Ty,Ty\right)\right),\psi\left(G_p\left(Ax,Sx,Sx\right)\right),\\ \psi\left(G_p\left(Ty,By,By\right)\right),\psi\left(G_p\left(Sx,By,By\right)\right),\psi\left(G_p\left(Ax,Ty,Ty\right)\right)\big)\leq 0,$$

which is inequality (4.1). Hence, the conditions of Theorem 4.1 are satisfied and Theorem 5.3 follows by Theorem 4.1. □

By Theorem 5.3 and Example 3.1 we obtain

THEOREM 5.4. *Let A,B,S and T be self mappings of a G_p-metric space (X,G_p) such that for all $x,y\in X$,*

$$\int_0^{G_p(Ax,By,By)} h\left(t\right)dt\leq k\max\Big\{\int_0^{G_p(Sx,Ty,Ty)} h\left(t\right)dt,\\ \int_0^{G_p(Ax,Sx,Sx)} h\left(t\right)dt,\int_0^{G_p(Ty,By,By)} h\left(t\right)dt,\\ \int_0^{G_p(Sx,By,By)} h\left(t\right)dt,\int_0^{G_p(Ax,Ty,Ty)} h\left(t\right)dt\Big\},$$

where $k\in[0,1)$ and $h\left(t\right)$ is as in Theorem 5.1. If (A,S) and T satisfy $CLR_{(A,S)T}$-property, then $\mathcal{C}\left(A,S\right)\neq\emptyset\neq\mathcal{C}\left(B,T\right)$.

Moreover, if A is pointwise S absorbing and B is pointwise T absorbing, then A,B,S and T have a unique common fixed point z with $G_p\left(z,z,z\right)=0$.

EXAMPLE 5.5 ([46]). Let $X=[0,\infty)$ and $G_p\left(x,y,z\right)=\max\left\{x,y,z\right\}$. Then (X,G_p) is a G_p-metric space. Consider the following mappings: $Ax=\frac{x}{2}$, $Sx=2x$, $Bx=0$, $Tx=x$. Then $S\left(X\right)=[0,\infty)$, $T\left(X\right)=[0,\infty)$, $S\left(X\right)\cap T\left(X\right)=[0,\infty)$. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty}x_n=0$. Then $\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}Sx_n=0=z\in S\left(X\right)\cap T\left(X\right)$ and $G_p\left(z,z,z\right)=0$. Hence, (A,S) and T satisfy $CLR_{(A,S)T}$-property. Note that

$$G_p\left(Sx,SAx,SAx\right)=2x,\quad G_p\left(Sx,Ax,Ax\right)=2x.$$

Hence,

$$G_p\left(Sx,SAx,SAx\right)\leq R_1G_p\left(Sx,Ax,Ax\right)\quad\text{for }R_1\geq 1.$$

Thus, A is S pointwise absorbing. We have also

$$G_p\left(Tx,TBx,TBx\right)=x,\quad G_p\left(Tx,Bx,Bx\right)=x.$$

Hence,

$$G_p(Tx, TBx, TBx) \leq R_2 G_p(Tx, Bx, Bx) \quad \text{for } R_2 \geq 1.$$

Thus, B is T pointwise absorbing. On the other hand,

$$G_p(Ax, By, By) = \frac{x}{2}, \quad G_p(Sx, Sx, Ax) = 2x.$$

Moreover

$$\int_0^{\frac{x}{2}} tdt \leq \int_0^{2x} tdt.$$

Thus, for $h(t) = t$ we obtain

$$\int_0^{G_p(Ax,By,By)} h(t)\,dt \leq k \int_0^{G(Sx,Sx,Ax)} h(t)\,dt,$$

where $\frac{1}{16} \leq k < 1$. Hence,

$$G_p(Ax, By, By) \leq k \max\left\{ \int_0^{G_p(Sx,Ty,Ty)} h(t)\,dt, \int_0^{G_p(Ax,Sx,Sx)} h(t)\,dt, \right.$$
$$\left. \int_0^{G_p(Ty,By,By)} h(t)\,dt, \int_0^{G_p(Sx,By,By)} h(t)\,dt, \int_0^{G_p(Ax,Ty,Ty)} h(t)\,dt \right\},$$

where $\frac{1}{16} \leq k < 1$.

By Theorem 5.4, A, B, S and T have a unique common fixed point $z = 0$ with $G_p(z, z, z) = 0$.

Remark 5.6. By Theorem 5.1 and Examples 3.2–3.8 we obtain new particular results.

5.2. Fixed points for mappings satisfying φ-contractive conditions in G_p-metric spaces

As in [24], let Φ be the set of all real continuous nondecreasing functions $\varphi\colon [0,\infty) \to [0,\infty)$ such that

1) $\varphi(t) < t$ for all $t > 0$,
2) $\varphi(t) = 0$ if and only if $t = 0$.

The following functions $F\colon \mathbb{R}_+^6 \to \mathbb{R}_+$ are in $\mathcal{F}$.

Example 5.7. $F(t_1, \dots, t_6) = t_1 - \varphi(\max\{t_2, t_3, t_4, t_5, t_6\})$.

Example 5.8. $F(t_1, \dots, t_6) = t_1 - \varphi\left(\max\left\{t_2, t_3, t_4, \frac{t_5+t_6}{2}\right\}\right)$.

Example 5.9. $F(t_1, \dots, t_6) = t_1 - \varphi\left(\max\left\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\right\}\right)$.

Example 5.10. $F(t_1, \dots, t_6) = t_1 - \varphi\left(\max\{t_2, \sqrt{t_3t_4}, \sqrt{t_3t_5}, \sqrt{t_3t_5}, \sqrt{t_4t_6}\}\right)$.

Example 5.11. $F(t_1, \dots, t_6) = t_1 - \varphi(at_2 + bt_3 + ct_4 + dt_5 + et_6)$, where $a, b, c, d, e \geq 0$ and $a + b + c + d + e \leq 1$.

Example 5.12. $F(t_1, \dots, t_6) = t_1 - \varphi\left(at_2 + \frac{b\sqrt{t_5t_6}}{1+t_3+t_4}\right)$, where $a, b \geq 0$ and $a + b \leq 1$.

Example 5.13. $F(t_1, \dots, t_6) = t_1 - \varphi\left(at_2 + b\max\{t_3, t_4\} + c\max\left\{\frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\right\}\right)$, where $a, b, c \geq 0$ and $a + b + c \leq 1$.

Example 5.14. $F(t_1, \dots, t_6) = t_1 - \varphi(at_2 + b\max\{2t_4 + t_5, 2t_4 + t_6, t_3 + t_5 + t_6\})$, where $a, b \geq 0$ and $a + b \leq 1$.

By Theorem 4.3 and Example 5.7 we obtain

Theorem 5.15. *Let A, B, S and T be self mappings of a G_p-metric space (X, G_p) such that*

$$G_p(Ax, By, By) \leq \varphi\big(\max\{G_p(Sx, Ty, Ty), G_p(Ax, Sx, Sx), G_p(Ty, By, By), G_p(Sx, By, By), G_p(Ax, Ty, Ty)\}\big)$$

for all $x, y \in X$, where $\varphi \in \Phi$.

If (A, S) and T satisfy $CLR_{(A,S)T}$-property, then $\mathcal{C}(A, S) \neq \emptyset \neq \mathcal{C}(B, T)$.

Moreover, if A is pointwise S absorbing and B is pointwise T absorbing, then A, B, S and T have a unique common fixed point z with $G_p(z, z, z) = 0$.

Example 5.16 ([46]). Let $X = [0, \infty)$ and $G_p(x, y, z) = \max\{x, y, z\}$. Then (X, G_p) is a G_p-metric space. Let A, B, S and T be as in Example 5.5. As in Example 5.5, (A, S) and T satisfy $CLR_{(A,S)T}$-property, A is S pointwise absorbing and B is T pointwise absorbing. Moreover

$$G_p(Ax, By, By) = \frac{x}{2}, \quad G_p(Ax, Sx, Sx) = 2x.$$

Put $\varphi(t) = \frac{t}{2}$. Then $\varphi \in \Phi$ and

$$\begin{aligned} G_p(Ax, By, By) &\le \frac{1}{2} G_p(Ax, Sx, Sx) \\ &\le \frac{1}{2}\max\{G_p(Sx, Ty, Ty), G_p(Ax, Sx, Sx), \\ &\quad G_p(Ty, By, By), G_p(Sx, By, By), G_p(Ax, Ty, Ty)\} \\ &= \varphi\big(\max\{G_p(Sx, Ty, Ty), G_p(Ax, Sx, Sx), \\ &\quad G_p(Ty, By, By), G_p(Sx, By, By), G_p(Ax, Ty, Ty)\}\big). \end{aligned}$$

By Theorem 5.15, A, B, S and T have a unique common fixed point $z = 0$ with $G_p(z, z, z) = 0$.

Remark 5.17. By Theorem 4.3 and Examples 5.7-5.14 we obtain new particular results.

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References

[1] M. Aamri and D. El Moutawakil, *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl. **270** (2002), no. 1, 181–188.

[2] M.R. Ahmadi Zand and A. Dehghan Nezhad, *A generalization of partial metric spaces*, J. Contemp. Appl. Math. **1** (2011), no. 1, 86–93.

[3] J. Ali and M. Imdad, *An implicit function implies several contractive conditions*, Sarajevo J. Math. **4(17)** (2008), no. 2, 269–285.

[4] I. Altun, F. Sola and H. Simsek, *Generalized contractions on partial metric spaces*, Topology Appl. **157** (2010), no. 18, 2778–2785.

[5] H. Aydi, E. Karapınar and P. Salimi, *Some fixed point results in GP-metric spaces*, J. Appl. Math. **2012**, Art. ID 891713, 16 pp.

[6] M.A. Barakat and A.M. Zidan, *A common fixed point theorem for weak contractive maps in G_p-metric spaces*, J. Egyptian Math. Soc. **23** (2015), no. 2, 309–314.
[7] N. Bilgili, E. Karapınar and P. Salimi, *Fixed point theorems for generalized contractions on GP-metric spaces*, J. Inequal. Appl. **2013**, 2013:39, 13 pp.
[8] A. Branciari, *A fixed point theorem for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci. **29** (2002), no. 9, 531–536.
[9] K.P. Chi, E. Karapınar and T.D. Thanh, *A generalized contraction principle in partial metric spaces*, Math. Comput. Modelling **55** (2012), no. 5–6, 1673–1681.
[10] L. Ćirić, B. Samet, H. Aydi and C. Vetro, *Common fixed points of generalized contractions on partial metric spaces and an application*, Appl. Math. Comput. **218** (2011), no. 6, 2398–2406.
[11] B.C. Dhage, *Generalised metric spaces and mappings with fixed point*, Bull. Calcutta Math. Soc. **8** (1992), no. 4, 329–336.
[12] B.C. Dhage, *Generalized metric spaces and topological structure. I*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **46** (2000), no. 1, 3–24.
[13] D. Gopal, A.S. Ranadive and U. Mishra, *On some open problems of common fixed point theorems of noncompatible mappings*, Proc. Math. Soc. BHU **20** (2004), 135–141.
[14] D. Gopal, A.S. Ranadive and R.P. Pant, *Common fixed points of absorbing maps*, Bull. Marathwada Math. Soc. **9** (2008), no. 1, 43–48.
[15] M. Imdad and S. Chauhan, *Employing common limit range property to prove unified metrical common fixed point theorems*, Int. J. Anal. **2013**, Art. ID 763261, 10 pp.
[16] M. Imdad, S. Chauhan and Z. Kadelburg, *Fixed point theorem for mappings with common limit range property satisfying generalized (ψ,ϕ)-weak contractive conditions*, Math. Sci. (Springer) **7** (2013), Art. 16, 8 pp.
[17] M. Imdad, B.D. Pant and S. Chauhan, *Fixed point theorems in Menger spaces using $CLR_{(S,T)}$ property and applications*, J. Nonlinear Anal. Optim. **3** (2012), no. 2, 225–237.
[18] M. Imdad, A. Sharma and S. Chauhan, *Unifying a multitude of common fixed point theorems in symmetric spaces*, Filomat **28** (2014), no. 6, 1113–1132.
[19] G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci. **9** (1986), no. 4, 771–779.
[20] Z. Kadelburg, H.K. Nashine and S. Radenović, *Fixed point results under various contractive conditions in partial metric spaces*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **107** (2013), no. 2, 241–256.
[21] E. Karapınar and U. Yüksel, *Some common fixed point theorems in partial metric spaces*, J. Appl. Math. **2011**, Art. ID 263621, 16 pp.
[22] M.S. Khan, M. Swaleh and S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Austral. Math. Soc. **30** (1984), no. 1, 1–9.
[23] Y. Liu, J. Wu and Z. Li, *Common fixed points of single-valued and multivalued maps*, Int. J. Math. Math. Sci. **2005**, no. 19, 3045–3055.
[24] J. Matkowski, *Fixed point theorems for mappings with a contractive iterate at a point*, Proc. Amer. Math. Soc. **62** (1977), no. 2, 344–348.
[25] S.G Matthews, *Partial metric topology*, in: S. Andima et al. (eds.), *Papers on General Topology and Applications*, Eighth Summer Conference at Queens College, Annals of the New York Academy of Sciences, 728, New York, 1994, pp. 183–197.
[26] U. Mishra and A.S. Ranadive, *Common fixed point of absorbing mappings satisfying implicit relations*. Preprint.
[27] Z. Mustafa, H. Obiedat and F. Awawdeh, *Some fixed point theorem for mapping on complete G-metric spaces*, Fixed Point Theory Appl. **2008**, Art. ID 189870, 12 pp.
[28] Z. Mustafa and B. Sims, *Some remarks concerning D-metric spaces*, in: J.G. Falset et al. (eds.), *International Conference on Fixed Point Theory and Applications*, Proceedings of the conference held in Valencia, July 13-19, 2003, Yokohama Publishers, Yokohama, 2004, pp. 189–198.
[29] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal. **7** (2006), no. 2, 289–297.

[30] R.P. Pant, *Common fixed point theorems for contractive maps*, J. Math. Anal. Appl. **226** (1998), no. 1, 251–258.
[31] R.P. Pant, *R-weak commutativity and common fixed points of noncompatible maps*, Ganita **49** (1998), no. 1, 19–27.
[32] R.P. Pant, *R-weak commutativity and common fixed points*, Soochow J. Math. **25** (1999), no. 1, 37–42.
[33] V. Parvanieh, J.R. Roshan and Z. Kadelburg, *On generalized weakly GP-contractive mappings in ordered GP-metric spaces*, Gulf J. Math. **1** (2013), no. 1, 78–97.
[34] V. Popa, *Fixed point theorems for implicit contractive mappings*, Stud. Cercet. Ştiinţ. Ser. Mat. Univ. Bacău No. **7** (1997), 127–133.
[35] V. Popa, *Some fixed point theorems for compatible mappings satisfying an implicit relation*, Demonstratio Math. **32** (1999), no. 1, 157–163.
[36] V. Popa, *A general fixed point theorem for occasionally weakly compatible mappings and applications*, Sci. Stud. Res. Ser. Math. Inform. **22** (2012), no. 1, 77–91.
[37] V. Popa, *Fixed point theorems for two pairs of mappings satisfying a new type of common limit range property*, Filomat **31** (2017), no. 11, 3181–3192.
[38] V. Popa and M. Mocanu, *Altering distance and common fixed points under implicit relations*, Hacet. J. Math. Stat. **38** (2009), no. 3, 329–337.
[39] V. Popa and A.-M. Patriciu, *A general fixed point theorem for pairs of weakly compatible mappings in G-metric spaces*, J. Nonlinear Sci. Appl. **5** (2012), no. 2, 151–160.
[40] V. Popa and A.-M. Patriciu, *Fixed point theorems for mappings satisfying an implicit relation in complete G-metric spaces*, Bul. Inst. Politeh. Iaşi. Secţ. Mat. Mec. Teor. Fiz. **59(63)** (2013), no. 2, 97–123.
[41] V. Popa and A.-M. Patriciu, *A general fixed point theorem for a pair of self mappings with common limit range property in G-metric spaces*, Facta Univ. Ser. Math. Inform. **29** (2014), no. 4, 351–370.
[42] V. Popa and A.-M. Patriciu, *Two general fixed point theorems for a sequence of mappings satisfying implicit relations in G_p-metric spaces*, Appl. Gen. Topol. **16** (2015), no. 2, 225–231.
[43] V. Popa and A.-M. Patriciu, *Well posedness of fixed point problem for mappings satisfying an implicit relation in G_p-metric spaces*, Math. Sci. Appl. E-Notes **3** (2015), no. 1, 108–117.
[44] V. Popa and A.-M. Patriciu, *Fixed point theorems for two pairs of mappings satisfying common limit range property in G-metric spaces*, Bul. Inst. Politeh. Iaşi. Secţ Mat. Mec. Teor. Fiz. **62(66)** (2016), no. 2, 19–42.
[45] V. Popa and A.-M. Patriciu, *Fixed point results for pairs of absorbing mappings in partial metric spaces*, Acta Univ. Apulensis Math. Inform. No. **50** (2017), 97–109.
[46] V. Popa and A.-M. Patriciu, *Fixed point theorems for two pairs of mappings satisfying a new type of common limit range property in G_p-metric spaces*, Ann. Math. Sil. **32** (2018), 295–312.
[47] W. Shatanawi, *Fixed point theory for contractive mappings satisfying Φ-maps in G-metric spaces*, Fixed Point Theory Appl. **2010**, Art. ID 181650, 9 pp.
[48] W. Sintunavarat and P. Kumam, *Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces*, J. Appl. Math. **2011**, Art. ID 637958, 14 pp.
[49] C. Vetro and F. Vetro, *Common fixed points of mappings satisfying implicit relations in partial metric spaces*, J. Nonlinear Sci. Appl. **6** (2013), no. 3, 152–161.

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