Annales Mathematicae Silesianae **34** (2020), no. 1, 45–50
DOI: 10.2478/amsil-2020-0014

# SOME KINDS OF SPARSENESS ON THE REAL LINE AND IDEALS ON $\omega$

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*Dedicated to Professor Zygfryd Kominek on the occasion of his 75th birthday*

**Abstract.** We show that a large class of summable ideals can be defined using a certain kind of "sparseness" of subsets of the line near zero, but it is still an open question whether this gives a characterization of the whole class.

## 1. Introduction

The idea of defining some families of subsets of the set of all natural numbers using a notion of "sparseness" of subsets of the real line near zero comes from [5]. In that paper a nice connection between a classical notion of a right-hand dispersion point of a measurable subset of $\mathbb{R}$ (see [4]) and a notion of a subset of $\omega$, i.e. of the set of all positive integers, having density zero is shown. Namely, it is proven that $A$ - a subset of $\omega$ has density zero (it belongs to the ideal of asymptotic density zero sets - for definitions see [1]) if and only if zero is a right-hand dispersion point of a set $D(A) := \bigcup_{i\in A}\left[\frac{1}{i+1}, \frac{1}{i}\right]$ or, which is equivalent, if zero is a density point of the complement of $D(A)$. Therefore having a "nice" notion of "sparseness" near zero of unions of intervals

*Received: 27.11.2019. Accepted: 03.06.2020. Published online: 09.07.2020.*
(2010) Mathematics Subject Classification: 28A99.
*Key words and phrases:* summable ideals, O'Malley points.

on the real line $\mathbb{R}$ we can define families of subsets of $\omega$. Another example is given by the simpliest ideal in $\omega$ – the ideal of finite sets.

In many situations "sparseness" is defined by different kinds of densities ([4]). Our aim is to find pairs of known ideals of subsets of $\omega$ and notions of density on $\mathbb{R}$ for which an analogous relationship holds. In this paper we associate in this sense summable ideals with O'Malley points of sets.

## 2. Basic notions

One kind of ideals of subsets of $\omega$ which are used quite often are summable ideals ([1]).

Definition 1. *For any divergent series of positive numbers $\sum_{n\in N} a_n$ a family*

$$\mathcal{I}_{a_n} := \left\{ A \subset \omega : \sum_{n\in A} a_n < \infty \right\}$$

*is an ideal on $\omega$ called a summable ideal.*

Clearly, if a sequence $\{a_n\}$ is bounded below by a positive constant then $\mathcal{I}_{a_n}$ is an ideal of finite sets. We focus attention on series with $\lim_{n\to+\infty} a_n = 0$. It is worth noting that for such a series its summable ideal $\mathcal{I}_{a_n}$ is tall, i.e. for every infinite set $A \subset \omega$ there exists an infinite set $B \subset A$ such that $B$ belongs to ideal ([1]). As a basic example of a summable ideal we consider here a summable ideal for the harmonic series $\mathcal{I}_{\frac{1}{n}}$.

Of course different series can generate the same ideal. We give a useful sufficient condition for this.

Lemma 2. *Let $\sum_{n\in N} a_n$ and $\sum_{n\in N} b_n$ be divergent series. If* $\liminf_{n\to\infty} \frac{a_n}{b_n} > 0$ *and* $\limsup_{n\to\infty} \frac{a_n}{b_n} < +\infty$ *then $\mathcal{I}_{a_n} = \mathcal{I}_{b_n}$.*

Proof. Obviously all finite sets belong to both ideals. Since there exist positive constants $p$ and $q$ such that for all sufficiently large "n" we have inequalities $pb_n < a_n < qb_n$, by comparison test we get for any set $A \subset \omega$ that

$$\sum_{n\in A} a_n < \infty \Longleftrightarrow \sum_{n\in A} b_n < \infty.$$

It means $\mathcal{I}_{a_n} = \mathcal{I}_{b_n}$. □

In [3] a notion of O'Malley point of a Lebesgue measurable subset of $\mathbb{R}$ was introduced.

Definition 3. *We say that* $0$ *is a O'Malley point of a Lebesgue measurable set* $B$ *if*

$$\int_0^1 \frac{1}{t}(\chi_{B'}(t) + \chi_{B'}(-t))\, dt < \infty,$$

*where* $\chi_{B'}$ *denotes a characteristic function of the complement of the set* $B$ *(denoted by* $B'$*). The integral here is considered as an improper Riemann integral. We write* $0 \in \Phi_{OM}(B)$*.*

Being a such point means that in each small neighbourhood of this point the majority of points, in some sense, comes from the considered set, so it can be treated as a kind of "density" point.

This notion was generalized in [2] by using instead of the function $t^{-1}$ a fixed function with similar properties.

Let $\mathcal{F}$ denote the family of all continuous, nonincreasing functions $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$, where $\mathbb{R}_+ := (0, +\infty)$, such that the improper integral $\int_0^1 \phi(t)\, dt$ is divergent.

Definition 4. *We say that* $0$ *is a* $\phi$*-O'Malley point of a Lebesgue measurable set* $B$ *if*

$$\int_0^1 \phi(t)(\chi_{B'}(t) + \chi_{B'}(-t))\, dt < \infty.$$

*We write* $0 \in \Phi_{\phi OM}(B)$*.*

Lemma 5. *Let* $f, g \in \mathcal{F}$*. If* $\liminf_{x\to 0^+} \frac{f(x)}{g(x)} > 0$ *and* $\limsup_{x\to 0^+} \frac{f(x)}{g(x)} < +\infty$ *then for any Lebesgue measurable set* $B$ *we have an equivalence*

$$0 \in \Phi_{fOM}(B) \quad \Leftrightarrow \quad 0 \in \Phi_{gOM}(B).$$

Proof. It is enough to observe that there exist $h, k, l > 0$ such that for any $x \in (0, h)$, $kg(x) < f(x) < lg(x)$. Therefore,

$$\int_0^1 f(t)(\chi_{B'}(t) + \chi_{B'}(-t))\, dt < \infty \Longleftrightarrow \int_0^1 g(t)(\chi_{B'}(t) + \chi_{B'}(-t))\, dt < \infty$$

for any Lebesgue measurable set $B$. $\square$

To each set $A \subset \omega$ we assign a set $D(A) := \bigcup_{i \in A} \left[\frac{1}{i+1}, \frac{1}{i}\right]$ and we define a family of subsets of $\omega$ as follows:

$$\mathcal{I}_{\phi OM} := \{A \subset \omega : 0 \in \Phi_{\phi OM}(\mathbb{R} \setminus D(A))\}.$$

For the function $\phi(t) := t^{-1}$ we write $\mathcal{I}_{OM}$ instead of $\mathcal{I}_{\phi OM}$.

Clearly

$$A \in \mathcal{I}_{\phi OM} \iff \int_0^1 \phi(t)\chi_{D(A)}(t)\, dt < +\infty$$

and it is not difficult to prove that $\mathcal{I}_{\phi OM}$ is an ideal.

## 3. Main results

Since, as we have noticed, 0 is a $\phi$-O'Malley point of a complement of the set $D(A)$ if and only if the improper integral $\int_0^1 \phi(t)\chi_{D(A)}(t)\, dt$ is convergent, it can be also expressed as a convergence of the series $\sum_{i \in A} \int_{\frac{1}{i+1}}^{\frac{1}{i}} \phi(t)\, dt$.

An obvious consequence of this is the following.

Theorem 6. *For any $\phi \in \mathcal{F}$ there exists a sequence $\{a_n\}$ such that the series $\sum_{n \in N} a_n$ is divergent and*

$$\mathcal{I}_{\phi OM} = \mathcal{I}_{a_n}.$$

Proof. It is enough to use $a_n := \int_{\frac{1}{n+1}}^{\frac{1}{n}} \phi(t)\, dt$. It is a divergent series by properties of $\phi$. $\square$

In the case of a classic O'Malley point using the sequence suggested in the proof we get

$$\mathcal{I}_{OM} = \mathcal{I}_{\ln \frac{n+1}{n}}.$$

In fact, we can use here also a harmonic series. Since $\lim_{n \to +\infty} n \ln \frac{n+1}{n} = 1$, Lemma 2 leads to

$$\mathcal{I}_{OM} = \mathcal{I}_{\frac{1}{n}}.$$

What is interesting not only each version of O'Malley point defines by our method a certain summable ideal but many summable ideals can be described this way. We give here a convenient condition for a series to find a corresponding kind of O'Malley point.

THEOREM 7. *For any divergent series* $\sum_{n\in\omega} a_n$ *satisfying* $\frac{a_{n+1}}{a_n} \geq \frac{n}{n+2}$ *for all sufficiently large* $n \in \omega$ *there exists a function* $\phi_{a_n} \in \mathcal{F}$ *such that*

$$\mathcal{I}_{a_n} = \mathcal{I}_{\phi_{a_n} OM}.$$

PROOF. There is no loss in generality in assuming that the inequality $\frac{a_{n+1}}{a_n} \geq \frac{n}{n+2}$ holds for all $n \in \omega$. It is enough to build a function $\phi_{a_n} \in \mathcal{F}$ satisfying the condition $\int_{\frac{1}{n+1}}^{\frac{1}{n}} \phi_{a_n}(t)\,dt = a_n$.

Let $\phi_{a_n}\left(\frac{1}{2n}\right) = \phi_{a_n}\left(\frac{1}{2n-1}\right) := a_{2n-1}(2n-1)2n$, let $\phi_{a_n}$ be constant on intervals $\left[\frac{1}{2n}, \frac{1}{2n-1}\right]$, and let it be continuous, nonincreasing on intervals $\left[\frac{1}{2n+1}, \frac{1}{2n}\right]$ and such that the average value of a function $\phi_{a_n}$ on this interval is equal to $a_{2n}(2n+1)2n$ for $n \in \omega$.

This construction is possible since $\phi_{a_n}\left(\frac{1}{2n+1}\right) = a_{2n+1}(2n+1)(2n+2) \geq a_{2n}(2n+1)2n \geq a_{2n-1}(2n-1)2n = \phi_{a_n}\left(\frac{1}{2n}\right)$ and the average value of $\phi_{a_n}$ on $\left[\frac{1}{n+2}, \frac{1}{n+1}\right]$ is greater than the average value of $\phi_{a_n}$ on $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ for $n \in \omega$.

The desired condition is satisfied. □

Observe that for the harmonic series we can define a suitable function $\phi_{\frac{1}{n}}$ as follows: $\phi_{\frac{1}{n}}(x) := 2n$ for $x \in \left[\frac{1}{2n}, \frac{1}{2n-1}\right]$ and $\phi_{\frac{1}{n}}(x) := -4n(2n+1)x + 6n + 2$ for $x \in \left[\frac{1}{2n+1}, \frac{1}{2n}\right]$.

It is easy to check that Lemma 5 gives an equivalence

$$0 \in \Phi_{\phi_{\frac{1}{n}} OM}(B) \Longleftrightarrow 0 \in \Phi_{OM}(B)$$

for any Lebesgue measurable set $B$.

Observe that for a given function $\phi \in \mathcal{F}$ a sequence $\{a_n\}$, where $a_n := \int_{\frac{1}{n+1}}^{\frac{1}{n}} \phi(t)\,dt$, suggested in the proof of Theorem 6 to obtain $\mathcal{I}_{\phi OM} = \mathcal{I}_{a_n}$, satisfies the condition from Theorem 7. Hence we can find a corresponding function $\phi_{a_n} \in \mathcal{F}$, not necessarily the same as $\phi$, to have $\mathcal{I}_{a_n} = \mathcal{I}_{\phi_{a_n} OM}$.

Note that if $\frac{a_{n+1}}{a_n} < \frac{n}{n+2}$ for all sufficiently large $n \in \omega$ then for sufficiently large $n$ $a_n < \frac{2}{(n+1)n}$ and, by the comparison test, $\sum_{n\in\omega} a_n$ is convergent.

However, there are divergent series not satisfying the condition from the last theorem but still we can define their summable ideals by some kind of O'Malley points.

For example, consider $B := \{n \in \omega \setminus \{1\} : \exists_{k\in\omega} n = k^2\}$ and define $b_n := \frac{1}{n}$ for $n \in \omega \setminus B$ and $b_n := \frac{2}{n+1}$ for $n \in B$. Then $\sum_{n\in\omega} b_n$ is divergent and, since there is no pair of successive natural numbers in $B$, $\frac{b_{n+1}}{b_n} = \frac{1}{2} < \frac{n}{n+2}$ for $n \in B$. Observe that $\mathcal{I}_{b_n} = \mathcal{I}_{\frac{1}{n}}$ since $B \in \mathcal{I}_{\frac{1}{n}}$ and $\sum_{n\in B} b_n = \sum_{k>1} \frac{2}{k^2+1} < +\infty$.

Hence $\sum_{n\in\omega} b_n$ does not satisfy the condition from Theorem 7 but $\mathcal{I}_{b_n}$ can be described by O'Malley points: $\mathcal{I}_{b_n} = \mathcal{I}_{OM}$.

There is still an open question whether each summable ideal $\lim_{n\to+\infty} a_n=0$ can be described by some kind of O'Malley points.

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