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GENERALIZATION OF THE HARMONIC WEIGHTED MEAN VIA PYTHAGOREAN INVARIANCE IDENTITY AND APPLICATION

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Dedicated to Professor Zygfryd Kominek on the occasion of his 75th birthday

Abstract. Under some simple conditions on the real functions f and g defined on an interval $I \subset (0,\infty)$, the two-place functions $A_f(x,y) = f(x) + y - f(y)$ and $G_g(x,y) = \frac{g(x)}{g(y)}y$ generalize, respectively, A and G, the classical weighted arithmetic and geometric means. In this note, basing on the invariance identity $G \circ (H,A) = G$ (equivalent to the Pythagorean harmony proportion), a suitable weighted extension $H_{f,g}$ of the classical harmonic mean H is introduced. An open problem concerning the symmetry of $H_{f,g}$ is proposed. As an application a method of effective solving of some functional equations involving means is presented.

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1. Introduction

Let $I \subset \mathbb{R}$ be an interval and let $f, \varphi\colon I \to \mathbb{R}$ be arbitrary functions. In [8] it was shown that the two-variable function $M\colon I^2 \to \mathbb{R}$ of the form

$$M(x,y) = f(x) + \varphi(y), \quad x, y \in I,$$

is a mean in I, i.e. that

$$\min(x,y) \leq M(x,y) \leq \max(x,y), \quad x, y \in I,$$

if, and only if, $\varphi = \mathrm{id}|_I - f$, and the functions f and $\mathrm{id}|_I - f$ are non-decreasing, and the function $M = A_f$ defined by $A_f(x,y) = f(x) + y - f(y)$, strictly related to addition, generalizes the weighted arithmetic mean. In particular, if A_f is symmetric then $A_f = A$, where $A(x,y) := \frac{x+y}{2}$ (see Proposition 1).

Let $I \subset (0,\infty)$ and let $g, \psi\colon I \to (0,\infty)$. In the present note we first observe that the function $M\colon I^2 \to \mathbb{R}$ of the form

$$M(x,y) = g(x)\,\psi(y), \quad x, y \in I,$$

is a mean in I, iff $\psi = \frac{\mathrm{id}|_I}{g}$ and both functions g, $\frac{\mathrm{id}|_I}{g}$ are non-decreasing. Moreover the mean $M = G_g$ defined by $G_g(x,y) = \frac{g(x)}{g(y)}y$, strictly related to multiplication, generalizes the weighted geometric mean. In particular, if G_g is symmetric then $G_g = G$, where $G(x,y) = \sqrt{xy}$ (see Proposition 2).

The generalizations of the weighted arithmetic and geometric means given by Proposition 1 and Proposition 2, are based, respectively, on relation of the classical arithmetic mean A to addition, and the geometric mean G to multiplication.

Having these generalizations, a legitimate question arises if the classical harmonic mean $H(x,y) = \frac{2xy}{x+y}$ can be also extended. The main result of the present paper, Theorem 1 in section 4, gives the affirmative answer. It turns out that, with the aid of the weighted extensions A_f and G_g of the arithmetic and geometric means, basing on the identity $G \circ (H, A) = G$, the invariance of the geometric mean G with respect to the mean-type mapping (H, A) (equivalent to the classical Pythagorean harmony proportion), one can obtain the means $H_{f,g}$ generalizing the harmonic mean. In section 5 the symmetry of the mean $H_{f,g}$ is considered and an open problem is proposed. In section 6, given $H_{f,g}$, we ask for its harmonically complementary mean $H_{\varphi,\psi}$, i.e. such that $H \circ (H_{f,g}, H_{\varphi,\psi}) = H$. In section 7 we apply Theorem 1 to obtain the effective form of the continuous solutions Φ of functional equations of form

$$\Phi(H_{f,g}(x,y), A_f(x,y)) = \Phi(x,y).$$

2. Generalization of weighted arithmetic mean

We begin with recalling the following

PROPOSITION 1 (see [8]). *Let $I \subset \mathbb{R}$ be an interval and let $f, \varphi \colon I \to \mathbb{R}$. Then the function*

$$M(x,y) = f(x) + \varphi(y), \quad x, y \in I,$$

is a mean iff $\varphi = \mathrm{id}|_I - f$, i.e. $M = A_f$, where $A_f \colon I^2 \to \mathbb{R}$ is defined by

$$A_f(x,y) := f(x) + y - f(y), \quad x, y \in I,$$

and the functions f and $\mathrm{id}|_I - f$ are non-decreasing.

Moreover

(i) *A_f is a mean iff the function f is non-decreasing and non-expansive;*
(ii) *A_f is a strict mean iff f and $\mathrm{id}|_I - f$ are strictly increasing, or equivalently, iff f is strictly increasing and strictly contractive;*
(iii) *A_f is symmetric iff $A_f = A$, or equivalently, iff the function $f(x) - \frac{x}{2}$ is constant in I.*

REMARK 1. Let a, b, c be positive real numbers and let $f\colon (0,\infty) \to \mathbb{R}$ be defined by

$$f(x) = \frac{ax^2 + bx}{cx + 1}, \quad x > 0.$$

Then

(i) f is strictly increasing;
(ii) if $a \le c$ and $b < 1$, or $a < c$ and $b \le 1$ then $\mathrm{id}|_{(0,\infty)} - f$ is strictly increasing.

PROOF. It is enough to note that, for all $x > 0$,

$$f'(x) = \frac{acx^2 + 2ax + b}{(cx+1)^2}, \quad \left(\mathrm{id}|_{(0,\infty)} - f\right)'(x) = \frac{(c-a)\,x\,(cx+2) + (1-b)}{(cx+1)^2},$$

are positive, so the assumptions of Proposition 1 are satisfied. □

Let us also note some general properties of functions of the form A_f, not assuming that A_f is a mean.

REMARK 2. Let $f\colon (0,\infty) \to \mathbb{R}$. The following conditions are equivalent:

(i) the function A_f is sub-homogeneous, i.e.

$$A_f(tx,ty) \le tA_f(x,y), \quad t,x,y>0;$$

(ii) the function A_f is super-homogeneous, i.e.

$$A_f(tx,ty) \ge tA_f(x,y), \quad t,x,y>0;$$

(iii) the function A_f is homogeneous, i.e.

$$A_f(tx,ty) = tA_f(x,y), \quad t,x,y>0;$$

(iv) the function f is linear, that is $f(x) = f(1)x$ for all $x>0$, and

$$A_f(x,y) = f(1)x + y - f(1)y, \quad x,y>0.$$

Indeed, if A_f is sub-homogeneous then, for all $t,x,y>0$, we have $f(tx) + ty - f(ty) \le t[f(x) + y - f(y)]$, whence, for all $t,x,y>0$, we have $f(tx) - tf(x) \le f(ty) - tf(y)$. This implies that there is a real constant b such that $f(tx) - tf(x) = b$ for all $t,x>0$. Taking here $x=1$ and setting $a=f(1)$ we obtain $f(t) = at+b$ for all $t>0$.

Similarly, applying the basic fact of additive functions (cf. [1], [4]) we get the following

REMARK 3. Let $f\colon \mathbb{R} \to \mathbb{R}$. The following conditions are equivalent:

(i) the function A_f is sub-translative, i.e.

$$A_f(x+t,y+t) \le A_f(x,y) + t, \quad t,x,y \in \mathbb{R};$$

(ii) the function A_f is super-translative, i.e.

$$A_f(x+t,y+t) \ge A_f(x,y) + t, \quad t,x,y \in \mathbb{R};$$

(iii) the function A_f is translative, i.e.

$$A_f(x+t,y+t) = A_f(x,y) + t, \quad t,x,y \in \mathbb{R};$$

(iv) there is additive function $\alpha\colon \mathbb{R} \to \mathbb{R}$ and $b \in \mathbb{R}$ such that the function $f(x) = \alpha(x) + b$ for all $x \in \mathbb{R}$, and

$$A_f(x,y) = \alpha(x-y) + y, \quad x,y \in \mathbb{R}.$$

3. Generalization of weighted geometric mean

It is easy to prove the following

Proposition 2. *Let $I \subset (0,\infty)$ be an interval and let $g,\psi\colon I \to (0,\infty)$. Then the function*

$$M(x,y) = g(x)\,\psi(y), \quad x,y \in I,$$

is a mean iff $\psi = \frac{\mathrm{id}|_I}{g}$ i.e. $M = G_g$, where $G_g\colon I^2 \to (0,\infty)$ is defined by

$$G_g(x,y) := \frac{g(x)}{g(y)}y, \quad x,y \in I,$$

and the functions g and $\frac{\mathrm{id}|_I}{g}$ are non-decreasing.

Moreover

(i) *G_g is a mean iff the function g is non-decreasing and*

$$1 \le \frac{g(y)}{g(x)} \le \frac{y}{x}, \quad x,y \in I,\ x<y;$$

(ii) *G_g is a strict mean iff g and $\frac{\mathrm{id}|_I}{g}$ are strictly increasing, or equivalently, iff*

$$1 < \frac{g(y)}{g(x)} < \frac{y}{x}, \quad x,y \in I,\ x<y;$$

(iii) *G_g is symmetric iff $G_g = G$, or equivalently, iff the function $\frac{g(x)}{\sqrt{x}}$ is constant in I.*

To determine a possible broad class of functions $g\colon (0,\infty) \to (0,\infty)$, being good candidates for generating the generalized geometric means, let us fix $p \in (0,1]$, write g in the form

$$g(x) = x^p\gamma(x), \quad x>0,$$

where $\gamma\colon (0,\infty) \to (0,\infty)$ should be chosen in such a way that the functions g and $\frac{\mathrm{id}|_I}{g}$ are increasing. Since

$$g'(x) = x^{p-1}\left[p\gamma(x) + x\gamma'(x)\right], \quad \left(\frac{\mathrm{id}|_I}{g}\right)'(x) = x^{-p}\left[(1-p)\gamma(x) - x\gamma'(x)\right],$$

the function γ should be such that

$$p\gamma(x) + x\gamma'(x) \geq 0 \quad (1-p)\gamma(x) - x\gamma'(x) \geq 0 \quad \text{for all } x > 0,$$

or equivalently, such that

$$0 \leq \frac{\gamma'(x)}{\gamma(x)} x + p \leq 1, \quad x > 0.$$

The homographic function $(0, \infty) \ni x \longmapsto \frac{ax+b}{cx+1}$ with positive parameters $a, b, c > 0$ satisfies the inequality

$$0 \leq \frac{ax+b}{cx+1} \leq 1, \quad x > 0,$$

iff

$$0 < a \leq c \quad \text{and} \quad 0 < b \leq 1.$$

Solving the differential equation

$$\frac{\gamma'(x)}{\gamma(x)} x + p = \frac{ax+b}{cx+1},$$

we get

$$\gamma(x) = dx^{b-p}(cx+1)^{c(a-bc)}, \quad x > 0,$$

whence

$$g(x) = dx^{b}(cx+1)^{c(a-bc)}$$

for some $d > 0$. Setting $q := -c(a-bc)$, $p = b$, $r = c$, and taking into account that $0 < a \leq c$, we hence get the following

Remark 4. If $d > 0$, $r > 0$, $p \in (0, 1]$, and q is such that

$$-(1-p)r^2 \leq q < pr^2,$$

and $g\colon (0, \infty) \to (0, \infty)$ is given by

$$g(x) = \frac{dx^p}{(rx+1)^q}, \quad x > 0,$$

then the functions g and $\frac{\mathrm{id}|_I}{g}$ are increasing, and strictly increasing if $p \in (0, 1)$. (Of course, without any loss of generality, one can take $d = 1$.)

Let us also note some general properties of functions of the form G_g, not assuming that G_g is a mean.

REMARK 5. Let $g\colon (0,\infty) \to (0,\infty)$. The following conditions are equivalent:

(i) the function G_g is sub-homogeneous, i.e.

$$G_g(tx,ty) \le tG_g(x,y), \quad t,x,y>0;$$

(ii) the function G_g is super-homogeneous, i.e.

$$G_g(tx,ty) \ge tG_g(x,y), \quad t,x,y>0;$$

(iii) the function G_g is homogeneous, i.e.

$$G_g(tx,ty) = tG_g(x,y), \quad t,x,y>0;$$

(iv) the function g is multiplicative, i.e.,

$$g(tx) = g(t)\,g(x), \quad t,x,y>0.$$

Moreover, if the graph of g is not dense in $(0,\infty)^2$ then there exists $p\in\mathbb{R}$ such that

$$g(x) = x^p, \quad x>0.$$

Indeed, assume that G_g is sub-homogeneous. Replacing x by $\frac{x}{t}$ and y by $\frac{y}{t}$ in (i), we get $\frac{1}{t}G_g(x,y) \le G_g\left(\frac{x}{t},\frac{y}{t}\right)$ for all t,x,y, which shows that G_g is super-homogeneous, so it is homogeneous. The implication (iii) $\Longrightarrow$ (iv) is obvious. For the "moreover" part see [1], [4].

4. Main result – generalization of weighted harmonic mean

The above considerations lead to the natural question if one can define a relevant counterpart of the harmonic mean

$$H(x,y) = \frac{2xy}{x+y}, \quad x,y>0.$$

In this section we show that, basing on the classical invariance identity $G \circ (H, A) = G$, and applying the above generalizations of the arithmetic and geometric means, one can give a positive answer. Namely, we prove the following

THEOREM 1. *Let $I \subset (0,\infty)$ be an interval and assume that $f\colon I \to \mathbb{R}$, $g\colon I \to (0,\infty)$ be such that the functions f, $\mathrm{id}|_I - f$, g and $\frac{\mathrm{id}|_I}{g}$ are strictly increasing. Then*

(i) *f, g are continuous, and the function $H_{f,g}\colon I^2 \to (0,\infty)$ defined by*

$$H_{f,g}(x,y) := g^{-1}\left(\frac{g(A_f(x,y))}{A_f(x,y)} G_g(x,y)\right), \quad x,y \in I, \tag{1}$$

is reflexive in I, that is

$$H_{f,g}(x,x) = x, \quad x \in I;$$

(ii) *if moreover the function*

$$I \ni t \longmapsto \frac{t}{[g(t)]^2} \quad \text{is nonincreasing} \tag{2}$$

then the function $H_{f,g}$ is a strict mean in I;

(iii) *if the function*

$$I^2 \ni (x,y) \longmapsto \frac{g(A_f(x,y))}{A_f(x,y)} \frac{g(x)}{g(y)} y \quad \text{is (strictly) increasing} \tag{3}$$

in both variables, then $H_{f,g}$ is a (strict) mean in I;

(iv) *if (2) or (3) holds, the mean G_g is invariant with respect to the mean-type mapping $(H_{f,g}, A_f) : I^2 \to I^2$, that is*

$$G_g \circ (H_{f,g}, A_f) = G_g; \tag{4}$$

and the sequence of iterates $((H_{f,g}, A_f)^n : n \in \mathbb{N})$ converges in I^2 pointwise to (G_g, G_g).

PROOF. (i) The continuity of the functions f and g follows from the assumed increasing monotonicity of the functions f, $\mathrm{id}|_I - f$, g and $\frac{\mathrm{id}|_I}{g}$. For every $x \in I$, by the definition of A_f and G_g, we have

$$H_{f,g}(x,x) = g^{-1}\left(\frac{g(A_f(x,x))}{A_f(x,x)} G_g(x,x)\right) = g^{-1}\left(\frac{g(x)}{x} x\right) = x,$$

so $H_{f,g}$ is reflexive.

(ii) Take $x, y \in I$. If $x < y$ then, as g is strictly increasing, this inequality is equivalent to

$$g(x) \leq \frac{g(A_f(x,y))}{A_f(x,y)} \frac{g(x)}{g(y)} y \leq g(y).$$

Since the function $\frac{\mathrm{id}|_I}{g}$ is increasing and $x < A_f(x,y) < y$ we have

$$\frac{A_f(x,y)}{g(A_f(x,y))} < \frac{y}{g(y)},$$

or equivalently

$$1 < \frac{g(A_f(x,y))}{A_f(x,y)} \frac{y}{g(y)}.$$

Multiplying both sides by $g(x)$ gives

$$g(x) < \frac{g(A_f(x,y))}{A_f(x,y)} \frac{g(x)}{g(y)} y$$

whence, by the monotonicity of g and the definition of $H_{f,g}$, we get

$$\min(x,y) = x < g^{-1}\left(\frac{g(A_f(x,y))}{A_f(x,y)} \frac{g(x)}{g(y)} y\right) = H_{f,g}(x,y).$$

Since $x < A_f(x,y) < y$, applying in turn the monotonicity of g and (2) we have

$$\frac{g(A_f(x,y))}{A_f(x,y)} g(x) < \frac{[g(A_f(x,y))]^2}{A_f(x,y)} \leq \frac{[g(y)]^2}{y},$$

so

$$\frac{g(A_f(x,y))}{A_f(x,y)} \frac{g(x)}{g(y)} y < g(y).$$

Hence, by the monotonicity of g and the definition of $H_{f,g}$,

$$H_{f,g}(x,y) = g^{-1}\left(\frac{g(A_f(x,y))}{A_f(x,y)} \frac{g(x)}{g(y)} y\right) < y = \max(x,y).$$

If $x > y$ the argument is similar, so we omit it. Thus, for all $x, y \in I$, $x \neq y$, we have

$$\min(x,y) < g^{-1}\left(\frac{g(A_f(x,y))}{A_f(x,y)}\frac{g(x)}{g(y)}y\right) < \max(x,y),$$

which shows that $H_{f,g}$ is a strict mean.

To prove (iii) take arbitrary $x, y \in I$. Without any loss of generality, we can assume that $x < y$, so that $x = \min(x,y)$ and $y = \max(x,y)$. Applying the reflexivity of $H_{f,g}$, the definitions of $H_{f,g}$, A_f, G_g, increasing strict monotonicity of the function (3) with respect to both variables and increasing strict monotonicity of g^{-1}, we obtain

$$\begin{aligned}
\min(x,y) = x = H_{f,g}(x,x) &= g^{-1}\left(\frac{g(f(x)+x-f(x))}{f(x)+x-f(x)}\frac{g(x)}{g(x)}x\right)\\
&< g^{-1}\left(\frac{g(f(x)+y-f(y))}{f(x)+y-f(y)}\frac{g(x)}{g(y)}y\right) = H_{f,g}(x,y)\\
&< g^{-1}\left(\frac{g(f(y)+y-f(y))}{f(y)+y-f(y)}\frac{g(y)}{g(y)}y\right) = y = \max(x,y),
\end{aligned}$$

so $H_{f,g}$ is a strict mean. The monotonicity of $H_{f,g}$ is obvious.

(iv) Note that, for all $x, y \in I$, by the definitions of G_g, A_f and (1), we have

$$\begin{aligned}
G_g(H_{f,g}(x,y), A_f(x,y)) &= \frac{g(H_{f,g}(x,y))}{g(A_f(x,y))}A_f(x,y)\\
&= \frac{g\left(g^{-1}\left(\frac{g(A_f(x,y))}{A_f(x,y)}G_g(x,y)\right)\right)}{g(A_f(x,y))}A_f(x,y)\\
&= \frac{\frac{g(A_f(x,y))}{A_f(x,y)}G_g(x,y)}{g(A_f(x,y))}A_f(x,y) = G_g(x,y),
\end{aligned}$$

so (4) holds, that is G_g is invariant with respect to the mean-type mapping $(H_{f,g}, A_f)$.

The remaining statement follows from the main result in [6] (cf. also [7]). □

To illustrate an application of this result recall that *if* a, b, c *are positive real numbers such that* $a \leq c$ *and* $b < 1$ *, then*, in view of Remark 1, the function A_f *with* $f\colon (0,\infty) \to \mathbb{R}$ *given by* $f(x) = \frac{ax^2+bx}{cx+1}$ *is a mean in* $(0,\infty)$.

Similarly, *if* $p \in (0,1]$, *and* q *is such that* $p-1 \leq q < p$, *then* in view of Remark 4, the function G_g *with* $g\colon (0,\infty) \to (0,\infty)$ *given by* $g(x) = \frac{x^p}{(x+1)^q}$, *is a mean in* $(0,\infty)$.

Using these functions and Theorem 1 we give two examples of generalizations of the classical harmonic mean.

Example 1. For the functions f and g with $a = 1$, $b = \frac{1}{2}$, $c = 2$, an arbitrary $p \in (0,1]$, and $q = 0$, we have

$$\frac{g(A_f(x,y))}{A_f(x,y)} \frac{g(x)}{g(y)} y = 2^{1-p} \frac{x^p y^{1-p}}{(x+y)^{1-p}}, \quad x,y > 0.$$

Since

$$\frac{\partial}{\partial x} \frac{x^p y^{1-p}}{(x+y)^{1-p}} = y^{1-p}(x+y)^{p-2}\left[(2p-1)x^p + p x^{p-1} y\right],$$

$$\frac{\partial}{\partial y} \frac{x^p y^{1-p}}{(x+y)^{1-p}} = (1-p)x^{1+p} y^{-p}(x+y)^{p-2},$$

the above function is increasing with respect to both variables if $p \geq \frac{1}{2}$. By Theorem 1, taking into account that $g^{-1}(x) = x^{1/p}$ for all $x > 0$, we conclude that, for every $p \in \left[\frac{1}{2}, 1\right]$, the function

$$H_{f,g}(x,y) = \left(2^{1-p} \frac{x^p y^{1-p}}{(x+y)^{1-p}}\right)^{1/p} = \frac{2^{\frac{1}{p}-1} x y^{\frac{1}{p}-1}}{(x+y)^{\frac{1}{p}-1}}, \quad x,y > 0,$$

is a mean in $(0,\infty)$. In particular, taking $p = \frac{1}{2}$, we obtain

$$H_{f,g}(x,y) = \frac{2xy}{x+y}, \quad x,y > 0,$$

so, in this case, $H_{f,g}$ coincides with the classical harmonic mean.

Example 2. Taking the functions f and g with $a = 1$, $b = \frac{1}{2}$, $c = 2$; $p \in \left[\frac{1}{2}, 1\right]$, and $q = p - \frac{1}{2}$, one can check that the function

$$\frac{g(A_f(x,y))}{A_f(x,y)} \frac{g(x)}{g(y)} y = 2^{1-p} \frac{x^p y^{1-p}}{(x+y)^{1-p}}, \quad x,y > 0.$$

is increasing with respect to each variable. Consequently, by Theorem 1, the function $H_{f,g}$ is a mean. In particular, in the case $p = 1$ we obtain

$$g(x) = \frac{x}{\sqrt{x+1}}, \quad g^{-1}(x) = \frac{x^2}{2}\left(1+\sqrt{1+\frac{4}{x^2}}\right), \quad x > 0,$$

$$\frac{g(A_f(x,y))}{A_f(x,y)} \frac{g(x)}{g(y)} y = \frac{\sqrt{2}x\sqrt{y+1}}{\sqrt{x+y+2}\sqrt{x+1}}, \quad x, y > 0,$$

and

$$H_{f,g}(x,y) = \frac{x^2(y+1)}{(x+1)(x+y+2)}\left(1+\sqrt{1+2\frac{(x+1)(x+y+2)}{x^2(y+1)}}\right), \quad x, y > 0.$$

Remark 6. Since, in view of Theorem 1,

$$H_{f,g}(x,y) = g^{-1}\left(g(A_f(x,y))\, g\left(g^{-1}\left(\frac{G_g(x,y)}{A_f(x,y)}\right)\right)\right), \quad x, y \in I,$$

the mean $H_{f,g}$ is a composition of the bivariable function $g^{-1}(g(u)\, g(v))$, the mapping $\left(u, g^{-1}\left(\frac{v}{u}\right)\right)$ and the means A_f,G_g (compare [3] where the compositions of quasi-arithmetic means are considered; see also [9]).

5. Symmetry of generalized weighted harmonic mean and an open question

It is easy to verify that the generalized weighted means A_f and G_g are symmetric iff they coincide with the classical arithmetic geometric means A and G, respectively. It is interesting that the problem of symmetry of the generalized weighted mean $H_{f,g}$ appears to be nontrivial. To show it we begin with the following

Proposition 3. *Let $I \subset (0,\infty)$ be an interval and $t \in (0,1)$. Assume that $f(x) := tx$ for $x \in I \to \mathbb{R}$, the function $g\colon I \to (0,\infty)$ is increasing, differentiable and the function $\frac{\mathrm{id}|_I}{g}$ is increasing.*

The mean $H_{f,g}\colon I^2 \to (0,\infty)$ defined by (1) is symmetric, i.e.

$$H_{f,g}(x,y) = H_{f,g}(y,x), \quad x, y \in I,$$

iff $A_f = A$, $G_g = G$ *and* $H_{f,g} = H$, *where* A, G, H *are the classical symmetric arithmetic, geometric and harmonic means.*

Proof. Since $A_f(x,y) = tx + (1-t)y$ is the standard weighted arithmetic mean in I, in view of formula (1),

$$H_{f,g}(x,y) = g^{-1}\left(\frac{g(tf(x) + (1-t)f(y))}{tf(x) + (1-t)f(y)}\frac{g(x)}{g(y)}y\right), \quad x, y \in I.$$

Thus $H_{f,g}$ is symmetric iff

$$\frac{g(tx + (1-t)y)}{tx + (1-t)y}\frac{g(x)}{g(y)}y = \frac{g((1-t)x + ty)}{(1-t)x + ty}\frac{g(y)}{g(x)}x, \quad x, y \in I,$$

which is equivalent to

$$\begin{aligned}&[(1-t)x + ty]\, g(tx + (1-t)y)\, [g(x)]^2 y \\ &\quad = [tx + (1-t)y]\, g((1-t)x + ty)\, [g(y)]^2 x, \quad x, y \in I.\end{aligned}$$

Differentiating both sides with respect to x and then setting $y = x$ we get

$$x[g(x)]^2\{(2t+1)g'(x)x - 2tg(x)\} = 0, \quad x > 0,$$

whence

$$\frac{g'(x)}{g(x)} = \frac{2t}{2t+1}\frac{1}{x}, \quad x > 0.$$

Solving this differential equation we obtain

$$g(x) = cx^{\frac{2t}{2t+1}}, \quad x > 0,$$

for some $c > 0$. Now, from the definition of $H_{f,g}$, after simple calculations, we get

$$H_{f,g}(x,y) = x\left(\frac{y}{tx + (1-t)y}\right)^{\frac{1}{2t}}, \quad x, y > 0.$$

It is obvious that $H_{f,g}$ is a symmetric mean iff $t = \frac{1}{2}$. Thus $f(x) = \frac{x}{2}$, $g(x) = c\sqrt{x}$ for all $x \in I$, and, consequently, $A_f(x,y) = \frac{x+y}{2} = A(x,y)$; $G_g(x,y) = \sqrt{xy} = G(x,y)$ and, $H_{f,g}(x,y) = \frac{2xy}{x+y} = H(x,y)$ for all $x, y \in I$. $\square$

Proposition 4. *Let $I \subset (0,\infty)$ be an interval and $t \in (0,1)$. Assume that $g(x) = x^t$ for $x \in I \to \mathbb{R}$, the functions $f\colon I \to (0,\infty)$ and $\mathrm{id}|_I - f$ are increasing.*

The mean $H_{f,g}\colon I^2 \to (0,\infty)$ defined by (1) is symmetric iff $A_f = A$, $G_g = G$ and $H_{f,g} = H$, where A, G, H are the classical symmetric arithmetic, geometric and harmonic means.

Proof. Since $G_g(x,y) = x^t y^{1-t}$ is the weighted arithmetic mean in I, making use of (1), the mean $H_{f,g}$ is symmetric iff

$$\frac{[A_f(x,y)]^t}{A_f(x,y)}\frac{x^t}{y^t}y = \frac{[A_f(y,x)]^t}{A_f(y,x)}\frac{y^t}{x^t}x, \quad x,y \in I,$$

or, equivalently, iff

$$A_f(x,y)\, x^{\frac{2t-1}{t-1}} = A_f(y,x)\, y^{\frac{2t-1}{t-1}}, \quad x,y \in I.$$

By the definition of A_f, setting here

$$p := \frac{2t-1}{t-1},$$

we get

$$[f(x) + y - f(y)]\, x^p = [f(y) + x - f(x)]\, y^p, \quad x,y \in I,$$

whence

$$f(x) - f(y) = \frac{xy^p - yx^p}{x^p + y^p}, \quad x,y \in I.$$

Thus f is of the class C^∞ in I and, clearly,

$$\frac{\partial^2}{\partial x \partial y}\frac{xy^p - yx^p}{x^p + y^p} = 0, \quad x,y \in I.$$

Since

$$\begin{aligned}
&\frac{\partial^2}{\partial x \partial y}\frac{xy^p - yx^p}{x^p + y^p} \\
&\qquad = p\frac{x^{p-1}y^{p-1}}{(x^p + y^p)^3}\left[(1-p)\left(x^{p+1} - y^{p+1}\right) + (1+p)\,xy\left(y^{p-1} - x^{p-1}\right)\right],
\end{aligned}$$

for all $x, y \in I$, it follows that $p = 0$ and, by the definition of p, we get $t = \frac{1}{2}$. Hence $g(x) = \sqrt{x}$ and $f(x) = \frac{x}{2} + c$ for some $c > 0$. It follows that $A_f = A$, $G_g = G$ and $H_{f,g} = H$. □

In this context a natural and open question arises:

PROBLEM 1. *Is it generally true that the generalized harmonic mean $H_{f,g}$ is symmetric if, and only if, $H_{f,g} = H$, $A_f = A$ and $G_g = G$?*

The above two propositions seem to suggest that the answer is affirmative. In this case it would strengthen the central position of the classical means in the rich family of means, and the very special role of the Pythagorean harmony proportion identity.

REMARK 7. The mean $H_{f,g}$ is symmetric iff

$$\frac{g(A_f(x,y))}{A_f(x,y)} \frac{g(x)}{g(y)} y = \frac{g(A_f(y,x))}{A_f(y,x)} \frac{g(y)}{g(x)} x, \quad x, y \in I,$$

or, equivalently, iff

$$A_f(y,x) g(A_f(x,y)) [g(x)]^2 y = A_f(x,y) g(A_f(y,x)) [g(y)]^2 x, \quad x, y \in I. \tag{5}$$

Assuming that f, g are three times differentiable, taking derivative of both sides in x, and then setting $y = x$, we get

$$[g(x)]^2 x \{x g'(x) - 2f'(x) [g(x) - x g'(x)]\} = 0, \quad x \in I,$$

whence

$$2f'(x) [g(x) - x g'(x)] = x g'(x), \quad x \in I.$$

If $g(x) - x g'(x) = 0$ then $x g'(x) = 0$ and, consequently, we would have $g(x) = 0$, contradicting the assumption that $g(x)$ is positive for every $x \in I$. It follows that

$$f'(x) = \frac{1}{2} \frac{x g'(x)}{g(x) - x g'(x)}, \quad x \in I,$$

and, obviously,

$$f''(x) = \frac{1}{2} \frac{g'(x) [g(x) - x g'(x) + x g''(x)]}{[g(x) - x g'(x)]^2}, \quad x \in I.$$

Differentiating both sides of (5) in x and y, and then setting $y = x$ gives no information about f and g. It turns out that the additional differentiation (with respect to x or y), setting $y = x$ and applications of the above formulas for f' and f'' lead to a rather complicated ordinary differential equation of the third order for the function g.

6. Harmonically complementary generalized harmonic means

For a real one-to-one and continuous function φ defined in an interval I and $t \in (0,1)$, let $M_t^{[\varphi]} \colon I^2 \to I$ be the weighted quasi-arithmetic mean given by

$$M_t^{[\varphi]}(x,y) := \varphi^{-1}\left(t\varphi(x) + (1-t)\varphi(y)\right), \quad x, y \in I,$$

(for this and other classes of means see, for instance, [2]). Of course, $M_t^{[\varphi]}$ is symmetric iff $t = \frac{1}{2}$, that is iff $M_t^{[\varphi]} = M^{[\varphi]}$, where

$$M^{[\varphi]}(x,y) := \varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right), \quad x, y \in I.$$

Note the following easy to verify

Remark 8. For every $t \in (0,1)$, the quasi-arithmetic mean $M^{[\varphi]}$ is invariant with respect to the weighted quasi-arithmetic mean-type mapping $\left(M_t^{[\varphi]}, M_{1-t}^{[\varphi]}\right)$, i.e.

$$M^{[\varphi]} \circ \left(M_t^{[\varphi]}, M_{1-t}^{[\varphi]}\right) = M^{[\varphi]}.$$

In particular, taking $I = (0,\infty)$ and $\varphi(x) := \frac{1}{x}$ for all $x \in I$, we have

$$H \circ (H_t, H_{1-t}) = H,$$

where

$$H_t(x,y) = \frac{xy}{(1-t)x + ty}, \quad H(x,y) = \frac{2xy}{x+y}, \quad x, y > 0,$$

are, respectively, the weighted harmonic mean and symmetric harmonic mean. Thus the harmonic weighted means H_t and H_{1-t} are *complementary* with respect to the harmonic mean H (cf. [5]).

In this connection, taking into account the key role played by the invariant means in effective finding the limits of the iterates of the mean-type mappings, as well as the considerations in the previous section, the following crucial question arises.

Given an interval $I \subset (0,\infty)$, determine all functions $f, g, \varphi, \psi : I \to (0,\infty)$, satisfying the suitable conditions of Theorem 1, such that

$$H \circ (H_{f,g}, H_{\varphi,\psi}) = H,$$

i.e. such that the generalized weighted means $H_{f,g}$ and $H_{\varphi,\psi}$ are complementary with respect to the harmonic mean H.

Assume that $H_{f,g}$ is a mean. Since H is continuous, symmetric and increasing in each variable, there exists a unique mean $N\colon I^2 \to I$ such that $H \circ (H_{f,g}, N) = H$ (see Remark 1 in [5]). So the question is if there are φ and ψ such that $N = H_{\varphi,\psi}$. Note that this equality is equivalent to the functional equation

$$(x+y)\, H_{f,g}(x,y)\, H_{\varphi,\psi}(x,y) = xy \left[H_{f,g}(x,y) + H_{\varphi,\psi}(x,y)\right], \quad x, y \in I.$$

7. An application

THEOREM 2. *Let $I \subset (0,\infty)$ be an interval. Assume that the functions $f\colon I \to \mathbb{R}$, $g\colon I \to (0,\infty)$ are such that f, $\mathrm{id}|_I - f$, g and $\frac{\mathrm{id}|_I}{g}$ are strictly increasing, and the function*

$$I^2 \ni (x,y) \longmapsto \frac{g\left(A_f(x,y)\right)}{A_f(x,y)} \frac{g(x)}{g(y)} y$$

is increasing with respect to each variable. Then:

(i) *A function $\Phi\colon I^2 \to \mathbb{R}$, continuous on the diagonal $\{(x,x) : x \in I\}$, satisfies the functional equation*

$$\Phi\left(H_{f,g}(x,y), A_f(x,y)\right) = \Phi(x,y), \quad x, y \in I, \tag{7.1}$$

if and only if there is a single-variable continuous function $\varphi\colon I \to \mathbb{R}$ such that

$$\Phi = \varphi \circ G_g. \tag{7.2}$$

(ii) *A mean* $M\colon I^2 \to I$ *satisfies the functional equation*

$$M\left(H_{f,g}\left(x,y\right), A_f\left(x,y\right)\right) = M\left(x,y\right), \quad x,y \in I,$$

if and only if

$$M = G_g.$$

Proof. (i) Assume that a continuous on the diagonal $\{(x,x) : x \in I\}$ function $\Phi\colon I^2 \to \mathbb{R}$ satisfies equation (7.1), and define $\varphi\colon I \to \mathbb{R}$ by

$$\varphi\left(u\right) := \Phi\left(u,u\right), \quad u \in I. \tag{7.3}$$

From (7.1), by induction, we get

$$\Phi\left(x,y\right) = \left(\Phi \circ \left(H_{f,g}, A_f\right)^n\right)\left(x,y\right), \quad x,y \in I,\ n \in \mathbb{N}, \tag{7.4}$$

where $\left(H_{f,g}, A_f\right)^n$ denotes the nth iterate of the mean-type mapping $\left(H_{f,g}, A_f\right)$. In view of Theorem 1 we have

$$\lim_{n\to\infty} \left(H_{f,g}, A_f\right)^n\left(x,y\right) = \left(G_g\left(x,y\right), G_g\left(x,y\right)\right), \quad x,y \in I.$$

Since for every $x,y \in I$, the point $\left(G_g\left(x,y\right), G_g\left(x,y\right)\right)$ belongs to the diagonal $\Delta := \{(x,x) : x \in I\}$, and the function Φ is continuous on Δ, it follows from (7.4) and (7.3) that

$$\begin{aligned}\Phi\left(x,y\right) &= \lim_{n\to\infty} \left(\Phi \circ \left(H_{f,g}, A_f\right)^n\right)\left(x,y\right)\\ &= \Phi\left(\left(G_g\left(x,y\right), G_g\left(x,y\right)\right)\right) = \varphi\left(G_g\left(x,y\right)\right),\end{aligned}$$

which proves (7.2).

To prove the converse implication, assume that there is a function $\varphi\colon I \to \mathbb{R}$ such that (7.2) holds. Then, by Theorem 1(iii), the mean G_g is invariant with respect to the mean-type mapping $\left(H_{f,g}, A_f\right)$, and, for all $x,y \in I$,

$$\Phi\left(x,y\right) = \varphi\circ G_g\left(x,y\right) = \varphi\circ\left(G_g\left(H_{f,g}, A_f\right)\right)\left(x,y\right) = \Phi\left(H_{f,g}\left(x,y\right), A_f\left(x,y\right)\right),$$

which proves that Φ satisfies functional equation (7.1).

(ii) Since every mean is reflexive and continuous on the diagonal, the result follows from (i). $\square$

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