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# ON ITERATION OF BIJECTIVE FUNCTIONS WITH DISCONTINUITIES

Harald Fripertinger

*Dedicated to Zygfryd Kominek on the occasion of his 75th birthday*

**Abstract.** We present three different types of bijective functions $f\colon I \to I$ on a compact interval $I$ with finitely many discontinuities where certain iterates of these functions will be continuous. All these examples are strongly related to permutations, in particular to derangements in the first case, and permutations with a certain number of successions (or small ascents) in the second case. All functions of type III form a direct product of a symmetric group with a wreath product. It will be shown that any iterative root $F\colon J \to J$ of the identity of order $k$ on a compact interval $J$ with finitely many discontinuities is conjugate to a function $f$ of type III, i.e., $F = \varphi^{-1} \circ f \circ \varphi$ where $\varphi$ is a continuous, bijective, and increasing mapping between $J$ and $[0, n]$ for some integer $n$.

## 1. Introduction

During the ISFE54 Zygfryd Kominek raised discussion about the behavior of iterates of real functions with discontinuities. *"Is it possible that the $k$-th iterate of such a function is continuous?"* During the problems and remarks

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sessions there were some remarks concerning this topic by Roman Ger, Peter Stadler and myself (cf. [6, Problem 2.5 and Problem 2.9]). Finally it turned out that only *surjective* functions are interesting.

In order to obtain nice results it will also be assumed that these functions are injective. In the present paper three different types of bijective functions defined on a compact interval with finitely many removable and/or jump discontinuities will be presented, where certain iterates of these functions will be continuous. As a matter of fact, functions of type III are generalizations of functions of type I or II. We will see that these examples of bijective functions are strongly related to permutations of finite sets. Therefore, we consider these functions also as discrete structures, and in addition to analyzing their properties we will also try to enumerate them. This way we obtain an overview on how many different types of these functions can be constructed.

## 2. Functions of type I

Let $n \geq 2$ be an integer, $I = [0, n+1]$ be the closed real interval, $f\colon I \to I$ be a bijective function with $n$ removable discontinuities in the points belonging to $n := \{1, \ldots, n\}$. From the context it should always be clear whether $n$ denotes a positive integer or a set of positive integers. Then $f$ is a function of type I, iff $f(x) = x$ for $x \in I \setminus n$. Since $f$ is bijective, for all $j \in n$ there exists some $i \in n$ such that $i \neq j$ and $f(i) = j$. Thus $f$ restricted to $n$ is a permutation $\pi = \pi_f \in S_n$, the symmetric group of $n$. It is free of fixed points, thus it is a derangement. We call it the derangement obtained from $f$. Conversely, to each derangement there corresponds exactly one function of type I.

Some relations between $f$ and $\pi$ are collected in

Lemma 2.1. *Let $f$ be a function of type I and $\pi$ the derangement obtained from $f$. Then*
1. *$f^k$ is continuous, iff $f^k = \mathrm{id}$.*
2. *$f^k(i) = \pi^k(i)$, $i \in n$, $k \in \mathbb{N}$.*
3. *$f^k$ is continuous, iff $\pi^k = \mathrm{id}$, iff the order $\mathrm{ord}(\pi)$ of $\pi$ is a divisor of $k$.*

There are various formulae known concerning the enumeration of derangements. Let $d_n$ be the number of derangements in $S_n$, then it is also the number of functions of type I having $n$ discontinuities. E.g., following [2, page 182 and 180] there is a recursive formula

$$d_0 = 1,\ d_1 = 0,\ d_n = (n-1)(d_{n-1} + d_{n-2}), \quad n \geq 2,$$

Table 1. Numerical values of $d_n$

| $n$ | $d_n$ | $d_n/n!$ |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 2 | 1 | 0.5 |
| 3 | 2 | 0.333333 |
| 4 | 9 | 0.375 |
| 5 | 44 | 0.366666 |
| 6 | 265 | 0.368055 |
| 7 | 1 854 | 0.367857 |
| 8 | 14 833 | 0.367881 |
| 9 | 133 496 | 0.367879 |
| 10 | 1 334 961 | 0.367879 |
| 11 | 14 684 570 | 0.367879 |
| 12 | 12 176 214 841 | 0.367879 |

and a formula based on the inclusion-exclusion principle

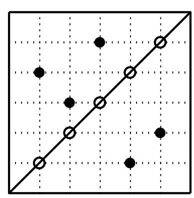
$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}, \quad n \geq 0.$$

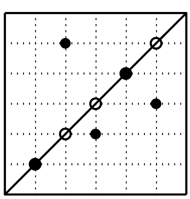
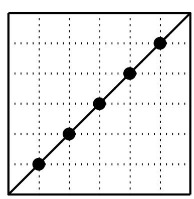
These numbers $d_n$ can be found as A000166 in the On-Line Encyclopedia of Integer Sequences (OEIS).

Some numerical values are presented in Table 1. Approximately 37% of all permutations are derangements. Actually, it is easy to prove that

$$\lim_{n\to\infty} \frac{d_n}{n!} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \mathrm{e}^{-1} \approx 0.367879.$$

For example consider the derangement

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix} = (1,4)(2,3,5)$$ and the function $f =$ 

then $f^2 =$  , ..., until $f^6 =$  .

If $\pi \in S_n$ decomposes into $a_i$ disjoint cycles of length $i$, for $i \in n$, we call $a = (a_1, \ldots, a_n)$ the cycle type of $\pi$. The order of $\pi$ depends only on the cycle type of $\pi$ since it is the least common multiple of all cycle lengths occurring in the decomposition of $\pi$. We can express $\operatorname{ord}(\pi)$ as the $\operatorname{lcm}\{i \in n \mid a_i \neq 0\}$ where $a = (a_1, \ldots, a_n)$ is the cycle type of $\pi$. In general, a sequence of $n$ non-negative integer $(a_1, \ldots, a_n)$ is a cycle type of a permutation $\pi$ in $S_n$, iff

$$\sum_{i \in n} i a_i = n.$$

Such sequences are sometimes called cycle types of $n$. From these considerations and the example above it is clear that the following lemma holds true.

LEMMA 2.2. *Consider a positive integer $k$. Let $f$ be a function of type I and $\pi$ the derangement obtained from $f$. The discontinuities of $f$ corresponding to any cycle of length $i$ of $\pi$ disappear in the $k$-th iterate $f^k$ of $f$, iff $i \mid k$. Therefore, the number of discontinuities of $f^k$ is*

$$n - \sum_{i \mid k} i a_i = \sum_{i \nmid k} i a_i.$$

We call the least positive integer $k$ such that $f^k$ is continuous the order of $f$ written as $\operatorname{ord}(f)$. Thus $\operatorname{ord}(f) = \operatorname{ord}(\pi_f)$ where $\pi_f$ is associated with $f$.

What is the maximum order of a function of type I with $n$ discontinuities? The maximum possible order of permutations in $S_n$ is given by the Landau function $g(n) := \max\{\operatorname{ord}(\pi) \mid \pi \in S_n\}$. It satisfies $g(n) \leq g(n+1)$ for all $n$. Furthermore, let $\tilde{g}(n) := \max\{\operatorname{ord}(\pi) \mid \pi \in S_n \text{ is a derangement}\}$ be the maximum order of a derangement of $n$. It satisfies $g(n-2) \leq \tilde{g}(n) \leq g(n)$ for all $n \geq 4$. Whenever $g(n-1) < g(n)$, then necessarily $\tilde{g}(n) = g(n)$. Obviously, $\tilde{g}(n)$ is the maximum order of a function of type I with $n$ discontinuities.

For example we list some values of $g(n)$ and $\tilde{g}(n)$ in Table 2. The numbers $g(n)$ and $\tilde{g}(n)$ can be fund in the OEIS as A000793 and A123131 respectively.

In order to get an overview over all functions of type I with $n$ discontinuities it is enough to study functions where the associated derangements belong to different conjugacy classes in $S_n$. The different conjugacy classes in $S_n$ correspond to the different cycle types of $n$. Consider two functions $f_i$, $i = 1, 2$, of type I where the associated derangements $\pi_i$, $i = 1, 2$, are conjugate in $S_n$. Then the $\pi_i$ have the same cycle types and according to Lemma 2.2 the number of discontinuities of $f_1^k$ and $f_2^k$, $k \geq 1$, coincide. Therefore functions of type I, the associated derangements are conjugate in $S_n$, show similar behavior. From Lemma 2.2 we deduce that the number of discontinuities of $f^k$ can be described in terms of the cycle type of $\pi_f$. Thus the behavior of $f$ depends only on the conjugacy class of $\pi_f$.

Table 2. Values of $g(n)$ and $\tilde{g}(n)$

| $n$ | $g(n)$ | $\tilde{g}(n)$ |
|---|---|---|
| 2 | 2 | 2 |
| 3 | 3 | 3 |
| 4 | 4 | 4 |
| 5 | 6 | 6 |
| 6 | 6 | 6 |
| 7 | 12 | 12 |
| 8 | 15 | 15 |
| 9 | 20 | 20 |
| 10 | 30 | 30 |
| 11 | 30 | 30 |
| 12 | 60 | 60 |
| 13 | 60 | 42 |
| 102 | 446 185 740 | 446 185 740 |
| 103 | 446 185 740 | 314 954 640 |
| 104 | 446 185 740 | 446 185 740 |

Table 3. Values of $p_n$ and $\tilde{p}_n$

| $n$ | $d_n$ | $\tilde{p}_n$ | $p_n$ |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 |
| 2 | 1 | 1 | 2 |
| 3 | 2 | 1 | 3 |
| 4 | 9 | 2 | 5 |
| 5 | 44 | 2 | 7 |
| 6 | 265 | 4 | 11 |
| 7 | 1 854 | 4 | 15 |
| 8 | 14 833 | 7 | 22 |
| 9 | 133 496 | 8 | 30 |
| 10 | 1 334 961 | 12 | 42 |
| 11 | 14 684 570 | 14 | 56 |
| 12 | 12 176 214 841 | 21 | 77 |

Cycle types of derangements in $S_n$ correspond to partitions of the integer $n$ having no parts of size 1. A partition of $n$ is a sequence $\alpha = (\alpha_1, \ldots, \alpha_h)$ of integers $\alpha_1 \geq \ldots \geq \alpha_h \geq 1$ with $\alpha_1 + \cdots + \alpha_h = n$.

E.g., the partitions of $n = 8$ with no parts of size 1 are $8 = 6+2 = 5+3 = 4+4 = 4+2+2 = 3+3+2 = 2+2+2+2$. These are 7 different types.

Given a positive integer $n$ the set of orders of functions of type I having $n$ discontinuities is finite. It is a subset of $\{2, \ldots, \tilde{g}(n)\}$. E.g., for $n = 1$ it is the empty set, for $n = 8$ it is $\{8, 6, 15, 4, 2\}$. There are no functions of type I with $n$ removable discontinuities such that $\operatorname{ord}(f) > \tilde{g}(n)$. E.g., there are no functions with 2 removable discontinuities such that $f^3$ is continuous.

Considering just conjugacy classes reduces the combinatorial complexity. The numbers $p_n$ of all partitions of $n$, and $\tilde{p}_n$, the partition numbers without 1, can be found in the OEIS as A002865 and A000041, see Table 3 for some values.

There are no functions of type I with exactly one removable discontinuity. The iterates $f^k$ have at most as many discontinuities as $f$.

## 3. Functions of type II

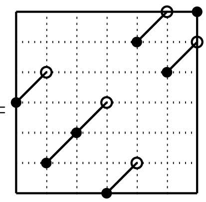
Now we consider bijective functions $f\colon [0,n] \to [0,n]$, $n \geq 2$, such that for each $i \in n$ there exists one $j \in n$ such that

$$f(t) = t - (i-1) + (j-1) = t - i + j, \quad t \in [i-1, i),$$

and $f(n) = n$. Therefore, $f$ is continuous in each interval $I_i := [i-1, i)$ (in $i-1$ continuous from the right), $i \in n$. Discontinuities can appear only in the positions $1, \ldots, n$.

Since $f$ is bijective, it defines a permutation $\pi \in S_n$ given by

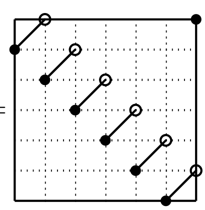
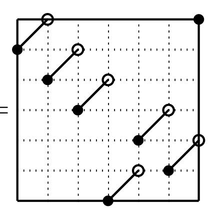
$$\pi(i) = j \iff f(I_i) = I_j.$$

For example $f =$  defines $\pi = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 4\,2\,3\,1\,6\,5 \end{pmatrix}$.

Lemma 3.1. *If $f$ is a function of type II and $\pi \in S_n$ is obtained from $f$, then:*

1. $f(t) = \pi(i) + t - i$, $t \in I_i$, $i \in n$.
2. *$f$ is continuous in $i$, iff* $\pi(i+1) = \pi(i) + 1$, $1 \le i < n$.
3. *$f$ is continuous in $n$, iff* $\pi(n) = n$.
4. *$f^k$ is continuous, iff* $f^k = \mathrm{id}$.
5. $f^k(t) = \pi^k(i) + t - i$, $t \in I_i$, $i \in n$.
6. *$f^k$ is continuous, iff* $\pi^k = \mathrm{id}$.

An element $i \in n-1$ is called a succession (or a small ascent) of $\pi$, iff $\pi(i+1) = \pi(i) + 1$. The $f$ above has exactly one succession namely 2. The number of discontinuities of $f$ among $\{1, \ldots, n-1\}$ is the number of $i$-s which are not successions of $\pi$. A permutation $\pi$ without successions satisfying $\pi(n) < n$ defines a function with $n$ discontinuities. These are the functions of type II having the maximum number of discontinuities. E.g., $\pi = (1, n)(2, n-1) \ldots$ or $\sigma = (1, n, 2, n-1, \ldots)$ lead to $n$ discontinuities of $f$, $n \ge 2$.

$\pi =$  $\sigma =$  for $n = 6$.

Hence, we try to enumerate permutations without successions. Let $a_n$ be the number of permutations in $S_n$ having no successions and $b_n$ the number of permutations in $S_n$ having exactly one succession, then it is easy to prove that

$$a_1 = 1, \quad a_2 = 1, \quad b_1 = 0, \quad b_2 = 1,$$

and

$$a_n = (n-1)a_{n-1} + b_{n-1}, \quad n \geq 2,$$

$$b_n = (n-1)a_{n-1}, \quad n \geq 2,$$

thus

$$a_n = (n-1)a_{n-1} + (n-2)a_{n-2} = b_n + b_{n-1}, \quad n \geq 3,$$

$$b_n = (n-1)(b_{n-1} + b_{n-2}), \quad n \geq 3.$$

Consequently, $b_n = d_n$, $n \geq 1$, the number of derangements of $n$ objects.

For $a_n$ see A000255 in the OEIS.

Let $c_n$ be the number of permutations $\pi$ in $S_n$ having no successions and satisfying $\pi(n) = n$. Then

$$c_n = a_{n-1} - c_{n-1}, \quad n \geq 2.$$

Therefore $a_{n-1} = c_n + c_{n-1}$, $n \geq 2$, and since $c_2 = b_1$ and $c_3 = b_2$ we deduce $c_n = b_{n-1}$, $n \geq 2$.

The number of permutations $\pi$ in $S_n$ having no successions and satisfying $\pi(n) < n$ is therefore

$$a_n - c_n = a_n - b_{n-1} = b_n = (n-1)a_{n-1}, \quad n \geq 2.$$

Corollary 3.2. *The number of functions $f\colon [0,n] \to [0,n]$, $n \geq 2$, of type II having $n$ discontinuities (in the points $1, \dots, n$) is $(n-1)a_{n-1} = b_n = d_n$ the number of derangements.*

It is also possible to enumerate permutations with prescribed number of successions (cf. [1, section 5.4]). Let $a_{n,k}$ be the number of permutations $\pi \in S_n$ having exactly $k$ successions, $0 \leq k < n$, then $a_{n,0} = a_n$ and $a_{n,1} = b_n$ and

$$a_{n,k} = \frac{(n-1)!}{k!} \sum_{j=0}^{n-k-1} (-1)^j \frac{n-k-j}{j!} = \binom{n-1}{k} a_{n-k}, \quad 0 \leq k \leq n-1.$$

Therefore,

$$n! = \sum_{k=0}^{n-1} a_{n,k} = \sum_{k=0}^{n-1} \binom{n-1}{k} a_{n-k}, \quad n \geq 1,$$

Table 4. Values of $a_{n,k}$

| $n$ | $a_{n,0}$ | $a_{n,1}$ | $a_{n,2}$ | $a_{n,3}$ | $a_{n,4}$ | $a_{n,5}$ | $a_{n,6}$ | $a_{n,7}$ | $a_{n,8}$ | $a_{n,9}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 3 | 2 | 1 | | | | | | | |
| 4 | 11 | 9 | 3 | 1 | | | | | | |
| 5 | 53 | 44 | 18 | 4 | 1 | | | | | |
| 6 | 309 | 265 | 110 | 30 | 5 | 1 | | | | |
| 7 | 2119 | 1854 | 795 | 220 | 45 | 6 | 1 | | | |
| 8 | 16687 | 14833 | 6489 | 1855 | 385 | 63 | 7 | 1 | | |
| 9 | 148329 | 133496 | 59332 | 17304 | 3710 | 616 | 84 | 8 | 1 | |
| 10 | 1468457 | 1334961 | 600732 | 177996 | 38934 | 6678 | 924 | 108 | 9 | 1 |

and by binomial inversion we obtain

$$a_n = \sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{n-1}{k} (k+1)!, \quad n \geq 1.$$

Some values of $a_{n,k}$ are collected in Table 4. See also A123513 in the OEIS.

E.g., there is only one permutation $\pi \in S_n$ having $n-1$ successions, namely $\pi = \mathrm{id}$, thus $a_{n,n-1} = 1$. Since $a_{n,n-2} = n-1$, the permutations $\pi^j$, $j \in n-1$, $n \geq 2$, $\pi = (1, \ldots, n)$, turn out to be the only permutations in $S_n$ having exactly $n-2$ successions.

In what follows we construct functions of type II with certain properties. Consider as above a cycle $\pi = (1, 2, \ldots, k) = \begin{pmatrix} 1 & 2 & \ldots & k-1 & k \\ 2 & 3 & \ldots & k & 1 \end{pmatrix}$ of length $k \geq 2$ with $k-2$ successions. Then for $1 \leq j < k$

$$\pi^j = \begin{pmatrix} 1 & 2 & \ldots & k-j & k-j+1 & \ldots & k \\ j+1 & j+2 & \ldots & k & 1 & \ldots & j \end{pmatrix}$$

has

$$\begin{cases} k-2 \text{ successions} & \text{if } k \not| \, j, \\ k-1 \text{ successions} & \text{if } k \mid j. \end{cases}$$

Let $f_{1,k}\colon [0,k] \to [0,k]$ be the function of type II determined by this $\pi$, then the iterates $f_{1,k}^j$ have

$$\begin{cases} 2 \text{ discontinuities} & \text{if } k \not| \, j, \\ 0 \text{ discontinuities} & \text{if } k \mid j, \end{cases}$$

where the two discontinuities of $f^j_{1,k}$ occur in $k-(j \bmod k)$ and $k$. By $j \bmod k$ we indicate the unique element $i \in \{0,\dots,k-1\}$ satisfying $i \equiv j \bmod k$.

The iterates $f^j_{s,k}$ of the functions $f_{s,k}\colon [0,sk] \to [0,sk]$ corresponding to the product of $s$ cycles of length $k$

$$(1,2,\dots,k)(k+1,k+2,\dots,2k)\cdots((s-1)k+1,\dots,sk)$$

have

$$\begin{cases} 2s \text{ discontinuities} & \text{if } k \nmid j,\\ 0 \text{ discontinuities} & \text{if } k \mid j, \end{cases}$$

where the $2s$ discontinuities of $f^j_{s,k}$ occur in $rk-(j \bmod k)$ and $rk$ for $1 \le r \le s$.

Similarly we consider the iterates $g^j_{s,k}$ of the functions $g_{s,k}\colon [0,sk+1] \to [0,sk+1]$ corresponding to the product of $s$ cycles and one fixed point

$$(1)(2,3,\dots,k+1)(k+2,k+3,\dots,2k+1)\cdots((s-1)k+2,\dots,sk+1).$$

They have

$$\begin{cases} 2s+1 \text{ discontinuities} & \text{if } k \nmid j,\\ 0 \text{ discontinuities} & \text{if } k \mid j, \end{cases}$$

where the $2s+1$ discontinuities of $g^j_{s,k}$ occur in $1$ and $rk+1-(j \bmod k)$ and $rk+1$ for $1 \le r \le s$.

Theorem 3.3. *For any $n \ge 2$ and $k \ge 2$ the iterates $f^j_{n/2,k}$ (for even $n$) or $g^j_{(n-1)/2,k}$ (for odd $n$) of the functions $f_{n/2,k}$, or $g_{(n-1)/2,k}$ have*

$$\begin{cases} n \textit{ discontinuities} & \textit{if } k \nmid j,\\ 0 \textit{ discontinuities} & \textit{if } k \mid j. \end{cases}$$

Now we define a concatenation of functions of type II. Given two functions $f\colon [0,n] \to [0,n]$ and $g\colon [0,m] \to [0,m]$ of type II, then $f \bullet g\colon [0,n+m] \to [0,n+m]$ is defined by

$$(f \bullet g)(t) = \begin{cases} f(t) & \text{if } t \in [0,n),\\ n+g(t-n) & \text{if } t \in [n,n+m]. \end{cases}$$

Since $f$ and $g$ are bijective and $f(n) = n$, the concatenation $f \bullet g$ is bijective, and $f \bullet g$ is of type II. If, furthermore, $f$ is continuous in $n$ and $g(0) = 0$, then $f \bullet g$ is continuous in $n$ since $g$ is continuous from the right side in 0. The function $f \bullet g$ is not continuous in $n$, iff $f$ is not continuous in $n$ or $g(0) \neq 0$.

THEOREM 3.4. *Consider $f$ and $g$ of type II having $r$ respectively $s$ discontinuities. Then the number of discontinuities of $f \bullet g$ is*

$$\begin{cases} r+s+1 & \text{if } f \text{ is continuous in } n \text{ and } g(0) \neq 0, \\ r+s & \text{else.} \end{cases}$$

REMARK 3.5.

1. Actually $f_{s,k} = f_{s-1,k} \bullet f_{1,k}$ and $g_{s,k} = g_{s-1,k} \bullet f_{1,k}$ for $s > 1$.
2. Even though $f_{1,k}(0) \neq 0$ the function $f_{s,k}$ has $2s$ (and $g_{s,k}$ has $2s+1$) discontinuities since $f_{s-1,k}$ and $g_{s-1,k}$ are not continuous at the end of their domains.
3. The functions $g_{s,k}$ satisfy $g_{s,k}(0) = 0$, thus the $j$-th iterate of the concatenation of $g_{s_1,k_1} \bullet \ldots \bullet g_{s_r,k_r}$ has
$$\sum_{i=1,\ k_i \nmid j}^{r} (2s_i + 1)$$
discontinuities. Concatenation of $g_{s,k}$ does not introduce new discontinuities.
4. Concatenation of the functions $f_{s,k}$ is more complicated, since $f_{s,k}(0) = 2 \neq 0$, and $f_{s,k}^{j}(0) = 0$ whenever $j$ is a multiple of $k$.

E.g., the numbers of discontinuities of $(f_{1,2} \bullet f_{1,3})^j$ and $(f_{1,3} \bullet f_{1,2})^j$ are

| $j$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| number of discontinuities of $(f_{1,2} \bullet f_{1,3})^j$ | 4 | 3 | 2 | 3 | 4 | 0 |
| number of discontinuities of $(f_{1,3} \bullet f_{1,2})^j$ | 4 | 2 | 3 | 2 | 4 | 0 |

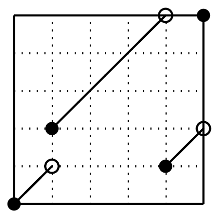
In the next examples we restrict ourselves to functions which are continuous in 0, the left end of their domains. We already know the functions $g_{s,k}$ with this property whose iterates have either $2s+1$ or no discontinuities. Hence we are looking for functions whose iterates either have an even number $\ell > 0$ or 0 discontinuities.

If $\ell$ is even, $\ell \geq 6$, then $\ell = (\ell - 3) + 3$, and the iterates $f^j$ of $f = g_{(\ell-4)/2,k} \bullet g_{1,k}$ have

$$\begin{cases} \ell \text{ discontinuities} & \text{if } k \not| j, \\ 0 \text{ discontinuities} & \text{if } k \mid j. \end{cases}$$

In other words for all $j \geq 0$ the $j$-th iterate of the function $g_{(\ell-4)/2,k} \bullet g_{1,k}$ has the same number of discontinuities as the $j$-th iterate of $f_{\ell/2,k}$, but $(g_{(\ell-4)/2,k} \bullet g_{1,k})(0) = 0$.

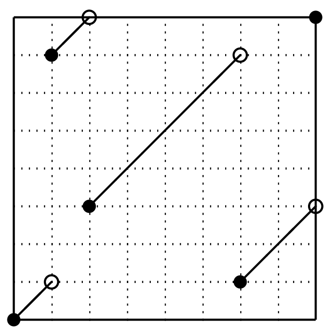
What about functions with 2 or 4 discontinuities among $1, \dots, n-1$? It is easy to see that there is no function $f\colon [0,n] \to [0,n]$ of type II such that $f(0) = 0$ having exactly two discontinuities. These functions have at least three discontinuities. An example for $n = 5$ is given by



Concerning permutations with exactly four discontinuities we obtain: The permutation $\pi = (1)(2,4)(3)$ has order 2 and yields 4 discontinuities.

There is no permutation of order 3 which yields 4 discontinuities.

A family $(\pi_k)_{k\geq 2}$ of permutations of order $\operatorname{ord}(\pi_k) = 2k+1$, $k \geq 2$, which yields functions $f$ having exactly 4 discontinuities is given by $\pi_2 = (1)(2,6,3,4,5)$, $\pi_3 = (1)(2,8,3,4,5,6,7)$, and $\pi_k = (1)(2,2k+2,3,4,\dots,2k+1)$.

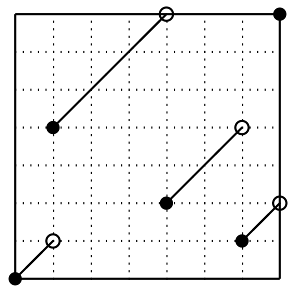


A new phenomenon occurs with these functions. There exist iterates of $f$ having more discontinuities than $f$ itself. The number of discontinuities of the iterates $f^j$, $f$ corresponding to $\pi_3$, are:

| $j$ | 1 | 2 | 3,4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| number of discontinuities of $f^j$ | 4 | 6 | 7 | 6 | 4 | 0 |

Probably these results can be generalized for arbitrary $k$.

A family $(\pi_k)_{k\geq 2}$ of permutations of order $\mathrm{ord}(\pi_k) = 2k+1$, $k \geq 2$, which yields functions $f$ having exactly 4 discontinuities is given by $\pi_2 = (1)(2,4,3,5)$, $\pi_3 = (1)(2,5,3,6,4,7)$, and $\pi_k = (1)(2, k+2, 3, k+3, \ldots, k+1, 2k+1)$.



The number of discontinuities of the iterates $f^j$, $f$ corresponding to $\pi_3$, are:

| $j$ | 1 | 2,3,4 | 5 | 6 |
|---|---|---|---|---|
| number of discontinuities of $f^j$ | 4 | 5 | 4 | 0 |

Probably these results can be generalized for arbitrary $k$.

For $\ell \in \{3,5,6,7,\ldots\}$ and $k \geq 2$ let

$$h_{\ell,k} := \begin{cases} g_{(\ell-1)/2,k} & \text{if } \ell \equiv 1 \bmod 2, \\ g_{(\ell-4)/2,k} \bullet g_{1,k} & \text{if } \ell \equiv 0 \bmod 2. \end{cases}$$

From Theorem 3.4 we deduce

THEOREM 3.6. *For $\ell \in \{3,5,6,7,\ldots\}$ and $k \geq 2$ the functions $h_{\ell,k}\colon [0,n] \to [0,n]$ are of type II. They satisfy $h_{\ell,k}(0) = 0$ and their iterates $h^j_{\ell,k}$ have*

$$\begin{cases} \ell \textit{ discontinuities} & \textit{if } k \not| \, j, \\ 0 \textit{ discontinuities} & \textit{if } k \mid j. \end{cases}$$

*Then the $j$-th iterate of the concatenation $h_{\ell_1,k_1} \bullet \ldots \bullet h_{\ell_r,k_r}$, $\ell_i \in \{3,5,6,7,\ldots\}$, $k_i \geq 2$, $1 \leq i \leq r$, has exactly*

$$\sum_{i=1,\ k_i \not|\, j}^{r} \ell_i \quad \textit{discontinuities.}$$

There are no functions $f$ of type II with exactly one discontinuity, or with exactly two discontinuities satisfying $f(0) = 0$.

As a generalization of functions of type I and type II we introduce

## 4. Functions of type III

A bijective function $f\colon [0, n] \to [0, n]$ is a function of type III, iff $f$ permutes the integers $\{0, 1, \ldots, n\}$, and for each $i \in n$ there exists exactly one $j \in n$ such that either

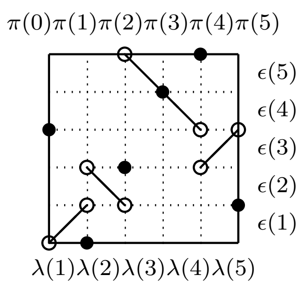
$$f(t) = t - (i - 1) + (j - 1) = t - i + j, \quad t \in (i - 1, i),$$

or

$$f(t) = j - (t - (i - 1)) = j + i - 1 - t, \quad t \in (i - 1, i).$$

This means that $f$ permutes the open intervals $I_i = (i - 1, i)$, $i \in n$. In the first case $f$ is strictly increasing on $I_i$, in the second case strictly decreasing on $I_i$.

The restriction of $f$ to $\{0, \ldots, n\}$ defines a permutation $\pi$. Let $\lambda \in S_n$ be the induced permutation $f(I_i) = I_{\lambda(i)}$, $i \in n$. Moreover, define $\varepsilon\colon n \to \{\pm 1\}$ by $\epsilon(i) = 1$, iff $f$ is increasing on $I_{\lambda^{-1}(i)}$. E.g.,




| | | |
|---|---|---|
| $\pi(0) = 3$ | | |
| $\pi(1) = 0$ | $\lambda(1) = 1$ | $\epsilon(1) = 1$ |
| $\pi(2) = 2$ | $\lambda(2) = 2$ | $\epsilon(2) = -1$ |
| $\pi(3) = 4$ | $\lambda(3) = 5$ | $\epsilon(3) = 1$ |
| $\pi(4) = 5$ | $\lambda(4) = 4$ | $\epsilon(4) = -1$ |
| $\pi(5) = 1$ | $\lambda(5) = 3$ | $\epsilon(5) = -1$ |

A function $f$ of type III is uniquely determined by $(\pi, (\epsilon, \lambda))$, $\pi \in S_{n+1}$, $\epsilon \in \{\pm 1\}^n$, $\lambda \in S_n$, since

$$f(t) = \lambda(i) - \frac{1}{2} + \epsilon(\lambda(i)) \left( t - i + \frac{1}{2} \right), \quad t \in I_i,\ i \in n,$$

and $f(j) = \pi(j)$, $j \in \{0, \ldots, n\}$. The value $\epsilon(\lambda(i))$ indicates whether $f$ is increasing or decreasing on the interval $I_i$, $i \in n$. We indicate this by $f \leftrightarrow (\pi, (\epsilon, \lambda))$.

The function $f$ is continuous in $i \in n-1$, iff either $\epsilon(\lambda(i)) = \epsilon(\lambda(i+1)) = 1$, $\lambda(i+1) = \lambda(i)+1$, and $\pi(i) = \lambda(i)$, or $\epsilon(\lambda(i)) = \epsilon(\lambda(i+1)) = -1$, $\lambda(i+1) = \lambda(i)-1$, and $\pi(i) = \lambda(i+1)$.

$f$ is continuous in 0, iff either $\epsilon(\lambda(1)) = 1$ and $\pi(0) = \lambda(1)-1$ or $\epsilon(\lambda(1)) = -1$ and $\pi(0) = \lambda(1)$. In a similar way the continuity of $f$ in $n$ can be described. There are two possibilities that the $k$-th iterate of a function $f$ of type III is continuous, either $f^k = \mathrm{id}$ or $f^k = n - \mathrm{id}$.

Now we show that the pairs $(\epsilon, \lambda)$ are elements of a wreath product. This is a particular form of a semidirect product (cf. [3, section 4.1] or [4, p. 37]).

Theorem 4.1 (Structure Theorem). *Consider two functions of type III, $f \leftrightarrow (\pi, (\epsilon, \lambda))$ and $f' \leftrightarrow (\pi', (\epsilon', \lambda'))$. Then their composition yields*

$$f \circ f' \leftrightarrow (\pi \circ \pi', (\epsilon\epsilon'_\lambda, \lambda \circ \lambda'))$$

*where*

$$\epsilon\epsilon'_\lambda(i) := \epsilon(i)\epsilon'(\lambda^{-1}(i)), \quad i \in n.$$

*Thus the set of all functions of type III is the direct product*

$$S_{n+1} \times (\{\pm 1\} \wr S_n)$$

*where the factor on the right side is a wreath product*

$$\{\pm 1\} \wr S_n = \{(\epsilon, \lambda) \mid \epsilon \in \{\pm 1\}^n,\ \lambda \in S_n\}$$

*of order $n! \cdot 2^n$ with $(\epsilon, \lambda)(\epsilon', \lambda') = (\epsilon\epsilon'_\lambda, \lambda \circ \lambda')$.*

Consequently, the number of functions of type III on $[0, n]$ is $n!(n+1)!2^n$, see Table 5. Functions of type I or type II are particular cases of these functions.

With each cycle of $\lambda = \prod_\nu (j_\nu, \lambda(j_\nu), \ldots, \lambda^{l_\nu - 1}(j_\nu))$ we associate the $\nu$-th cycle product $h_\nu(\epsilon, \lambda) = \epsilon(j_\nu)\epsilon(\lambda^{-1}(j_\nu)) \cdots \epsilon(\lambda^{-l_\nu+1}(j_\nu)) = \epsilon\epsilon_\lambda \cdots \epsilon_{\lambda^{l_\nu - 1}}(j_\nu)$. This value indicates the direction of $f^{l_\nu}$ on the intervals $I_j$ for $j \in \{j_\nu, \lambda(j_\nu), \ldots, \lambda^{l_\nu-1}(j_\nu)\}$. Then, $f^k = \mathrm{id}$, iff $(\pi^k, (\epsilon, \lambda)^k) = (\mathrm{id}, (1, \mathrm{id}))$, iff $\pi^k = \mathrm{id}$, $\lambda^k = \mathrm{id}$, (thus $l_\nu \mid k$ for all $\nu$) and $h_\nu^{k/l_\nu}(\epsilon, \lambda) = 1$ for all $\nu$. Thus $k$ is a multiple of $\mathrm{ord}(\pi)$ in $S_{n+1}$ and of $\mathrm{ord}(\epsilon, \lambda)$ in $\{\pm 1\} \wr S_n$. The latter is either $\mathrm{ord}(\lambda)$ or $2\mathrm{ord}(\lambda)$.

The smallest positive $k$ with these properties is the order of $f$.

Table 5. Number of functions of type III on $[0, n]$

| $n$ | $n!(n+1)!2^n$ |
|---|---|
| 0 | 1 |
| 1 | 4 |
| 2 | 48 |
| 3 | 1152 |
| 4 | 46080 |
| 5 | 2764800 |
| 6 | 232243200 |
| 7 | 26011238400 |
| 8 | 3745618329600 |
| 9 | 674211299328000 |
| 10 | 148326485852160000 |

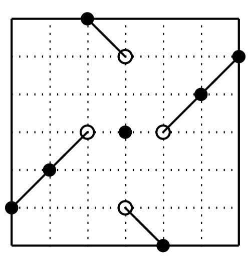
Corollary 4.2. *Let $f \leftrightarrow (\pi, (\epsilon, \lambda))$ be a function of type III, then*

$$\operatorname{ord}(f) = \operatorname{lcm}(\operatorname{ord}(\pi), \operatorname{ord}(\epsilon, \lambda)).$$

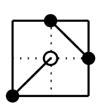
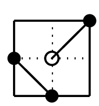
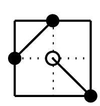
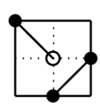
In the situation of functions of type III it is possible to find decreasing continuous iterates. They must be of the form $n - \mathrm{id}$. Let $f \leftrightarrow (\pi, (\epsilon, \lambda))$ be a function of type III, then $(\pi^k, (\lambda^k, \tilde{\epsilon})) \leftrightarrow f^k = n - \mathrm{id}$, iff $\pi^k = (0, n)(1, n-1)\ldots$, $\lambda^k = (1, n)(2, n-1)\ldots$, and $\tilde{\epsilon} = -1$. Thus $\pi$ and $\lambda$ are roots of order $k$ of permutations of cycle type

$$\begin{cases} (0, \frac{n+1}{2}) \text{ and } (1, \frac{n-1}{2}) \text{ resp.} & \text{if } n+1 \equiv 0 \bmod 2, \\ (1, \frac{n}{2}) \text{ and } (0, \frac{n}{2}) \text{ resp.} & \text{if } n+1 \equiv 1 \bmod 2. \end{cases}$$

They can be enumerated and also constructed.

E.g., $f =$  satisfies $f^3 = 6 - \mathrm{id}$.

There exist also functions of type III with exactly one discontinuity. We distinguish four different forms:

 or  or  or  .

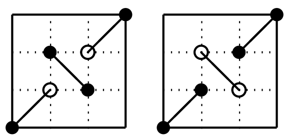
Actually the discontinuity need not be exactly in 1, it can be in any position $n_1$ so that $0 \leq n_1 < n$. In other words the interval $[0, n]$ can be partitioned into two intervals $[0, n_1]$ and $[n_1, n]$ such that $f$ is strictly monotonic on both intervals. Let $n_2 = n - n_1$.

A function of the *first form* is the concatenation of $\mathrm{id}_{n_1}$ and $n_2 - \mathrm{id}_{n_2}$ both of which are continuous, and the discontinuity disappears with the second iteration.

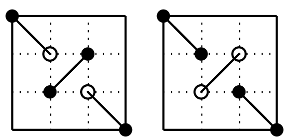
Also the discontinuity of functions of the *second form* disappears with the second iteration.

The behaviour of functions of the *third* and *forth form* can be studied.

There exist functions of type III with exactly two discontinuities in the interior of the interval. They can be constructed from the four different forms of functions with exactly one discontinuity. E.g., the functions



have exactly two discontinuities in the interior of the interval and are of order 2. They correspond to the first and second form. The functions



have exactly two discontinuities in the interior of the interval and are of order 2. They correspond to the third and fourth form. These examples again can be generalized by partitioning the interval $[0, n]$ into 3 parts $[0, n_1]$, $[n_1, n_2]$, $[n_2, n]$ with $0 < n_1 < n_2 < n \in \mathbb{N}$.

Finally we study some relations between functions of type III and iterative roots of the identity. Let $k$ be the order of a function $f$ of type III. Then $f^k = \mathrm{id}$, and $f$ is an iterative root of the identity. By applying a continuous, bijective, and increasing function $\varphi$ we obtain

THEOREM 4.3. *Let $J$ be a compact interval, $\varphi\colon J \to [0, n]$ be continuous, bijective, and increasing, and $f\colon [0, n] \to [0, n]$ be of type III with $r$ discontinuities and* $\mathrm{ord}(f) = k$, *then*

$$F := \varphi^{-1} \circ f \circ \varphi\colon J \to J$$

*is bijective, has $r$ discontinuities, and satisfies $F^k = \mathrm{id}_J$, thus $F$ is an iterative root of the identity of order $k$.*

Conversely, consider an iterative root $F\colon J \to J$ of the identity of order $k$ on a compact interval $J$ with finitely many discontinuities. We will prove that it is always possible to find some $n \in \mathbb{N}$, a continuous, bijective, and increasing function $\varphi\colon J \to [0,n]$ and a function $f\colon [0,n] \to [0,n]$ of type III so that $F = \varphi^{-1} \circ f \circ \varphi$.

It is obvious that if $J$ is a compact interval and $F\colon J \to J$ is a bijective mapping with finitely many discontinuities, then they must be removable or jump discontinuities.

In general the integer $n$ is not uniquely determined, so we are looking for the smallest $n$ possible. Assume that $F^k = \mathrm{id}$ and $F$ has $r$ discontinuities $\xi_1, \dots, \xi_r \in J = [a,b]$. Consider the union of orbits

$$U = \{a,b\} \cup \bigcup_{j=1}^{r} \{F^i(\xi_j) \mid 1 \le i \le k\},$$

then $U$ is finite and we determine $n$ by

$$n = |U| - 1.$$

This particular $n$ will be called $n(F)$. The $n+1$ elements of $U$ will be labeled by $a = x_0 < \ldots < x_n = b$. Since $F(U) = U$ we have $F(x_i) \in U$ for all $0 \le i \le n$, thus $F$ is a permutation of $U$. Let $J_i$ be the open interval $(x_{i-1}, x_i)$, $i \in n$, then

$$[a,b] = U \cup J_1 \cup \cdots \cup J_n.$$

For all $i \in n$ it is obvious that $F$ is continuous on $J_i$, and there exists some $j \in n$ so that $F(J_i) = J_j$, thus $F$ permutes the intervals $J_i$.

The function $\varphi$ will be constructed in two steps: First we determine some $\varphi\colon J \to [0,n]$ so that $\varphi(J_i) = (i-1, i)$ for $i \in n$. Let $\varphi(x_i) := i$, $i \in n$. For $x \in J_i = (x_{i-1}, x_i)$ let

$$\varphi(x) := i - 1 + \frac{x - x_{i-1}}{x_i - x_{i-1}}\,,$$

then $\varphi$ is continuous in $J_i$, and $\lim_{x \to x_{i-1}^+} \varphi(x) = i - 1 = \varphi(x_{i-1})$ and $\lim_{x \to x_i^-} \varphi(x) = i = \varphi(x_i)$. Therefore, $\varphi$ is continuous on $J$. Moreover, $\varphi$ is strictly increasing and bijective. If $\tilde{F}$ denotes the function $\varphi \circ F \circ \varphi^{-1}\colon [0,n] \to [0,n]$, then

- $\tilde{F}$ is bijective,
- $\tilde{F}^j = \mathrm{id}_{[0,n]}$, iff $F^j = \mathrm{id}_J$,
- $\tilde{F}$ is an iterative root of the identity of order $k$,

- $\tilde{F}$ has discontinuities in $\varphi(\xi_i)$, $i \in r$,
- $\tilde{F}(i) \in \{0, \ldots, n\}$, $i \in \{0, \ldots, n\}$, $\tilde{F}$ permutes these elements,
- $\tilde{F}$ is continuous on $I_i = (i-1, i)$, $i \in n$,
- $\tilde{F}$ is a permutation of the intervals $I_i$, $i \in n$,
- $\tilde{F}$ is increasing on $I_i$, iff $F$ is increasing on $J_i$, $i \in n$.

In a second step we try to find some $\psi\colon [0,n] \to [0,n]$ so that $\psi \circ \tilde{F} \circ \psi^{-1}$ is affine on each interval $I_i = (i-1, i)$, $i \in n$.

Lemma 4.4. *Assume that $f := \tilde{F}|_{I_i}$ is a mapping $I_i \to I_j$ for $i \neq j$, $i, j \in n$.*

*If $f$ is strictly increasing, then there exists some $\psi_j\colon I_j \to I_j$ bijective and increasing, so that $\psi_j(f(x)) = j + x - i$, $x \in I_i$.*

*If $f$ is strictly decreasing, then there exists some $\psi_j\colon I_j \to I_j$ bijective and increasing, so that $\psi_j(f(x)) = j - x + i - 1$, $x \in I_i$.*

Proof. 1. Let $\psi_j(x) = j + f^{-1}(x) - i$, for $x \in I_j$, then $\psi_j$ is a bijective and increasing mapping $I_j \to I_j$, and $\psi_j(f(x)) = j + f^{-1}(f(x)) - i = j + x - i$, $x \in I_i$.

2. Let $\psi_j(x) = j - f^{-1}(x) + i - 1$, for $x \in I_j$, then $\psi_j$ is a bijective and increasing mapping $I_j \to I_j$, and $\psi_j(f(x)) = j - f^{-1}(f(x)) + i - 1$, $x \in I_i$. $\square$

Let $\psi_j(x) = x$ for $x \notin I_j$, then $\psi_j$ is bijective and increasing on $[0, n]$.

We had just seen that $\tilde{F}$ is a permutation of the intervals $I_i$, $i \in n$. Consider a cycle $I_{i_1} \to I_{i_2} \to \ldots \to I_{i_\ell} \to I_{i_1}$ of length $\ell \geq 1$. Then $F^\ell(I_{i_j}) = I_{i_j}$, $j \in \ell$.

Composition of two increasing or two decreasing functions yields an increasing function, composition of one increasing and one decreasing function yields a decreasing function. Therefore, if $\tilde{F}$ is decreasing on an even number of intervals in this cycle, then $\tilde{F}^\ell$ is increasing on all $I_{i_j}$, otherwise $\tilde{F}^\ell$ is decreasing on all $I_{i_j}$.

Since $\psi_j$ restricted to $I_i$ is a bijective mapping $I_i \to I_i$, $i \in n$, the restriction $\psi_j \circ \tilde{F} \circ \psi_j^{-1}$ to $I_i$ involves only $\tilde{F}|_{I_i}$.

Continuous iterative roots of the identity on an interval $I$ are continuous solutions of the Babbage equation. According to [5, Theorem 11.7.1] they are either the identity on $I$ or they are strictly decreasing involutions. The graph of a strictly decreasing involution of an interval is symmetric with respect to the line $\{(x, x) \mid x \in \mathbb{R}\}$ (cf. [5, Theorem 11.7.2]).

In the *first case* we assume that $\tilde{F}$ contains a cycle of intervals of length $\ell$ with an even number of decreasing functions in this cycle, then $\tilde{F}^\ell$ is continuous and strictly increasing on each of these intervals which means that $\tilde{F}^\ell|_{I_{i_j}} = \mathrm{id}|_{I_{i_j}}$ for all $j \in \ell$.

In the case $\ell = 1$ the function $\tilde{F}$ is already the identity on $I_{i_1}$. Assume that $\ell \geq 2$. Then the function $\tilde{F}|_{I_{i_1}}$ maps $I_{i_1} \to I_{i_2}$. According to Lemma 4.4 there exists a bijective and increasing mapping $\psi_{i_2}$ on $[0,n]$ so that $\psi_{i_2}\circ\tilde{F}|_{I_{i_1}}$ is affine, i.e. it is either $x\mapsto i_2+x-i_1$ or $x\mapsto i_2-1+i_1-x$. Then also $\psi_{i_2}\circ\tilde{F}\circ\psi_{i_2}^{-1}$ is affine on $I_{i_1}$ since $\psi_{i_2}^{-1}$ does not influence the function restricted to $I_{i_1}$.

If $\ell>2$, then the function $\psi_{i_2}\circ\tilde{F}\circ\psi_{i_2}^{-1}|_{I_{i_2}}$ maps $I_{i_2}\to I_{i_3}$. According to Lemma 4.4 there exists a bijective and increasing mapping $\psi_{i_3}$ on $[0,n]$ so that $\psi_{i_3}\circ\psi_{i_2}\circ\tilde{F}\circ\psi_{i_2}^{-1}$ is affine on $I_{i_2}$. Then also $\psi_{i_3}\circ\psi_{i_2}\circ\tilde{F}\circ\psi_{i_2}^{-1}\circ\psi_{i_3}^{-1}$ is affine on $I_{i_j}$, $j=1,2$.

Continuing in the same way, the function $\psi_{i_{\ell-1}}\circ\cdots\circ\psi_{i_2}\circ\tilde{F}\circ\psi_{i_2}^{-1}\circ\cdots\circ\psi_{i_{\ell-1}}^{-1}|_{I_{i_{\ell-1}}}$ maps $I_{i_{\ell-1}}\to I_{i_\ell}$. There exists a bijective and increasing mapping $\psi_{i_\ell}$ on $[0,n]$ so that $\psi_{i_\ell}\circ\psi_{i_{\ell-1}}\circ\cdots\circ\psi_{i_2}\circ\tilde{F}\circ\psi_{i_2}^{-1}\circ\cdots\circ\psi_{i_{\ell-1}}^{-1}$ is affine on $I_{i_{\ell-1}}$. Then also $\psi_{i_\ell}\circ\cdots\circ\psi_{i_2}\circ\tilde{F}\circ\psi_{i_2}^{-1}\circ\cdots\circ\psi_{i_\ell}^{-1}$, is affine on $I_{i_j}$, $j\in\ell-1$.

The mapping $\psi=\psi_{i_\ell}\circ\cdots\circ\psi_{i_2}$ is bijective and increasing on $[0,n]$, $\psi\circ\tilde{F}\circ\psi^{-1}|_{I_{i_j}}$ is affine, $j\in\ell-1$, and $\psi(x)=x$ for $x\in I_{i_1}$.

We have $\mathrm{id}|_{I_{i_1}}=\tilde{F}^\ell|_{I_{i_1}}=\tilde{F}|_{I_{i_\ell}}\circ\cdots\circ\tilde{F}|_{I_{i_1}}$. Therefore $\mathrm{id}|_{I_{i_1}}=\psi\circ\mathrm{id}\circ\psi^{-1}|_{I_{i_1}}=\psi\circ\tilde{F}^\ell\circ\psi^{-1}|_{I_{i_1}}=(\psi\circ\tilde{F}\circ\psi^{-1})^\ell|_{I_{i_1}}=(\psi\circ\tilde{F}\circ\psi^{-1})\circ\Big[(\psi\circ\tilde{F}\circ\psi^{-1})\circ\cdots\circ(\psi\circ\tilde{F}\circ\psi^{-1})\Big]\Big|_{I_{i_1}}=(\psi\circ\tilde{F}\circ\psi^{-1})|_{I_{i_\ell}}\circ\Big[(\psi\circ\tilde{F}\circ\psi^{-1})|_{I_{i_{\ell-1}}}\circ\cdots\circ(\psi\circ\tilde{F}\circ\psi^{-1})|_{I_{i_1}}\Big]$. The term between [ and ] is a composition of affine functions, thus it is affine, whence also $\psi\circ\tilde{F}\circ\psi^{-1}|_{I_{i_\ell}}$ is affine. Consequently $\psi\circ\tilde{F}\circ\psi^{-1}$ is affine on $I_{i_j}$ for each $j\in\ell$.

In the *second case* assume that $\tilde{F}$ contains a cycle of intervals of length $\ell$ with an odd number of decreasing functions in this cycle, then $\tilde{F}^\ell$ restricted to $I_{i_j}$ is a decreasing involution on $I_{i_j}$, $j\in\ell$, but it need not be affine on these intervals. Then there exists a bijective and increasing function $\tilde{\psi}$ so that $(\tilde{\psi}\circ\tilde{F}^\ell\circ\tilde{\psi}^{-1})|_{I_{i_j}}=(\tilde{\psi}\circ\tilde{F}\circ\tilde{\psi}^{-1})^\ell|_{I_{i_j}}$ is also affine, i.e. of the form $x\mapsto 2i_j-1-x$, $x\in I_{i_j}$ for each $j\in\ell$. Without loss of generality we assume that $\tilde{F}^\ell|_{I_{i_j}}$ is affine for each $j\in\ell$.

Similar to the first case, there exists $\psi=\psi_{i_\ell}\circ\ldots\circ\psi_{i_2}$ so that $\psi\circ\tilde{F}\circ\psi|_{I_{i_j}}$ is affine on $I_{i_j}$ for $j\in\ell-1$. By construction $\psi(x)=x$ for $x\in I_{i_1}$.

Therefore we have $\psi\circ\tilde{F}^\ell\circ\psi^{-1}|_{I_{i_1}}=\psi(2i_1-1-\psi^{-1}(x))=2i_1-1-x=\psi\circ\tilde{F}\circ\psi^{-1}|_{I_{i_\ell}}\circ\Big[\psi\circ\tilde{F}\circ\psi^{-1}|_{I_{i_{\ell-1}}}\circ\cdots\circ\psi\circ\tilde{F}\circ\psi^{-1}|_{I_{i_1}}\Big](x)$. The term between [ and ] is a composition of affine functions, thus it is affine, whence also $\psi\circ\tilde{F}|_{I_{i_\ell}}\circ\psi^{-1}|_{I_{i_\ell}}$ is affine. Consequently, $\psi\circ\tilde{F}|_{I_{i_j}}\circ\psi^{-1}$ is affine on $I_{i_j}$ for each $j\in\ell$.

This finishes the proof of

Theorem 4.5. *Let $J$ be a compact interval, $F\colon J\to J$ an iterative root of the identity of order $k$ with finitely many discontinuities. Then there exists* *some positive integer $n$ and a continuous, bijective, and increasing function $\varphi\colon J \to [0,n]$ so that $f = \varphi \circ F \circ \varphi^{-1}$ is a function of type III with* $\mathrm{ord}(f) = k$.

Two bijective functions $F_1\colon J_1 \to J_1$ and $F_2\colon J_2 \to J_2$ defined on compact intervals $J_1$ and $J_2$ are considered to be equivalent

$$F_1 \sim F_2$$

iff there exists a bijective increasing function $\varphi\colon J_1 \to J_2$ so that

$$F_2 = \varphi \circ F_1 \circ \varphi^{-1}.$$

It is easy to prove that for all bijective functions $F_i\colon J_i \to J_i$, $1 \le i \le 3$, we have $F_1 \sim F_1$, $F_1 \sim F_2$ iff $F_2 \sim F_1$, and if $F_1 \sim F_2$ and $F_2 \sim F_3$ then $F_1 \sim F_3$.

THEOREM 4.6. *Consider $f_1, f_2$ functions of type III corresponding to elements of $S_{n+1} \times (\{\pm 1\} \wr S_n)$, with $n = n(f_1) = n(f_2)$. Then*

$$f_1 \sim f_2 \Leftrightarrow f_1 = f_2.$$

THEOREM 4.7. *Consider an iterative root $F\colon J \to J$ of the identity of order $k$ on a compact interval $J$ with finitely many discontinuities. Let $n = n(F)$. Then there exists exactly one function $f \in S_{n+1} \times (\{\pm 1\} \wr S_n)$ of type III so that*

$$F \sim f.$$

Consider $f \in S_{n+1} \times (\{\pm 1\} \wr S_n)$ with $m = n(f) < n$. Then there exists $f' \in S_{m+1} \times (\{\pm 1\} \wr S_m)$ so that $f \sim f'$. It is possible that there exists $f'' \in S_{n+1} \times (\{\pm 1\} \wr S_n)$, $f'' \neq f$, so that $f \sim f''$.

How many functions $f$ of type III exist with $n = n(f)$? Their number is the number of non-equivalent functions $f$ of type III with $n = n(f)$. So far the author does not know an explicit formula in order to enumerate them. For small values of $n$ it is possible to check all functions of type III. Table 6 contains numerical data (computed with SYMMETRICA [7]) comparing the numbers of all functions of type III for small $n$, with the numbers of functions with $n(f) < n$ and $n(f) = n$.

Consider e.g. functions of type III which are of the form

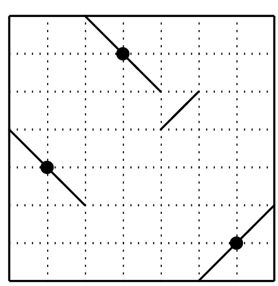
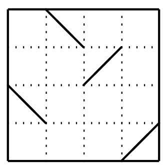


Table 6. Comparison of numbers of functions of type III

| $n$ | $n!(n+1)!2^n$ | $n(f) < n$ | $n(f) = n$ |
|---|---|---|---|
| 0 | 1 | | |
| 1 | 4 | | |
| 2 | 48 | 4 | 44 |
| 3 | 1152 | 40 | 1112 |
| 4 | 46080 | 892 | 45188 |
| 5 | 2764800 | 37708 | 2727092 |
| 6 | 232243200 | 2337808 | 229905392 |
| 7 | 26011238400 | 201311920 | 25809926480 |
| 8 | 3745618329600 | 22951808356 | 3722666521244 |
| 9 | 674211299328000 | | |
| 10 | 148326485852160000 | | |

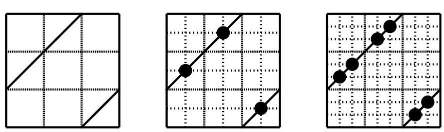
Here we have a permutation $\lambda$ for $n = 7$, where neither 1, 3 nor 6 occur in the orbit of a discontinuity of $f$ since $f(1) = 3$, $f(3) = 6$ and $f(6) = 1$. Thus these values could be omitted in order to get a function $g$ on $n = 4$.



Depending on $f(i) \in \{0, 2, 4, 5, 7\}$ for $i \in \{0, 2, 4, 5, 7\}$ there are $(8 - 3)!$ functions of this particular form with $n(f) = 4$.

A method for constructing functions of type III with $n(f) < n$ is the following:

Divide all intervals $(i - 1, i)$ belonging to a cycle of length $\ell$ of $\lambda$ into $k$ intervals of length $1/k$, and stretch each of these shorter intervals to length 1, then we obtain $k \cdot \ell$ intervals instead of $\ell$ intervals. Since the original function is continuous in $(i-1, i)$ the "stretched function" is continuous on $k$ consecutive intervals. E.g. from a function $f$ with $n(f) = n = 3$, where $\lambda = (1, 2, 3)$, and $\varepsilon = 1$, we obtain for $k = 1$, $k = 2$, $k = 3$, functions of type III with $n = 3$, $n = 6$, $n = 9$.



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University of Graz
Institute of Mathematics and Scientific Computing
NAWI-Graz
Heinrichstr. 36/4
A–8010 Graz
Austria
e-mail: harald.fripertinger@uni-graz.at