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CONVERGENCE IN MEASURE AND IN CATEGORY

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Dedicated to my friend Professor Zygfryd Kominek

Abstract. W. Orlicz in 1951 has observed that if $\{f_n(\cdot, y)\}_{n\in N}$ converges in measure to $f(\cdot, y)$ for each $y \in [0, 1]$, then $\{f_n\}_{n\in N}$ converges in measure to f on $[0, 1] \times [0, 1]$. The situation is different for the convergence in category even if we assume the convergence in category of sequences $\{f_n(\cdot, y)\}_{n\in N}$ for each $y \in [0, 1]$ and $\{f_n(x, \cdot)\}_{n\in N}$ for each $x \in [0, 1]$.

Recall that a sequence $\{f_n\}_{n\in N}$ of measurable real-valued functions defined on $[0, 1]$ ($[0, 1]\times[0, 1]$, respectively) converges in measure to $f\colon [0, 1] \to \mathbb{R}$ ($f\colon [0, 1] \times [0, 1] \to \mathbb{R}$, respectively) if and only if for each increasing sequence $\{n_m\}_{m\in N}$ of positive integers there exists a subsequence $\{n_{m_p}\}_{p\in N}$ such that $f_{n_{m_p}} \xrightarrow[p\to\infty]{} f$ almost everywhere in $[0, 1]$ ($[0, 1] \times [0, 1]$, respectively). This characterization is due to F. Riesz (see, for example [1, Th.9.2.1, p.226]). Following Wagner [5] we say that a sequence $\{f_n\}_{n\in N}$ of real-valued functions defined on $[0, 1]$ ($[0, 1] \times [0, 1]$, resp.) having the Baire property converges in category to $f\colon [0, 1] \to \mathbb{R}$ ($f\colon [0, 1] \times [0, 1] \to \mathbb{R}$, resp.) if and only if for each increasing sequence $\{n_m\}_{m\in N}$ of positive integers there exists a subsequence $\{n_{m_p}\}_{m\in N}$ such that $f_{n_{m_p}} \xrightarrow[p\to\infty]{} f$ except on a set of the first category on the real line (on the plane, resp.).

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It is well known that if $\{f_n\}_{n\in N}$ is a sequence of real-valued measurable functions defined on the unit square, then the set of convergence of $\{f_n\}_{n\in N}$, i.e. the set $A = \{(x,y)\colon \lim\limits_{n\to\infty} f_n(x,y) \text{ exists}\}$ is Lebesgue measurable. Hence if for each $y \in [0,1]$ a sequence $\{f_n(\cdot,y)\}_{n\in N}$ converges almost everywhere on $[0,1]$, then from Fubini theorem it follows immediately that $\{f_n\}_{n\in N}$ converges almost everywhere on $[0,1]\times[0,1]$.

Similarly, if $\{f_n\}_{n\in N}$ is a sequence of real-valued functions defined on the unit square and having the Baire property, then the set of convergence of $\{f_n\}_{n\in N}$ has the Baire property. So if for each $y \in [0,1]$ a sequence $\{f_n(\cdot,y)\}_{n\in N}$ converges I-almost everywhere on $[0,1]$ (which means except on a set of the first category), then from Kuratowski-Ulam theorem ([4, p.56]) it follows that $\{f_n\}_{n\in N}$ converges I-almost everywhere on $[0,1]\times[0,1]$.

When we are dealing with the convergence everywhere except on a set belonging to the σ-ideal of small sets (sets of measure zero, sets of the first category) the behaviour of measurable functions and functions having the Baire property is similar. Below we show that the situation is quite different for convergence in measure and in category.

W. Orlicz in [3] has proved the following theorem showing the relation between convergence in measure of a sequence of functions of two variables and convergence in measure of its sections.

Theorem (in original from). *Let Q be the Cartesian product of two bounded sets A and B of positive measure, and let the functions $f_i(x,y)$ be measurable in Q. If for every $x \in A$ $f_i(x,y) \xrightarrow[B]{as} f(x,y)$, then the sequence $f_i(x,y)$ converges asymptotically in the set Q to a function $\bar{f}(x,y)$ which, for almost every $x \in A$, is equal to $f(x,y)$ almost everywhere in B.*

If $f_i(x,y) \xrightarrow[Q]{as} f(x,y)$, then there exists a sequence $\{i_k\}$ of indices such that $f_{i_k}(x,y) \xrightarrow[B]{as} f(x,y)$ almost everywhere in A.

In the above theorem asymptotic convergence means the convergence in measure.

The proof makes an essential use of the fact that (all functions defined on B) if $\varrho(f,g) = \int\limits_B \frac{|f-g|}{1+|f-g|}dx$, then $\varrho(f_n,f) \xrightarrow[n\to\infty]{} 0$ if and only if $f_n \xrightarrow[n\to\infty]{} f$ in measure (see, for example [2, p.183]). Observe also that if $\{f_n\}_{n\in N}$ fulfills the assumption of the theorem, then also for almost each $y \in [0,1]$ the sequence $\{f_i(\cdot,y)\}_{i\in N}$ converges in measure to the function equivalent to $\bar{f}$. It follows immediately from the theorem of Vitali concerning double and iterated integrals.

In the case of functions having the Baire property the situation is quite different.

THEOREM 1. *There exists a sequence of real functions having the Baire property $\{f_n\}_{n\in N}$ defined on the unit square such that for each $y \in [0,1]$ the sequence $\{f_n(\cdot,y)\}_{n\in N}$ converges in category to* 0 *but $\{f_n\}_{n\in N}$ does not converge in category.*

PROOF. For $n \in N$ let

$$I_i^n = \left(\frac{i}{n}, \frac{i+1}{n}\right) \quad \text{for } i \in \{0,1,\ldots,n-1\} \text{ and}$$

$$J_i^n = \left(\frac{i}{n^2}, \frac{i+1}{n^2}\right) \quad \text{for } i \in \{0,1,\ldots,n^2-1\}.$$

Put $A_n = \bigcup_{i=0}^{n-1} (I_i^n \times \bigcup_{j=0}^{n-1} J_{j\cdot n+i}^n)$ and $f_n = \mathcal{X}_{A_n}$ for $n \in N$.

If $y \in [0,1]$, then $f_n(\cdot,y)$ is either equal to 0 or is a characteristic function of the interval of length $\frac{1}{n}$, so for each $y \in [0,1]$ the sequence $\{f_n(\cdot,y)\}_{n\in N}$ converges to 0 in category, because from each subsequence one can choose the subsequence convergent to 0 everywhere or except on one-point set.

Observe now that for each increasing sequence $\{n_m\}_{m\in N}$ of positive integers and for each $p \in N$ both sets $\bigcup_{m=p}^{\infty} A_{n_m}$ and $\bigcup_{m=p}^{\infty} (([0,1]\times[0,1]) \setminus A_{n_m})$ include open dense sets, from which it easily follows that $\{f_n\}_{n\in N}$ does not converge in category to any function. □

Observe also that if $x \in [0,1]$ is irrational, then for $n \in N$ the function $f_n(x,\cdot)$ is a characteristic function of the union of n open internals equidistributed on $[0,1]$. If we denote for such x $B_n^x = \{y\colon f_n(x,y) = 1\}$, then for each increasing sequence $\{n_m\}_{m\in N}$ and for each $p \in N$ we see that $\bigcup_{m=p}^{\infty} B_{n_m}^x$ and $\bigcup_{m=p}^{\infty} [0,1] \setminus B_{n_m}^x$ include open dense sets, so $\{f_n(x,\cdot)\}_{n\in N}$ does not converge in category to any function for irrational $x \in [0,1]$. The next theorem will show that even if the sequence $\{f_n\}_{n\in N}$ converges in category to 0 on each segment connecting points of the boundary of $[0,1]\times[0,1]$ (with respect to the topology of the segment), then it can happen that $\{f_n\}_{n\in N}$ does not converge in category.

THEOREM 2. *There exists a sequence $\{g_n\}_{n\in N}$ of real functions having the Baire property defined on the unit square such that for each segment D connecting points of the boundary of $[0,1]\times[0,1]$ the sequence $\{g_n|D\}_{n\in N}$ (treated as the sequence of functions of one variable) consists of functions* *having the Baire property on D and converges in category to* 0 *but* $\{g_n\}_{n\in N}$ *does not converges in category.*

In the proof of Theorem 2 we shall need the following lemma, in which $K(x,r) = \{p\colon d(p,x) \leqslant r\}$:

LEMMA. *If $E \subset [0,1] \times [0,1]$ is a finite set such that no three points of E are collinear, then there exists $k \in N$ such that each segment connecting points of the boundary of* $[0,1]\times[0,1]$ *has points in common with at most two circles from the family $\mathcal{E}_k = \{\{K(x,k^{-1})\}\colon x \in E\}$.*

PROOF. Suppose that this not the case, so for each $k \in N$ there exists a segment D_k connecting points of the boundary of $[0,1]\times[0,1]$ and intersecting at least three discs belonging to $\mathcal{E}_k$. Then there exist three different points $p_1, p_2, p_3 \in E$ and an increasing sequence $\{k_m\}_{m\in N}$ of positive integers such that $D_{k_m} \cap K(\pi_i, k_m^{-1}) \neq 0$ for $i = 1,2,3$ and for each $m \in N$. Observe that $D_{k_m} \xrightarrow[m\to\infty]{} D_0$ in Hausdorff metric in $\mathbb{R}^2$, where D_0 is a segment connecting points of the boundary of $[0,1]\times[0,1]$, and then $p_1, p_2, p_3 \in D_0$, (are collinear) – a contradiction. $\square$

Let's return to the

PROOF OF THEOREM 2. Let $E_n = \{p_0, p_1, \ldots, p_{n^2-1}\} \subset [0,1] \times [0,1]$ be a set such that card $E_n \cap ((\frac{i}{n}, \frac{i+1}{n}) \times (\frac{j}{n}, \frac{j+1}{n})) = 1$ for each $i, j \in 0, 1, \ldots, n-1$ and no three points of E_n are collinear.

Let m_n (for each $n \in N$) be a number described in the Lemma. Obviously $m_n \xrightarrow[n\to\infty]{} \infty$. Put $A_n = \bigcup_{i=0}^{n^2-1} K(p_i, m_n^{-1})$ and $g_n = \mathcal{X}_{A_n}$.

If D is an arbitrary segment connecting points of the boundary of the unit square, then $g_n|D$ is equal to 0 or is a characteristic function of the set consisting of one or two intervals, each of the length less than $2 \cdot m_m^{-1}$, so $\{g_n|D\}_{n\in N}$ converges in category to 0.

The proof that $\{g_n\}_{n\in N}$ does not converge in category is similar to that for $\{f_n\}_{n\in N}$ from Theorem 1. $\square$

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