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ON A FUNCTIONAL EQUATION APPEARING ON THE MARGINS OF A MEAN INVARIANCE PROBLEM

Justyna Jarczyk, Witold Jarczyk

Dedicated to Professor Zygfryd Kominek on the occasion of his 75th birthday

Abstract. Given a continuous strictly monotonic real-valued function α, defined on an interval I, and a function $\omega\colon I \to (0,+\infty)$ we denote by B^{α}_{ω} the Bajraktarević mean generated by α and weighted by ω:

$$B^{\alpha}_{\omega}(x,y) = \alpha^{-1}\left(\frac{\omega(x)}{\omega(x)+\omega(y)}\alpha(x) + \frac{\omega(y)}{\omega(x)+\omega(y)}\alpha(y)\right), \quad x,y \in I.$$

We find a necessary integral formula for all possible three times differentiable solutions (φ,ψ) of the functional equation

$$r(x)B^{\varphi}_{s}(x,y) + r(y)B^{\psi}_{t}(x,y) = r(x)x + r(y)y,$$

where $r,s,t\colon I \to (0,+\infty)$ are three times differentiable functions and the first derivatives of φ,ψ and r do not vanish. However, we show that not every pair (φ,ψ) given by the found formula actually satisfies the above equation.

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1. Introduction

Given an interval I, a continuous strictly monotonic function $\alpha\colon I \to \mathbb{R}$, and any $\kappa\colon I^2 \to (0,1)$ we define the *quasi-arithmetic mean* $A^{\alpha}_{\kappa}\colon I^2 \to I$ *generated by* α and *weighted by* κ:

$$A^{\alpha}_{\kappa}(x,y) = \alpha^{-1}\left(\kappa(x,y)\alpha(x) + (1-\kappa(x,y))\alpha(y)\right).$$

In [9] the first author of the present paper studied the functional equation

$$\lambda(x,y)A^{\varphi}_{\mu}(x,y) + (1-\lambda(x,y))\,A^{\psi}_{\nu}(x,y) = \lambda(x,y)x + (1-\lambda(x,y))\,y \tag{1}$$

with given $(0,1)$-valued functions λ, μ and ν defined on I^2, and unknown increasing continuous functions φ and ψ defined on I.

Let us note that (1) is the invariance equation of $A_\lambda := A^{\mathrm{id}}_{\lambda}$ with respect to the pair $\left(A^{\varphi}_{\mu}, A^{\psi}_{\nu}\right)$ expressed as

$$A_\lambda \circ \left(A^{\varphi}_{\mu}, A^{\psi}_{\nu}\right) = A_\lambda. \tag{2}$$

In the case of scalars λ, μ, ν equation (2) was investigated by several authors (cf. [13], [4], [5], [6]). The final solution was found by the first author (see [8]; cf. also [12] and [2]).

In this note we exploit the results of [9]–[11] for considering the special case of equation (1) concerned with the so-called *fraction weights,* i.e. functions of the form like

$$\lambda(x,y) = \frac{r(x)}{r(x)+r(y)},$$

where $r\colon I \to (0,+\infty)$ is said to be the *generator of* the *weight* λ. Assuming that λ, μ, ν are generated by functions $r, s, t\colon I \to (0,+\infty)$, respectively. Then the functional equation (1) can be rewritten as

$$\begin{aligned}\frac{r(x)}{r(x)+r(y)}\varphi^{-1}&\left(\frac{s(x)}{s(x)+s(y)}\varphi(x) + \frac{s(y)}{s(x)+s(y)}\varphi(y)\right)\\ &+ \frac{r(y)}{r(x)+r(y)}\psi^{-1}\left(\frac{t(x)}{t(x)+t(y)}\psi(x) + \frac{t(y)}{t(x)+t(y)}\psi(y)\right)\\ &= \frac{r(x)}{r(x)+r(y)}x + \frac{r(y)}{r(x)+r(y)}y.\end{aligned}$$

Given a continuous strictly monotonic function $\alpha\colon I \to \mathbb{R}$ and any function $\omega\colon I \to (0,+\infty)$ we define the Bajraktarević mean $B_\omega^\alpha\colon I^2 \to I$ by

$$B_\omega^\alpha(x,y) = \alpha^{-1}\left(\frac{\omega(x)}{\omega(x)+\omega(y)}\alpha(x) + \frac{\omega(y)}{\omega(x)+\omega(y)}\alpha(y)\right)$$

(see [1] and [3]; also [11]). Finally equation (1) takes the form

$$r(x)B_s^\varphi(x,y) + r(y)B_t^\psi(x,y) = r(x)x + r(y)y. \tag{3}$$

A special case of equation (3), namely if $I \subset (0,+\infty)$, r is constant and $s = t = \mathrm{id}|_I$, was considered in [7]. Another case, when $s = t$ and s satisfies the harmonic oscillator equation, was considered by the first present author in [10]. Recently Bajraktarević means and their generalizations are again extensively investigated. This concerns both the comparison problem (see [16]) and various invariance problems (see [14], [15] and [17]).

2. Main result

In what follows we need the below technical lemma which can be easily derived from [10, Remark 1] and [9, Lemmas 1 and 3]. Making this remember that every mean is reflexive, that is takes the value x at diagonal points (x,x).

Lemma 1. *Let $\lambda\colon I^2 \to (0,1)$ be a fraction weight generated by a function $\omega\colon I \to (0,+\infty)$ and let $\alpha\colon I \to \mathbb{R}$ be a continuous strictly monotonic function. If ω and α are differentiable and $\alpha'(x) \neq 0$ for each $x \in I$, then*

$$\partial_1\lambda(x,x) = \frac{\omega'(x)}{4\omega(x)} \tag{4}$$

and

$$\partial_1 B_\omega^\alpha(x,x) = \lambda(x,x) = \frac{1}{2} \tag{5}$$

for all $x \in I$. Moreover, if ω and α are twice differentiable, then

$$\partial_{1,1}^2\lambda(x,x) = \frac{\omega''(x)\omega(x) - \omega'(x)^2}{4\omega(x)^2} \tag{6}$$

and

$$\partial^2_{1,1} B^\alpha_\omega(x,x) = \frac{\alpha''(x)}{4\alpha'(x)} + \frac{\omega'(x)}{2\omega(x)} \tag{7}$$

for all $x \in I$. If, in addition, ω and α are three times differentiable, then

$$\partial^3_{1,1,1}\lambda(x,x) = \frac{3\omega'(x)^3 - 6\omega''(x)\omega'(x)\omega(x) + 2\omega'''(x)\omega(x)^2}{8\omega(x)^3}$$

and

$$\partial^3_{1,1,1} B^\alpha_\omega(x,x) = \frac{3}{8}\frac{\alpha'''(x)\alpha'(x) - \alpha''(x)^2}{\alpha'(x)^2} + \frac{3}{4}\frac{\omega''(x)\omega(x) - \omega'(x)^2}{\omega(x)^2} \tag{8}$$

for all $x \in I$.

The next fact follows immediately from [9, Lemma 2].

Lemma 2. *Let $r, s, t\colon I \to (0,1)$ be twice differentiable functions. If $\varphi\colon I \to \mathbb{R}$ and $\psi\colon I \to \mathbb{R}$ are twice differentiable functions with non-vanishing first derivatives and the pair (φ,ψ) satisfies equation* (3)*, then*

$$\frac{\varphi''(x)}{\varphi'(x)} + \frac{\psi''(x)}{\psi'(x)} = 4\frac{r'(x)}{r(x)} - 2\frac{s'(x)}{s(x)} - 2\frac{t'(x)}{t(x)}$$

for all $x \in I$.

The following result will play a fundamental role in next considerations.

Proposition 3. *Let $r, s, t\colon I \to (0,+\infty)$ be three times differentiable functions. If $\varphi\colon I \to \mathbb{R}$ and $\psi\colon I \to \mathbb{R}$ are three times differentiable functions with non-vanishing first derivatives and the pair (φ,ψ) is a solution of equation* (3)*, then the equality*

$$r'(x)\left(\frac{\varphi''(x)}{\varphi'(x)} - 2\frac{r'(x)}{r(x)} + 2\frac{s'(x)}{s(x)}\right) = 0 \tag{9}$$

holds for all $x \in I$.

Proof. Keep in mind that each mean equals x at the diagonal points (x, x). Let λ be a fraction weight with a generator r. According to equalities (5)

$$\partial_1 B_s^{\varphi}(x, x) = \partial_1 B_t^{\psi}(x, x) = \frac{1}{2}, \quad x \in I,$$

so, taking into account the second formula of the proof of [9, Theorem 2], we see that

$$\begin{aligned} &3\partial_1 \lambda(x, x) \left(\partial_{11}^2 B_s^{\varphi}(x, x) - \partial_{11}^2 B_t^{\psi}(x, x)\right) \\ &\qquad + \frac{1}{2} \left(\partial_{111}^3 B_s^{\varphi}(x, x) + \partial_{111}^3 B_t^{\psi}(x, x)\right) = 3\partial_{11}^2 \lambda(x, x) \end{aligned}$$

for all $x \in I$. Hence, making use of (4) and (6)–(8) and multiplying the obtained equality by 16, we get

$$\begin{aligned} &3\frac{r'(x)}{r(x)} \left(\frac{\varphi''(x)}{\varphi'(x)} + 2\frac{s'(x)}{s(x)} - \frac{\psi''(x)}{\psi'(x)} - 2\frac{t'(x)}{t(x)}\right) \\ &\quad + 3\left(\left(\frac{\varphi''(x)}{\varphi'(x)}\right)' + 2\left(\frac{s'(x)}{s(x)}\right)' + \left(\frac{\psi''(x)}{\psi'(x)}\right)' + 2\left(\frac{t'(x)}{t(x)}\right)'\right) = 12\left(\frac{r'(x)}{r(x)}\right)' \end{aligned}$$

for every $x \in I$. Using Lemma 2 we eliminate the term $\psi''(x)/\psi'(x)$ and for all $x \in I$ we obtain

$$\begin{aligned} &3\frac{r'(x)}{r(x)} \left(\frac{\varphi''(x)}{\varphi'(x)} + 2\frac{s'(x)}{s(x)} + \frac{\varphi''(x)}{\varphi'(x)} - 4\frac{r'(x)}{r(x)} + 2\frac{s'(x)}{s(x)} + 2\frac{t'(x)}{t(x)} - 2\frac{t'(x)}{t(x)}\right) \\ &\qquad + 3\left(\frac{\varphi''(x)}{\varphi'(x)}\right)' + 6\left(\frac{s'(x)}{s(x)}\right)' - 3\left(\frac{\varphi''(x)}{\varphi'(x)}\right)' + 12\left(\frac{r'(x)}{r(x)}\right)' \\ &\qquad - 6\left(\frac{s'(x)}{s(x)}\right)' - 6\left(\frac{t'(x)}{t(x)}\right)' + 6\left(\frac{t'(x)}{t(x)}\right)' = 12\left(\frac{r'(x)}{r(x)}\right)', \end{aligned}$$

which is (9). □

In what follows we consider the case when the equation $r'(x) = 0$ has no roots, postponing the research in the remaining case to another paper.

The following theorem provides some necessary conditions on the generating functions φ and ψ and the weights r, s, t under three times differentiability assumptions with non-vanishing first derivatives of the functions φ, ψ and r.

THEOREM 4. *Let $r, s, t\colon I \to (0, +\infty)$ be three times differentiable functions and assume that the equation $r'(x) = 0$ has no roots. If $\varphi\colon I \to \mathbb{R}$ and $\psi\colon I \to \mathbb{R}$ are three times differentiable functions with non-vanishing first derivatives and the pair (φ, ψ) satisfies equation* (3)*, then there exist numbers $c, d \in \mathbb{R} \setminus \{0\}$ such that*

$$\varphi'(x) = c\left(\frac{r(x)}{s(x)}\right)^2, \quad x \in I, \tag{10}$$

and

$$\psi'(x) = d\left(\frac{r(x)}{t(x)}\right)^2, \quad x \in I. \tag{11}$$

PROOF. By Proposition 3 we claim that φ satisfies equation (9). Therefore, since $r'(x) \neq 0$ for all $x \in I$ we have

$$\frac{\varphi''(x)}{\varphi'(x)} = 2\left(\frac{r'(x)}{r(x)} - \frac{s'(x)}{s(x)}\right), \quad x \in I,$$

hence

$$(\log|\varphi'(x)|)' = \left(\log r(x)^2\right)' - \left(\log s(x)^2\right)', \quad x \in I,$$

and, consequently,

$$|\varphi'(x)| = |c_0|\left(\frac{r(x)}{s(x)}\right)^2, \quad x \in I,$$

with some nonzero c_0. Thus, since φ' is continuous and does not vanish, we come to (10) either with $c = |c_0|$, or with $c = -|c_0|$. Using Lemma 2 we obtain

$$\frac{\psi''(x)}{\psi'(x)} = 2\left(\frac{r'(x)}{r(x)} - \frac{t'(x)}{t(x)}\right), \qquad x \in I.$$

Repeating the argument used to φ' we come to (11) with some nonzero d. $\square$

The opposite implication in general is not true, that is unfortunately not every pair (φ, ψ) with φ and ψ satisfying (10) and (11), respectively, is a solution of equation (3). This can be observed in the example below.

Example 5. Put $I = \mathbb{R}$ and define functions $r, s, t\colon I \to (0, +\infty)$ by

$$r(x) = \mathrm{e}^x, \qquad s(x) = t(x) = 1,$$

and functions $\varphi, \psi\colon I \to \mathbb{R}$ by the equalities

$$\varphi(x) = \psi(x) = \mathrm{e}^{2x}.$$

Then (10) and (11) are satisfied with $c = d = 2$. If (φ, ψ) were a solution of equation (3), then we would have

$$\frac{1}{2}\mathrm{e}^x \log\left(\frac{1}{2}\mathrm{e}^{2x} + \frac{1}{2}\mathrm{e}^{2y}\right) + \frac{1}{2}\mathrm{e}^y \log\left(\frac{1}{2}\mathrm{e}^{2x} + \frac{1}{2}\mathrm{e}^{2y}\right) = \mathrm{e}^x x + \mathrm{e}^y y,$$

hence

$$(\mathrm{e}^x + \mathrm{e}^y) \log \frac{\mathrm{e}^{2x} + \mathrm{e}^{2y}}{2} = 2(\mathrm{e}^x x + \mathrm{e}^y y)$$

for all $x, y \in \mathbb{R}$. Tending here with y to $-\infty$ and dividing both sides of the equality by e^x we get

$$\log \frac{\mathrm{e}^{2x}}{2} = 2x, \quad x \in \mathbb{R},$$

that is

$$\frac{\mathrm{e}^{2x}}{2} = \mathrm{e}^{2x}, \quad x \in \mathbb{R},$$

which is impossible.

Example 5 shows that we are still far from sufficient conditions for the pair (φ, ψ) to be a solution of equation (3). Theorem 4 provides the form of its derivative (φ', ψ'). Having it we can find the forms of φ and ψ containing, in general, the integral operator. Consequently, it is usually hard to determine the exact form of the inverses φ^{-1}, ψ^{-1} and to verify if equality (3) holds true everywhere in I.

Open Problem. *Find further necessary conditions for the pair* (φ, ψ) *to satisfy equation* (3) *under higher differentiability conditions.*

References

[1] M. Bajraktarević, *Sur une équation fonctionelle aux valeurs moyennes*, Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske. Ser. II **13** (1958), 243–248.

[2] P. Burai, *A Matkowski-Sutô type equation*, Publ. Math. Debrecen **70** (2007), 233–247.

[3] Z. Daróczy, Gy. Maksa and Zs. Páles, *On two variable means with variable weights*, Aequationes Math. **67** (2004), 154–159.

[4] Z. Daróczy and Zs. Páles, *On means that are both quasi-arithmetic and conjugate arithmetic*, Acta Math. Hungar. **90** (2001), 271–282.

[5] Z. Daróczy and Zs. Páles, *Gauss-composition of means and the solution of the Matkowski-Sutô problem*, Publ. Math. Debrecen **61** (2002), 157–218.

[6] Z. Daróczy and Zs. Páles, *A Matkowski-Sutô problem for weight quasi-arithmetic means*, Ann. Univ. Sci. Budapest. Sci. Comput. **22** (2003), 69–81.

[7] J. Domsta and J. Matkowski, *Invariance of the arithmetic mean with respect to special mean-type mappings*, Aequationes Math. **71** (2006), 70–85.

[8] J. Jarczyk, *Invariance in the class of weighted quasi-arithmetic means with continuous generators*, Publ. Math. Debrecen **71** (2007), 279–294.

[9] J. Jarczyk, *Invariance of quasi–arithmetic means with function weights*. J. Math. Anal. Appl. **353** (2009), 134–140.

[10] J. Jarczyk, *Invariance in a class of Bajraktarević means*, Nonlinear Anal. **72** (2010), 2608–2619.

[11] J. Jarczyk and W. Jarczyk, *Invariance of means*, Aequationes Math. **92** (2018), 801–872.

[12] J. Jarczyk and J. Matkowski, *Invariance in the class of weighted quasi-arithmetic means*, Ann. Polon. Math. **88** (2006), 39–51.

[13] J. Matkowski, *Invariant and complementary quasi-arithmetic means*. Aequationes Math. **57** (1999), 87–107.

[14] J. Matkowski, *Invariance of Bajraktarević mean with respect to quasi-arithmetic means*, Publ. Math. Debrecen **80** (2012), 441–45.

[15] J. Matkowski, *Invariance of Bajraktarević means with respect to the Beckenbach-Gini means*, Math. Slovaca **63** (2013), 493–502.

[16] Zs. Páles and A. Zakaria, *On the local and global comparison of generalized Bajraktarević means*, J. Math. Anal. Appl. **455** (2017), 792–815.

[17] Zs. Páles and A. Zakaria, *On the invariance equation for two-variable weighted nonsymmetric Bajraktarević means*, Aequationes Math. **93** (2019), 37–57.

Justyna Jarczyk
Institute of Mathematics
University of Zielona Góra
Szafrana 4A
65-516 Zielona Góra
Poland
e-mail: j.jarczyk@wmie.uz.zgora.pl

Witold Jarczyk
Institute of Mathematics, Informatics
and Landscape Architecture
John Paul II Catholic University of Lublin
Konstantynów 1h
20-708 Lublin
Poland
e-mail: wjarczyk@kul.lublin.pl