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# LIMITS OF SEQUENCES OF FEEBLY-TYPE CONTINUOUS FUNCTIONS

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*Dedicated to Professor Zygfryd Kominek on his 75th birthday*

**Abstract.** We consider the following families of real-valued functions defined on $\mathbb{R}^2$: feebly continuous functions (FC), very feebly continuous functions (VFC), and two-feebly continuous functions (TFC). It is known that the inclusions $\mathrm{FC} \subset \mathrm{VFC} \subset \mathrm{TFC}$ are proper. We study pointwise and uniform limits of sequences with terms taken from these families.

## 1. Introduction

In the paper, we continue our former investigations from [1] on various classes of feebly-like continuous real-valued functions in two variables. Let us recall basic definitions.

According to [3], we say that a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is *feebly continuous at a point* $\langle x, y\rangle \in \mathbb{R}^2$ if there exist sequences $x_n \searrow x$ and $y_m \searrow y$ such that

$$\lim_{n\to\infty} \lim_{m\to\infty} f(x_n, y_m) = f(x, y). \tag{1}$$

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Here the symbol $x_n \searrow x$ means that the sequence $(x_n)$ is strictly decreasing and convergent to $x$. The equality (1) means that, for every $n$, a limit $\lim_{m\to\infty} f(x_n, y_m) = a_n$ exists and $\lim_{n\to\infty} a_n = f(x, y)$.

In [3], Leader considered also another notion which is weaker than feebly continuity. Namely, $f\colon \mathbb{R}^2 \to \mathbb{R}$ is called *very feebly continuous at* a point $\langle x, y\rangle$ if there exist a sequence $x_n \searrow x$ and, for each $n \in \mathbb{N}$, a sequence $y_m^{(n)} \searrow y$ such that

$$\lim_{n\to\infty} \lim_{m\to\infty} f(x_n, y_m^{(n)}) = f(x, y).$$

In [1], we proposed a related notion. We say that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is *two-feebly continuous at* $\langle x, y\rangle$ if there exist sequences $x_n \searrow x$ and $y_n \searrow y$ such that $\lim_{n\to\infty} f(x_n, y_n) = f(x, y)$.

We say that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is feebly (very feebly, two-feebly) continuous whenever it has this property at every point of $\mathbb{R}^2$. The families of feebly (very feebly, two-feebly) continuous functions will be denoted by FC (VFC, TFC). It follows that

$$\mathrm{C} \subset \mathrm{FC} \subset \mathrm{VFC} \subset \mathrm{TFC}$$

and the inclusions are proper; see [1]. Here C denotes, as usual, the family of all continuous functions. Note that Leader in [3] constructed, under the Continuum Hypothesis, a function which is nowhere feebly continuous. However, such functions are neither measurable nor with Baire property, see [1, Theorem 1].

Our purpose in this paper is to study pointwise and uniform limits of sequences with terms taken from the classes FC, VFC and TFC.

## 2. Pointwise limits

Given functions $f$ and $f_n$, $n \in \mathbb{N}$, from $\mathbb{R}^2$ to $\mathbb{R}$, the symbol $f_n \longrightarrow f$ will stand for the pointwise convergence of a sequence $(f_n)$ to $f$.

Proposition 1. *Every function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a pointwise limit of sequences $(f_k)$, $(g_k)$ and $(h_k)$ with terms taken from* FC, VFC $\setminus$ FC, *and* TFC $\setminus$ VFC, *respectively. If, moreover, $f$ is Borel (Lebesgue, Baire) measurable, then all functions $f_k$, $g_k$, $h_k$ have the same property. Consequently, each of the sets* FC, VFC $\setminus$ FC, *or* TFC $\setminus$ VFC *is dense in the space $\mathbb{R}^{\mathbb{R}^2}$ with the topology of pointwise convergence.*

PROOF. Fix $f\colon \mathbb{R}^2 \to \mathbb{R}$.

I. First, we will construct a sequence $(f_k)$, with terms in FC, and such that $f_k \longrightarrow f$. Write $\mathbb{Q} = \{q_n\colon n \in \mathbb{N}\}$. Let $\{A_{k,n}\colon k, n \in \mathbb{N}\}$ be a family of pairwise disjoint countable subsets of $\mathbb{R}$, each dense in $\mathbb{R}$. Define $f_k\colon \mathbb{R}^2 \to \mathbb{R}$ by the formula

$$f_k(x,y) := \begin{cases} q_n & \text{if } x \in A_{k,n},\ n \in \mathbb{N}, \\ f(x,y) & \text{otherwise.} \end{cases}$$

Clearly, each $f_k$ is feebly continuous, $f_k \longrightarrow f$, and the functions $f_k$ are measurable whenever $f$ has this property.

II. We will define a sequence $(g_k)$, with terms in VFC $\setminus$ FC, pointwise convergent to $f$. We use some ideas from [1] (cf. Lemma 16 and Theorem 17). Choose distinct reals $v_n^k$, for $k, n \in \mathbb{N}$, such that each set $\{v_n^k\colon n \in \mathbb{N}\}$, $k \in \mathbb{N}$, is dense in $\mathbb{R}$. Let $\{D_{n,m}^k\colon k, n, m \in \mathbb{N}\}$ be a family of pairwise disjoint countable dense subsets of $\mathbb{R} \setminus \mathbb{N}$.

Fix $k \in \mathbb{N}$. List $D_{n,1}^k = \{d_{n,m}^k\colon m \in \mathbb{N}\}$ and put $D_n^k := \bigcup_m \{d_{n,m}^k\} \times D_{n,m}^k$, $D^k := \bigcup_n D_n^k$. Then define functions $f^k$ and $g_k$ from $\mathbb{R}^2$ to $\mathbb{R}$ by the formulas

$$f^k := \sum_{n\in\mathbb{N}} v_n^k \chi_{D_n^k},$$

$$g_k(x,y) := \begin{cases} f^k(x,y) & \text{for } \langle x,y\rangle \in D^k, \\ k+1 & \text{for } \langle x,y\rangle = \langle k,k\rangle, \\ \max(\min(f(x,y),k),-k) & \text{otherwise.} \end{cases}$$

Note that $g_k$ is well-defined because $\langle k,k\rangle \notin D^k$. It follows from [1, Lemma 16] that every extension of $f^k|D^k$ to the whole plane is very feebly continuous. Hence $g_k|D^k = f^k|D^k$ implies that $g_k \in$ VFC. Clearly, $g_k \longrightarrow f$.

Now, observe that $g_k$ is not feebly continuous at the point $\langle k,k\rangle$. This is a consequence of the fact that $g_k(k,k) \neq 0$ while the sets $A_x := \{y > k : g_k(x,y) \neq 0\} \subset D_{n,m}^k$, where $x = d_{n,m}^k$, and the sets $D_{n,m}^k$ are pairwise disjoint.

Finally, assume that the function $f$ is measurable. Then the function $\hat{f}_k = \max(\min(f(x,y),k),-k)$ is measurable, too. Since $g_k = \hat{f}_k$ on a co-countable set, $g_k$ is measurable, too.

III. Finally, we construct a sequence $(h_k)$, with terms in TFC $\setminus$ VFC, pointwise convergent to $f$. By the first assertion, $f$ is the pointwise limit of a sequence $(f_k)$ with terms in FC. For $k \in \mathbb{N}$, set $S_k := [k - \frac{1}{4}, k + \frac{1}{4}) \times [k - \frac{1}{4}, k + \frac{1}{4})$ and define $h_k\colon \mathbb{R}^2 \to \mathbb{R}$ as

$$h_k(x,y) := \begin{cases} f_k(x,y) & \text{if } \langle x,y\rangle \in \mathbb{R}^2 \setminus S_k, \\ x & \text{if } \langle x,y\rangle \in S_k \text{ and } x = y, \\ 0 & \text{if } \langle x,y\rangle \in S_k \text{ and } x \neq y. \end{cases}$$

Then $\lim_k h_k = \lim_k f_k = f$. Moreover, each $h_k$ is in TFC and it is not very feebly continuous at the point $\langle k,k\rangle$. Finally, it is easy to observe that $h_k$ is measurable whenever $f$ has this property. □

Remark 2. It is easy to observe that sequences $(f_n)$, $(g_n)$ and $(h_n)$ defined in the proof of Proposition 1 converge discretely to the function $f$. Recall that $(f_n)$ is *discretely convergent* to a function $f\colon X \to \mathbb{R}$ if for every $x \in X$ there is $N$ with $f_n(x) = f(x)$ for $n > N$. This notion was introduced by Császár and Laczkovich [2]. It is much stronger than pointwise convergence.

## 3. Uniform limits

Recall that the topology of uniform convergence in the space $\mathbb{R}^{\mathbb{R}^2}$ of all functions from $\mathbb{R}^2$ into $\mathbb{R}$ is metrizable by the metric

$$d(f,g) := \min\left(1, \sup_{x\in\mathbb{R}^2} |f(x) - g(x)|\right).$$

Proposition 3. *The families* VFC *and* TFC *are closed in the topology of uniform convergence in* $\mathbb{R}^{\mathbb{R}^2}$.

Proof. Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ is the uniform limit of a sequence $(f_n)$ with terms in VFC. We may assume that $d(f_n, f) < \frac{1}{n}$ for each $n \in \mathbb{N}$. Fix $\langle x,y\rangle \in \mathbb{R}^2$. We will show that $f$ is very feebly continuous at $\langle x,y\rangle$. For a given $n \in \mathbb{N}$, since $f_n$ is very feebly continuous at $\langle x,y\rangle$, there are: $x_n \in \mathbb{R}$ and $y_m^{(n)} \searrow y$ such that

(i) $|x_n - x| < \frac{1}{n}$;

(ii) $|\lim_m f_n(x_n, y_m^{(n)}) - f_n(x,y)| < \frac{1}{n}$.

The condition (ii) implies that the sequence $\big(f_n(x_n, y_m^{(n)})\big)_m$ is bounded, hence $\big(f(x_n, y_m^{(n)})\big)_m$ is bounded too. Let $k_m \nearrow \infty$ be such that $\big(f(x_n, y_{k_m}^{(n)})\big)_m$ is convergent to some $\lambda_n$. Then $|\lambda_n - f(x,y)| < \frac{2}{n}$, and for $z_m^{(n)} := y_{k_m}^{(n)}$ we have $z_m^{(n)} \searrow y$. We may assume that $x_n \searrow x$. Then $\lim_n \lim_m f(x_n, z_m^{(n)}) = f(x,y)$, so $f$ is very feebly continuous at $\langle x,y\rangle$.

The argument for TFC is similar. Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is the uniform limit of a sequence $(f_n)$ with terms in TFC, and assume that $d(f_n, f) < \frac{1}{n}$ for each $n \in \mathbb{N}$. Fix $\langle x, y\rangle \in \mathbb{R}^2$. For a given $n \in \mathbb{N}$, since $f_n$ is two-feebly continuous at $\langle x, y\rangle$, pick sequences $x_m^{(n)} \searrow x$ and $y_m^{(n)} \searrow y$ such that $\lim_m f_n(x_m^{(n)}, y_m^{(n)}) = f_n(x, y)$. Then choose inductively a sequence $m_n \nearrow \infty$ such that for every $n \in \mathbb{N}$,

$$|f_n(x_{m_n}^{(n)}, y_{m_n}^{(n)}) - f_n(x, y)| < \frac{1}{n}.$$

This implies that $|f(x_{m_n}^{(n)}, y_{m_n}^{(n)}) - f(x, y)| < \frac{3}{n}$ for every $n$ which shows that $f \in \mathrm{TFC}$. $\square$

In the sequel, we will use the following notation. We will write $p \in FC(f)$ whenever $f\colon \mathbb{R}^2 \to \mathbb{R}$ is feebly continuous at a point $p \in \mathbb{R}^2$ (similarly, for very feebly continuity and two-feebly continuity).

We need the following lemma which results directly from the definition of very feebly continuity.

Lemma 4 ([1, Lemma 3]). *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$. A point $z = \langle x, y\rangle$ belongs to $\mathbb{R}^2 \setminus VFC(f)$ if and only if there exist an interval $(p, q)$ containing $f(z)$, a real $t > 0$ and real numbers $r_s > 0$, chosen for every $s \in (0, t)$, such that $f$ does not attain values in $(p, q)$ at any point of the set*

$$G(z) := \{\langle x + a, y + b\rangle \colon 0 < a < t,\ 0 < b < r_a\}.$$

Proposition 5. *The family* TFC $\setminus$ VFC *is uniformly dense in* TFC. *Consequently,* VFC *is nowhere dense in* TFC *with the topology of uniform convergence.*

Proof. Let $f \in \mathrm{TFC}$. For a given $\varepsilon > 0$, we will find a function $g \in \mathrm{TFC} \setminus \mathrm{VFC}$ with $d(f, g) \leq 2\varepsilon$. Since $f \in \mathrm{TFC}$, there are sequences $\hat{x}_n \searrow 0$, $\hat{y}_n \searrow 0$ with $\lim_n f(\hat{x}_n, \hat{y}_n) = f(0, 0)$. For any $n \in \mathbb{N}$, let $L_n$ denote the closed segment with end-points $\langle \hat{x}_n, \hat{y}_n\rangle$, $\langle \hat{x}_{n+1}, \hat{y}_{n+1}\rangle$, and let $L := \{\langle 0, 0\rangle\} \cup \bigcup_n L_n$. Then define

$$T := \left\{\langle x, y\rangle \in \mathbb{R}^2 \colon x \in (0, \hat{x}_1)\ \&\ y \in \left[0, \frac{1}{2}L(x)\right]\right\},$$

where $L(x)$ denotes the unique $y$ with $\langle x, y\rangle \in L$. For every $k \in \mathbb{N}$, put

$$W := \{\langle x, y\rangle \in T \colon |f(x, y) - f(0, 0)| < \varepsilon\},$$

$$W^+ := \{\langle x, y\rangle \in W\colon f(0,0) < f(x,y) < f(0,0)+\varepsilon\}\,,$$

$$W^- := \{\langle x, y\rangle \in W\colon f(0,0) > f(x,y) > f(0,0)-\varepsilon\}\,.$$

Moreover, decompose the set $V := W \cap f^{-1}[\{f(0,0)\}]$ into two parts. Let $V^+$ be the set of all $\langle x, y\rangle \in V$ for which there are sequences $x_n \searrow x$, $y_n \searrow y$ with $f(x_n, y_n) \geq f(0,0)$ and $\lim_n f(x_n, y_n) = f(x,y)$, and set $V^- := V \setminus V^+$. Note that for any $\langle x, y\rangle \in V^-$, there are $x_n \searrow x$, $y_n \searrow y$ such that $f(x_n, y_n) < f(0,0)$ and $\lim_n f(x_n, y_n) = f(x,y)$.

Now, define $g\colon \mathbb{R}^2 \to \mathbb{R}$ by the formula

$$g(x,y) := \begin{cases} f(0,0)+\varepsilon & \text{for } \langle x,y\rangle \in W^+ \cup V^+, \\ f(0,0)-\varepsilon & \text{for } \langle x,y\rangle \in W^- \cup V^-, \\ f(x,y) & \text{for } \langle x,y\rangle \in \mathbb{R}^2 \setminus W. \end{cases}$$

Then obviously, $d(g,f) \leq 2\varepsilon$.

Let us verify that $g \in$ TFC. Since $g(\hat{x}_n, \hat{y}_n) = f(\hat{x}_n, \hat{y}_n)$ for all $n \in \mathbb{N}$, we have $\langle 0,0\rangle \in TFC(g)$. For every point $\langle x,y\rangle \notin Z := \{\langle 0,0\rangle\} \cup T$ there is a right-hand open square $S$ centered at $\langle x,y\rangle$ and disjoint with $Z$, so from $g = f$ on $\mathbb{R}^2 \setminus Z$ and $f \in$ TFC it follows that $\mathbb{R}^2 \setminus Z \subset TFC(g)$. Now, assume that $\langle x,y\rangle \in T$. Since $\langle x,y\rangle \in TFC(f)$, there are $x_n \searrow x$, $y_n \searrow y$ with $\lim_n f(x_n, y_n) = f(x,y)$. We consider a few cases.

1. First, suppose that $|f(x,y) - f(0,0)| > \varepsilon$. Then $|f(x_n,y_n) - f(0,0)| > \varepsilon$ for almost all $n$, hence $g(x_n,y_n) = f(x_n,y_n)$, so $\lim_n g(x_n,y_n) = \lim_n f(x_n,y_n) = f(x,y) = g(x,y)$ and therefore, $\langle x,y\rangle \in TFC(g)$.

2. Suppose that $\langle x,y\rangle \in W^+$. Then $\langle x_n,y_n\rangle \in W^+$ for almost all $n$, and we have $\lim_n g(x_n,y_n) = f(0,0)+\varepsilon = g(x,y)$. Similarly, if $\langle x,y\rangle \in W^-$.

3. Now, let $f(x,y) = f(0,0)+\varepsilon$. Then there is a sequence $i_n \nearrow \infty$ such that either $f(x_{i_n}, y_{i_n}) > f(0,0)+\varepsilon$ for every $n$, or $f(x_{i_n}, y_{i_n}) = f(0,0)+\varepsilon$ for all $n$, or $\langle x_{i_n}, y_{i_n}\rangle \in W^+$ for each $n$. In each of such cases $\lim_n g(x_{i_n}, y_{i_n}) = f(0,0)+\varepsilon = g(x,y)$. Similarly, if $f(x,y) = f(0,0)-\varepsilon$.

4. Finally, suppose that $\langle x,y\rangle \in V^+$. We may assume that $\langle x_n,y_n\rangle \in W^+ \cup V^+$ for all $n$. Then $\lim_n g(x_n,y_n) = f(0,0)+\varepsilon = g(x,y)$, hence $\langle x,y\rangle \in TFC(g)$. Similarly, if $\langle x,y\rangle \in V^-$.

Also, $g$ is not very feebly continuous at the point $\langle 0,0\rangle$ by Lemma 4.

Thus the above argument leads to the first assertion stating that TFC $\setminus$ VFC is uniformly dense in TFC. Now, the second assertion follows since VFC is closed by Proposition 3. □

Proposition 6. *The family* FC *is not closed in the space* $\mathbb{R}^{\mathbb{R}^2}$ *with the topology of uniform convergence.*

Proof. Let $\{p_k \colon k \in \mathbb{N}\}$ be the set of all prime numbers in $\mathbb{N}$. For any triple $\langle k, i, j\rangle \in \mathbb{N}^3$, let $S(k, i, j)$ denote a right-hand open square in $\mathbb{R}^2$ centered at the point $\langle \frac{1}{p_k^i}, \frac{1}{p_k^j}\rangle$, $S(k, i, j) := I(k, i, j) \times J(k, i, j)$, where $I(k, i, j)$ and $J(k, i, j)$ are intervals of the form $[a, b)$ with $a, b \in \mathbb{R}$, $a < b$. We may require that the closures of different intervals $J(k, i, j)$ and $J(k', i', j')$ are disjoint.

For $k \in \mathbb{N}$, define a function $f_k \colon \mathbb{R}^2 \to \mathbb{R}$ as follows

$$f_k(x, y) := \begin{cases} 1 - \frac{1}{k} & \text{for } \langle x, y\rangle \in \{\langle 0, 0\rangle\} \cup \bigcup_{n>k} \bigcup_{i,j\in\mathbb{N}} S(n, i, j), \\ 1 - \frac{1}{n} & \text{for } \langle x, y\rangle \in \bigcup_{i,j\in\mathbb{N}} S(n, i, j),\ n \le k, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that each $f_k$ is feebly continuous. In fact we only need to check that $\langle 0, 0\rangle \in FC(f)$ but this can be provided by the sequences $(x_n) := \left(\frac{1}{p_k^n}\right)$ and $(y_m) := \left(\frac{1}{p_k^m}\right)$.

Moreover, the sequence $(f_k)$ is uniformly convergent to the function

$$f(x, y) := \begin{cases} 1 & \text{for } \langle x, y\rangle = \langle 0, 0\rangle, \\ 1 - \frac{1}{n} & \text{for } \langle x, y\rangle \in \bigcup_{i,j\in\mathbb{N}} S(n, i, j),\ n \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

which is not feebly continuous at the point $\langle 0, 0\rangle$. □

Proposition 7. *The family* FC *is not uniformly dense in the class* VFC*.*

Proof. Choose reals $r_n$, for $n \in \mathbb{N}$, with the following properties:

- $0 < r_n < 1/n$ for every $n \in \mathbb{N}$;
- $\frac{r_n}{r_m} \notin \mathbb{Q}$ for $n \neq m$.

For all $n, m \in \mathbb{N}$, define $x_n := \frac{1}{n}$ and $y_m^{(n)} := \frac{r_n}{m}$, and let $S(n, m)$ be a right-hand open square centered at the point $\langle x_n, y_m^{(n)}\rangle$, $S(n, m) := I(n, m) \times J(n, m)$, where $\mathrm{cl}(J(n, m)) \cap \mathrm{cl}(J(i, j)) = \emptyset$ whenever $\langle n, m\rangle \neq \langle i, j\rangle$. Let $\widetilde{f} \colon \mathbb{R}^2 \to \mathbb{R}$ be the following modification of the function $f \colon \mathbb{R}^2 \to \mathbb{R}$ from [1, Example 8]:

$$\widetilde{f} := \chi_B, \quad \text{where } B := \{\langle 0, 0\rangle\} \cup \bigcup_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} S(n, m).$$

Observe that the function $\widetilde{f}$ is very feebly continuous, while there is no $g \in \mathrm{FC}$ with $d(g, \widetilde{f}) < \frac{1}{2}$. □

PROPOSITION 8. *The family* VFC $\setminus$ FC *is uniformly dense in* VFC.

PROOF. For given $f \in$ FC and $\varepsilon > 0$, we will construct $g \in$ VFC $\setminus$ FC with $d(f,g) \le \varepsilon$. Clearly, we may assume that $f(0,0) = 0$. Since $f$ is feebly continuous at the point $\langle 0,0\rangle$, there are sequences $x_n \searrow 0$, $y_m \searrow 0$ with $\lim_n \lim_m f(x_n, y_m) = 0$. Divide the set $\mathbb{N}$ into infinitely many sets which are ordered as increasing sequences $(i^k_m)_m$, $k \in \mathbb{N}$. Let $y^{(n)}_m := y_{i^n_m}$. For any $n, m \in \mathbb{N}$, let $S(n,m) := I(n,m) \times J(n,m)$ be a right-side open square centered at the point $\langle x_n, y^{(n)}_m\rangle$. We may assume that, for all pairs $\langle i,j\rangle$, $\langle n,m\rangle$ of natural numbers, the following conditions hold:

(1) if $i < n$ then $\sup I(i,j) < \inf I(n,m)$;

(2) if $j < m$ then $\sup J(n,m) < \inf J(n,j)$.

Let $S := \{\langle 0,0\rangle\} \cup \bigcup_{n,m} S(n,m)$. Decompose the set $Z := \mathbb{R}^2 \setminus S$ into three parts $A_+$, $A_-$, and $A_0$, where

$$A_+ : = Z \cap f^{-1}((0,+\infty)),$$

$$A_- : = Z \cap f^{-1}((-\infty,0)),$$

$$A_0 : = Z \setminus (A_+ \cup A_-).$$

Moreover, divide the set $A_0$ into two subsets $A_0^+$ and $A_0^-$ where $A_0^+$ is the set of all points $\langle x,y\rangle \in A_0$ for which there are $s_n \searrow x$, $t_m \searrow y$ such that $f(s_n,t_m) \ge 0$ and $\lim_n \lim_m f(s_n,t_m) = 0$, and let $A_0^- := A_0 \setminus A_0^+$. Note that, if $\langle x,y\rangle \in A_0^-$, then there are $s_n \searrow x$, $t_m \searrow y$ such that $f(s_n,t_m) \in A_0^-$ and $\lim_n \lim_m f(s_n,t_m) = 0$.

Now, we are ready to define the function $g\colon \mathbb{R}^2 \to \mathbb{R}$. Set

$$g(x,y) := \begin{cases} f(x,y) & \text{for } \langle x,y\rangle \in S, \\ f(x,y) + \varepsilon & \text{for } \langle x,y\rangle \in A_+ \cup A_0^+, \\ f(x,y) - \varepsilon & \text{for } \langle x,y\rangle \in A_- \cup A_0^-. \end{cases}$$

Then $g$ is as we need. Indeed, it is clear that $d(f,g) \le \varepsilon$. To see that $g$ is very feebly continuous at the point $\langle 0,0\rangle$, consider the sequences $(x_n)$, $(y^{(n)}_m)_m$. For every $n$, $(y^{(n)}_m)_m$ is a subsequence of $(y_m)$, therefore $\lim_m g(x_n, y^{(n)}_m) = \lim_m f(x_n, y_m)$. Hence $\lim_n \lim_m g(x_n, y^{(n)}_m) = \lim_n \lim_m f(x_n, y_m) = f(0,0) = g(0,0)$.

Now, we will verify that $\langle 0,0\rangle \notin FC(g)$. Suppose to the contrary that exist $s_n \searrow 0$, $t_m \searrow 0$ such that $\lim_n \lim_m g(s_n,t_m) = g(0,0) = 0$. We can assume that $|\lim_m g(s_n,t_m)| < \frac{\varepsilon}{2}$. This means that $\langle s_n, t_m\rangle \in \bigcup_{i,j} S(i,j)$, so $t_m$ belongs to infinitely many intervals $J(i,j)$ which is impossible. Finally, similarly as in the proof of Proposition 5, one can check that $g$ is feebly continuous at each point $\langle x, y\rangle \neq \langle 0, 0\rangle$. $\square$

Problem.

1. *Is the set* $\mathrm{VFC} \setminus \mathrm{FC}$ *residual (non-meager, or Borel) in the space* VFC *with the topology of uniform convergence?*
2. *Can every Borel (Lebesgue, or Baire) measurable function* $f \in \mathrm{VFC}$ *be a uniform limit of a sequence of Borel (Lebesgue, or Baire) measurable functions from the class* $\mathrm{VFC} \setminus \mathrm{FC}$*?*

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