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# REMARKS CONNECTED WITH THE WEAK LIMIT OF ITERATES OF SOME RANDOM-VALUED FUNCTIONS AND ITERATIVE FUNCTIONAL EQUATIONS

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*Dedicated to Professor Zygfryd Kominek on his 75th birthday*

**Abstract.** The paper consists of two parts. At first, assuming that $(\Omega, \mathcal{A}, P)$ is a probability space and $(X, \varrho)$ is a complete and separable metric space with the $\sigma$-algebra $\mathcal{B}$ of all its Borel subsets we consider the set $\mathcal{R}_c$ of all $\mathcal{B} \otimes \mathcal{A}$-measurable and contractive in mean functions $f\colon X \times \Omega \to X$ with finite integral $\int_\Omega \varrho(f(x,\omega),x)\,P(d\omega)$ for $x \in X$, the weak limit $\pi^f$ of the sequence of *iterates* of $f \in \mathcal{R}_c$, and investigate continuity-like property of the function $f \mapsto \pi^f, f \in \mathcal{R}_c$, and Lipschitz solutions $\varphi$ that take values in a separable Banach space of the equation

$$\varphi(x) = \int_\Omega \varphi(f(x,\omega))\,P(d\omega) + F(x).$$

Next, assuming that $X$ is a real separable Hilbert space, $\Lambda\colon X \to X$ is linear and continuous with $\|\Lambda\| < 1$, and $\mu$ is a probability Borel measure on $X$ with finite first moment we examine continuous at zero solutions $\varphi\colon X \to \mathbb{C}$ of the equation

$$\varphi(x) = \hat{\mu}(x)\varphi(\Lambda x)$$

which characterizes the limit distribution $\pi^f$ for some special $f \in \mathcal{R}_c$.

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Fix a probability space $(\Omega, \mathcal{A}, P)$ and a complete and separable metric space $(X, \varrho)$. Let $\mathcal{B}$ denote the $\sigma$-algebra of all Borel subsets of $X$.

We say that $f\colon X \times \Omega \to X$ is a *random-valued* function (an *rv-function* for short) if it is measurable with respect to the product $\sigma$-algebra $\mathcal{B} \otimes \mathcal{A}$. The iterates of such an rv-function are given by

$$f^0(x, \omega_1, \omega_2, \ldots) = x, \quad f^n(x, \omega_1, \omega_2, \ldots) = f(f^{n-1}(x, \omega_1, \omega_2, \ldots), \omega_n)$$

for $x$ from $X$ and $(\omega_1, \omega_2, \ldots)$ from $\Omega^\infty$ defined as $\Omega^{\mathbb{N}}$. Note that $f^n\colon X \times \Omega^\infty \to X$ is an rv-function on the product probability space $(\Omega^\infty, \mathcal{A}^\infty, P^\infty)$. More exactly, the $n$-th iterate $f^n$ is $\mathcal{B} \otimes \mathcal{A}_n$-measurable, where $\mathcal{A}_n$ denotes the $\sigma$-algebra of all sets of the form

$$\{(\omega_1, \omega_2, \ldots) \in \Omega^\infty : (\omega_1, \ldots, \omega_n) \in A\}$$

with $A$ from the product $\sigma$-algebra $\mathcal{A}^n$. (See [7, Sec. 1.4], [5].)

A simple criterion for the convergence in law of $(f^n(x, \cdot))_{n\in\mathbb{N}}$ to a random variable independent of $x \in X$ was proved in [1] and applied to the equation

$$\varphi(x) = \int_\Omega \varphi\left(f(x, \omega)\right) P(d\omega) + F(x) \tag{1}$$

with $\varphi$ as the unknown function. This criterion reads.

(H) There exists a $\lambda \in (0, 1)$ such that

$$\int_\Omega \varrho\left(f(x, \omega), f(z, \omega)\right) P(d\omega) \le \lambda \varrho(x, z) \quad \text{for } x, z \in X$$

and

$$\int_\Omega \varrho\left(f(x, \omega), x\right) P(d\omega) < \infty \quad \text{for } x \in X. \tag{2}$$

Thus, denoting by $\pi_n^f(x, \cdot)$ the distribution of $f^n(x, \cdot)$, i.e.,

$$\pi_n^f(x, B) = P^\infty\left(f^n(x, \cdot) \in B\right) \quad \text{for } n \in \mathbb{N} \cup \{0\},\ x \in X \text{ and } B \in \mathcal{B},$$

hypothesis (H) guarantees the existence of a probability Borel measure $\pi^f$ on $X$ such that

$$\lim_{n\to\infty} \int_X u(z) \pi_n^f(x, dz) = \int_X u(z) \pi^f(dz)$$

for $x \in X$ and for any continuous and bounded $u\colon X \to \mathbb{R}$; moreover, as observed in [3] (see also [6]),

$$\int_X \varrho(x,z)\pi^f(dz) < \infty \quad \text{for } x \in X. \tag{3}$$

In [2] we considered continuity-like property of the function $f \mapsto \pi^f$. In [4] we characterized the limit distribution $\pi^f$ via a functional equation for its characteristic function for some special rv-functions in Hilbert spaces. In the present paper we are strengthening the result of [2], apply it to equation (1) and consider also the equation used for the above mentioned characterization of the limit distribution.

1. Assuming that $(\Omega, \mathcal{A}, P)$ is a probability space and $(X, \varrho)$ is a complete and separable metric space, consider the set $\mathcal{R}_c$ of all rv-functions $f\colon X \times \Omega \to X$ such that

$$\int_\Omega \varrho(f(x,\omega), f(z,\omega))P(d\omega) \le \lambda_f \varrho(x,z) \quad \text{for } x, z \in X$$

with a $\lambda_f \in [0,1)$ and (2) holds. Put also

$$d(f,g) = \sup\left\{\int_\Omega \varrho(f(x,\omega), g(x,\omega))P(d\omega)\colon x \in X\right\} \quad \text{for } f, g \in \mathcal{R}_c.$$

The theorem in [2] says that if $f, g \in \mathcal{R}_c$, then

$$\left|\int_X u d\pi^f - \int_X u d\pi^g\right| \le \frac{1}{1-\min\{\lambda_f, \lambda_g\}} d(f,g) \tag{4}$$

for every non-expansive $u\colon X \to [-1,1]$. In fact the above inequality was proved there for every non-expansive and bounded $u\colon X \to \mathbb{R}$. But if $f \in \mathcal{R}_c$, then (3) holds and so every Lipschitz function mapping $X$ into a separable Banach space is Bochner integrable with respect to $\pi^f$. Therefore we can ask whether (4) holds also for such a function. The theorem reads as follows.

Theorem 1. *If $f, g \in \mathcal{R}_c$, then*

$$\left\|\int_X u d\pi^f - \int_X u d\pi^g\right\| \le \frac{1}{1-\min\{\lambda_f, \lambda_g\}} d(f,g) \tag{5}$$

*for every non-expansive $u$ mapping $X$ into a separable Banach space.*

Proof. Let $u$ be a non-expansive mapping of $X$ into a separable Banach space $Y$. To show that (5) holds we may assume that $Y$ is a real space.

Fix $y^* \in Y^*$ such that $\|y^*\| \leq 1$ and

$$\left\| \int_X u d\pi^f - \int_X u d\pi^g \right\| = y^* \left( \int_X u d\pi^f - \int_X u d\pi^g \right). \tag{6}$$

For every $k \in \mathbb{N}$ the function $\tau_k : \mathbb{R} \to \mathbb{R}$ given by $\tau_k(t) = -k$ for $t \in (-\infty, -k)$, $\tau_k(t) = t$ for $t \in [-k, k]$, $\tau_k(t) = k$ for $t \in (k, \infty)$ is non-expansive and $|\tau_k(t)| \leq |t|$ for $t \in \mathbb{R}$. Consequently, since (4) holds for every non-expansive and bounded $u \colon X \to \mathbb{R}$, for every $k \in \mathbb{N}$ we have

$$\left| \int_X \tau_k \circ y^* \circ u d\pi^f - \int_X \tau_k \circ y^* \circ u d\pi^g \right| \leq \frac{1}{1 - \min\{\lambda_f, \lambda_g\}} d(f, g) \tag{7}$$

and

$$|\tau_k(y^* u(z))| \leq \|u(z)\| \quad \text{for } z \in X \text{ and } k \in \mathbb{N}.$$

Hence, applying the Lebesgue dominated convergence theorem and passing with $k$ to the limit in (7) we get

$$\left| \int_X y^* \circ u d\pi^f - \int_X y^* \circ u d\pi^g \right| \leq \frac{1}{1 - \min\{\lambda_f, \lambda_g\}} d(f, g)$$

and (5) follows now from (6). □

The following example shows that both sides of (5) can be equal and non-zero.

Example 1. If $\Omega = \{0, 1\}, P(\{\omega\}) = 1/2$ for $\omega \in \{0, 1\}$ and $f_\alpha(x, \omega) = (x + \alpha\omega)/2$ for $x \in \mathbb{R},\ \omega \in \{0, 1\}$ and $\alpha \in (0, \infty)$, then $f_\alpha \in \mathcal{R}_c$ with $\lambda_{f_\alpha} = \frac{1}{2}$ and (see [4, Example 1])

$$\pi^{f_\alpha}(B) = \frac{1}{\alpha} \lambda_1(B \cap [0, \alpha]) \quad \text{for Borel } B \subset \mathbb{R} \text{ and } \alpha \in (0, \infty),$$

where $\lambda_1$ denotes the one-dimensional Lebesgue measure. Hence

$$\int_{\mathbb{R}} u d\pi^{f_\alpha} = \frac{1}{\alpha} \int_0^\alpha u(x) dx$$

for every $\alpha \in (0,\infty)$ and Lipschitz $u\colon \mathbb{R} \to \mathbb{R}$. In particular,

$$\left|\int_{\mathbb{R}} x\pi^{f_\alpha}(dx) - \int_{\mathbb{R}} x\pi^{f_\beta}(dx)\right| = \frac{1}{2}|\alpha-\beta| = \frac{1}{1-\min\{\lambda_{f_\alpha},\lambda_{f_\beta}\}} d(f_\alpha, f_\beta)$$

for $\alpha$, $\beta \in (0,\infty)$.

Denoting by $\|F\|_L$ the smallest Lipschitz constant for a Lipschitz function $F$ we have the following corollary concerning Lipschitz solutions $\varphi$ of (1).

Corollary 1. *Assume $F$ is a Lipschitz mapping of $X$ into a separable Banach space $Y$. If $f,g \in \mathcal{R}_c$ and*

$$\frac{\|F\|_L}{1-\min\{\lambda_f,\lambda_g\}} d(f,g) < \left\|\int_X F d\pi^g\right\|, \tag{8}$$

*then equation* (1) *has no Lipschitz solution* $\varphi\colon X \to Y$.

Proof. It follows from Theorem 1 and (8) that

$$\left\|\int_X F d\pi^f - \int_X F d\pi^g\right\| \le \frac{\|F\|_L}{1-\min\{\lambda_f,\lambda_g\}} d(f,g) < \left\|\int_X F d\pi^g\right\|,$$

whence $\int_X F d\pi^f \neq 0$ and according to [4, Theorem 2.1] equation (1) has no Lipschitz solution $\varphi\colon X \to Y$. □

The following example shows that under the assumptions of Corollary 1 equation (1) may have a continuous solution. (Cf. [1, Example 4.2].)

Example 2. Assume $p_1, p_2$ are positive reals, $p_2 < \frac{1}{9}$ and $p_1 + p_2 = 1$, reals $L_1, L_2$ satisfy

$$p_1 L_1^2 < 1, \quad p_1|L_1| + 3\sqrt{p_2(1-p_1L_1^2)} < 1, \quad |L_2| = \sqrt{(1-p_1L_1^2)/p_2},$$

and $a \in \mathbb{R}\setminus\{0\}$. Define $F\colon \mathbb{R} \to \mathbb{R}$ by

$$F(x) = -ap_2(2L_2x + a).$$

Putting $\Omega = \{1,2\}$ and $P(\{k\}) = p_k$ for $k \in \{1,2\}$ consider the $rv$-functions $f, g\colon \mathbb{R} \times \Omega \to \mathbb{R}$ given by

$$f(x,1) = L_1x, \quad f(x,2) = L_2x + a, \quad g(x,k) = L_kx \quad \text{for } k \in \{1,2\}.$$

Then $f, g \in \mathcal{R}_c$ with

$$\lambda_f = \lambda_g = p_1|L_1| + p_2|L_2| = p_1|L_1| + \sqrt{p_2(1 - p_1L_1^2)} < 1,$$

and equation (1) takes the form

$$\varphi(x) = p_1\varphi(L_1x) + p_2\varphi(L_2x + a) + F(x).$$

Since $p_1L_1^2 + p_2L_2^2 = 1$, the function $x \mapsto x^2$, $x \in \mathbb{R}$, solves it. Moreover, $g(0, \omega) = 0$ for $\omega \in \Omega$, whence also $g^n(0, \omega) = 0$ for $\omega \in \Omega^\infty$ and $n \in \mathbb{N}$. Consequently, $\pi^g$ is the Dirac measure $\delta_0$ and

$$\int_{\mathbb{R}} F d\pi^g = F(0) = -p_2a^2.$$

Finally, $\|F\|_L = 2p_2|aL_2|$, $d(f, g) = p_2|a|$, and so

$$\frac{\|F\|_L}{1 - \min\{\lambda_f, \lambda_g\}} d(f, g) = \frac{2p_2^2a^2|L_2|}{1 - \left(p_1|L_1| + \sqrt{p_2(1 - p_1L_1^2)}\right)} < p_2a^2 = \left|\int_{\mathbb{R}} F d\pi^g\right|.$$

Consider also a special case of (1), viz.

$$\varphi(x) = \frac{1}{n}\sum_{k=0}^{n-1} \varphi\left(\frac{1}{n}x + a_k\right) + F(x). \tag{9}$$

Corollary 2. *Assume $n \geq 2$ is an integer and $F$ is a Lipschitz mapping of $\mathbb{R}$ into a separable Banach space $Y$ with $\int_0^1 F(x)dx \neq 0$. If reals $a_0, \dots, a_{n-1}$ satisfy*

$$\|F\|_L \sum_{k=0}^{n-1} \left|\frac{k}{n} - a_k\right| < (n - 1)\left\|\int_0^1 F(x)dx\right\|,$$

*then equation (9) has no Lipschitz solution $\varphi\colon \mathbb{R} \to Y$.*

Proof. Put $\Omega = \{0, 1, \dots, n - 1\}, P(\{k\}) = \frac{1}{n}$ for $k \in \Omega$ and define $f, g\colon \mathbb{R} \times \Omega \to \mathbb{R}$ by

$$f(x, k) = \frac{1}{n}x + a_k, \quad g(x, k) = \frac{1}{n}x + \frac{k}{n}.$$

Clearly $f, g \in \mathcal{R}_c$ with $\lambda_f = \lambda_g = \frac{1}{n}$ and (see [4, Example 1]) $\pi^g(B) = \lambda_1(B \cap [0,1])$ for Borel $B \subset \mathbb{R}$. Moreover,

$$d(f,g) = \frac{1}{n}\sum_{k=0}^{n-1}\left|\frac{k}{n} - a_k\right|.$$

By Corollary 1 equation (9) has no Lipschitz solution $\varphi\colon \mathbb{R} \to Y$. □

2. Assuming now that $X$ is a real separable Hilbert space, $X \neq \{0\}$, $\Lambda\colon X \to X$ is linear and continuous with $\|\Lambda\| < 1$, and $\mu$ is a probability Borel measure on $X$, consider the equation

$$\varphi(x) = \hat{\mu}(x)\varphi(\Lambda x), \tag{10}$$

where $\hat{\mu}$ denotes the Fourier transform of $\mu$,

$$\hat{\mu}(x) = \int_X e^{i(x|z)}\mu(dz) \quad \text{for } x \in X.$$

Theorem 2. *If $\mu$ has a finite first moment, then there exists a probability Borel measure $\nu$ on $X$ with a finite first moment such that $\hat{\nu}$ solves* (10), *and for any continuous at zero solution $\varphi\colon X \to \mathbb{C}$ of* (10) *we have*

$$\varphi = \varphi(0)\hat{\nu};$$

*in particular, every continuous at zero solution $\varphi\colon X \to \mathbb{C}$ of* (10) *is of class $C^1$ and Lipschitz.*

Remind that a probability Borel measure $\nu$ on $X$ has a finite first moment provided the integral $\int_X \|x\|\nu(dx)$ is finite. We shall prove Theorem 2 later on, together with the next one and with the following remark.

Remark. If $\mu$ has a finite first moment and $\Lambda$ is injective, then for every $c \in \mathbb{C}$ the set of all discontinuous at zero solutions $\varphi\colon X \to \mathbb{C}$ of (10) such that $\varphi(0) = c$ and $\varphi\,|_{X\setminus\{0\}}$ is of class $C^1$ and Lipschitz has the cardinality of the continuum.

Theorem 2 implies that for every Borel and integrable with respect to $\mu$ function $\xi\colon X \to X$ the equation

$$\varphi(x) = \varphi(\Lambda x)\int_X e^{i(x|\xi(z))}\mu(dz) \tag{11}$$

has exactly one continuous at zero solution $\varphi^\xi\colon X \to \mathbb{C}$ such that $\varphi^\xi(0) = 1$, and it is of class $C^1$ and Lipschitz. Consequently, we have the operator $\xi \mapsto \varphi^\xi$, $\xi \in L^1(\mu, X)$, and a kind of its continuity gives the following theorem.

THEOREM 3. *If $\xi, \eta\colon X \to X$ are Borel and integrable with respect to $\mu$, then*

$$\left|\varphi^\xi(x) - \varphi^\eta(x)\right| \le \frac{\|x\|}{1 - \|\Lambda\|} \int_X \|\xi(z) - \eta(z)\|\mu(dz) \quad \textit{for } x \in X.$$

PROOFS. Consider the probability space $(X, \mathcal{B}, \mu)$ and, given Borel $\xi\colon X \to X$ integrable with respect to $\mu$, the rv-function $f$ on it defined by

$$f(x,\omega) = \Lambda^* x + \xi(\omega) \quad \text{for } (x,\omega) \in X \times X,$$

as well as the limit distribution $\pi^f$. Put $\pi_\xi = \pi^f$. According to [4, Theorem 3.1] $\hat{\pi_\xi}$ solves (11). Since the first moment of $\pi_\xi$ is finite, $\hat{\pi_\xi}$ is of class $C^1$ and Lipschitz.

To prove Theorem 2 put $\nu = \pi_{id_X}$ and let $\varphi\colon X \to \mathbb{C}$ be a continuous at zero solution of (10). Then

$$\varphi(x) = \varphi(\Lambda^n x) \prod_{k=0}^{n-1} \int_X e^{i(\Lambda^k x|z)} \mu(dz) \quad \text{for } n \in \mathbb{N},\ x \in X,$$

and $\lim_{n\to\infty} \Lambda^n x = 0$ for $x \in X$. Since

$$\left|\int_X e^{i(\Lambda^k x|z)} \mu(dz)\right| \le \int_X \left|e^{i(\Lambda^k x|z)}\right| \mu(dz) = 1$$

for $k \in \mathbb{N} \cup \{0\}$ and $x \in X$, it shows that if $\varphi(0) = 0$, then $\varphi = 0$, and if $\varphi(0) \neq 0$, then

$$\varphi(x) = \varphi(0) \prod_{n=0}^{\infty} \int_X e^{i(\Lambda^n x|z)} \mu(dz) \quad \text{for } x \in X.$$

Consequently, for every $c \in \mathbb{C}$ equation (10) has at most one continuous at zero solution $\varphi\colon X \to \mathbb{C}$ satisfying $\varphi(0) = c$ and by the first part of the proof $c\hat{\nu}$ is such a solution.

To get Theorem 3 it is enough to observe that since $\varphi^\xi = \hat{\pi_\xi}$, $\varphi^\eta = \hat{\pi_\eta}$ and

$$\left|e^{i(x|z_1)} - e^{i(x|z_2)}\right| \le \|x\|\|z_1 - z_2\| \quad \text{for } x, z_1, z_2 \in X,$$

by Theorem 1 for every $x \in X$ we have

$$\begin{aligned}\left|\varphi^{\xi}(x) - \varphi^{\eta}(x)\right| &= \left|\int_X e^{i(x|z)}\pi_{\xi}(dz) - \int_X e^{i(x|z)}\pi_{\eta}(dz)\right| \\ &\le \frac{\|x\|}{1-\|\Lambda\|}\int_X \|\xi(z) - \eta(z)\|\mu(dz).\end{aligned}$$

To verify the Remark given $c \in \mathbb{C}$ for every $a \in \mathbb{C}\setminus\{c\}$ define $\varphi_a \colon X \to \mathbb{C}$ by

$$\varphi_a(x) = a\hat{\nu}(x) \quad \text{for } x \in X \setminus \{0\}, \quad \varphi_a(0) = c,$$

and note that it solves (10), it is discontinuous at zero and $\varphi_a|_{X\setminus\{0\}}$ is of class $C^1$ and Lipschitz. □

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