

STABILITY OF FUNCTIONAL EQUATIONS AND PROPERTIES OF GROUPS

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Abstract. Investigating Hyers–Ulam stability of the additive Cauchy equation with domain in a group G , in order to obtain an additive function approximating the given almost additive one we need some properties of G , starting from commutativity to others more sophisticated. The aim of this survey is to present these properties and compare, as far as possible, the classes of groups involved.

1. Introduction

We begin by considering the equation of homomorphisms (Cauchy additive equation) from a group G to \mathbb{R} :

$$g(xy) = g(x) + g(y);$$

note that assuming that the range of g is in \mathbb{R} instead of a general Banach space is not a restriction, as proved in [5]. It is well known the following theorem which originates from the very first result of Hyers (see [5] and [10]):

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THEOREM 1.1. *Let G be a group and let $f: G \rightarrow \mathbb{R}$ be a function such that for some $K \geq 0$*

$$(1.1) \quad |f(xy) - f(x) - f(y)| \leq K \quad \text{for all } x, y \in G.$$

Then, for every $x \in G$, the limit

$$g(x) = \lim_{n \rightarrow \infty} \frac{f(x^{2^n})}{2^n}$$

exists, the function g is a solution of the functional equation

$$(1.2) \quad g(x^2) = 2g(x) \quad \text{for all } x \in G$$

and satisfies the inequality

$$(1.3) \quad |f(x) - g(x)| \leq K.$$

Moreover g is the unique function satisfying equation (1.2) and inequality (1.3).

In order to get Hyers–Ulam stability of the equation of homomorphisms, we should prove that g is additive, i.e.,

$$g(xy) = g(x) + g(y)$$

for all $x, y \in G$.

The additivity of g depends on properties of the group G . In the following we intend to deal with this problem, presenting various results and open problems.

A complete characterization of the groups where the Cauchy equation is stable has been obtained by Roman Badora ([1]).

DEFINITION 1.2. We say that a group G belongs to the class \mathcal{G} if for every subadditive functional $p: G \rightarrow \mathbb{R}$, i.e.,

$$p(xy) \leq p(x) + p(y), \quad x, y \in G,$$

there exists an additive function $a: G \rightarrow \mathbb{R}$ such that

$$a(x) \leq p(x), \quad x \in G.$$

Badora's result reads as follows:

THEOREM 1.3. *The function g is additive if and only if $G \in \mathcal{G}$.*

We immediately meet a problem: the definition of the class \mathcal{G} is not directly related to “algebraic” properties of their groups hence it is not immediately understandable which known classes of groups belong to it.

Another simple condition yielding the additivity of the function g is given by the following:

THEOREM 1.4. *The function g given in (1.2) is additive if and only if for every $x, y \in G$ we have that the related function f satisfies the condition*

$$(1.4) \quad \lim_{n \rightarrow \infty} 2^{-n} [f(x^{2^n} y^{2^n}) - f([xy]^{2^n})] = 0.$$

PROOF. Assume g is additive. We have

$$\begin{aligned} |f(x^{2^n} y^{2^n}) - f([xy]^{2^n})| &\leq |f(x^{2^n} y^{2^n}) - f(x^{2^n}) - f(y^{2^n})| \\ &\quad + |f([xy]^{2^n}) - f(x^{2^n}) - f(y^{2^n})| \\ &\leq K + |f([xy]^{2^n}) - f(x^{2^n}) - f(y^{2^n})|. \end{aligned}$$

If we divide by 2^n and take the limit, the right-hand side goes to $|g(xy) - g(x) - g(y)| = 0$, thus the property is proved.

Assume now that (1.4) holds. Setting in (1.1) x^{2^n} and y^{2^n} instead of x and y , dividing by 2^n and taking the limit, we obtain

$$\lim_{n \rightarrow \infty} 2^{-n} f(x^{2^n} y^{2^n}) = g(x) + g(y).$$

By the hypothesis (1.4)

$$\lim_{n \rightarrow \infty} 2^{-n} f(x^{2^n} y^{2^n}) = \lim_{n \rightarrow \infty} 2^{-n} f([xy]^{2^n}) = g(xy). \quad \square$$

Also this condition is not completely satisfactory since it refers to the function f and not directly to the group G , however we will see that from it we can deduce some useful properties of the group G . First, we note the obvious fact that in Abelian groups condition (1.4) is satisfied.

Note that condition (1.4) is equivalent to the following one: there exists a subsequence $\{m(n)\}$ of \mathbb{N} such that

$$(1.5) \quad \lim_{n \rightarrow \infty} 2^{-m(n)} [f(x^{2^{m(n)}} y^{2^{m(n)}}) - f([xy]^{2^{m(n)}})] = 0.$$

This depends on the fact that both limits

$$\lim_{n \rightarrow \infty} 2^{-n} f(x^{2^n} y^{2^n}) \quad \text{and} \quad \lim_{n \rightarrow \infty} 2^{-n} f([xy]^{2^n})$$

exist and are finite.

Another important and long studied functional equation is the quadratic one:

$$(1.6) \quad q(xy) + q(xy^{-1}) = 2q(x) + 2q(y),$$

where $q: G \rightarrow \mathbb{R}$ and G is a group.

As for the additive equation, we start from a function $f: G \rightarrow \mathbb{R}$ satisfying the inequality

$$(1.7) \quad |f(xy) + f(xy^{-1}) - 2f(x) - 2f(y)| \leq H$$

for all $x, y \in G$ and for some $H \geq 0$,

and we obtain the following (see, for instance, [20]):

THEOREM 1.5. *Let G be a group and let $f: G \rightarrow \mathbb{R}$ be a function satisfying inequality (1.7). Then, for every $x \in G$, the limit*

$$q(x) = \lim_{n \rightarrow \infty} \frac{f(x^{2^n})}{2^{2n}}$$

exists, the function q satisfies the conditions

$$(1.8) \quad q(x^2) = 4q(x), \quad |f(x) - q(x)| \leq H/2 \quad \text{for all } x \in G.$$

Moreover q is the unique function satisfying the functional equation and the inequality in (1.8).

Again, we are requested to find conditions on the group G in order that the function q be a quadratic function, i.e., satisfy (1.6). We have the following simple analogous result as in the case of the Cauchy equation.

THEOREM 1.6. *The function q is quadratic if and only if for every $x, y \in G$ we have*

$$(1.9) \quad \lim_{n \rightarrow \infty} 2^{-2n} [f(x^{2^n} y^{2^n}) + f(x^{2^n} y^{-2^n}) - f([xy]^{2^n}) - f([xy^{-1}]^{2^n})] = 0.$$

Again condition (1.9) is obviously satisfied if G is Abelian; moreover, it is equivalent to the following: there exists a subsequence $\{m(n)\}$ of \mathbb{N} such that

$$\lim_{n \rightarrow \infty} \frac{f(x^{2^{m(n)}} y^{2^{m(n)}}) + f(x^{2^{m(n)}} y^{-2^{m(n)}}) - f([xy]^{2^{m(n)}}) - f([xy^{-1}]^{2^{m(n)}})}{2^{-2m(n)}} = 0.$$

2. Amenability and invariant mean method

László Székelyhidi in a Remark given during the 22nd International Symposium on Functional Equations ([22]) replaced Hyers' proof on the stability of the additive Cauchy equation with a more general one using the notion of invariant mean.

Let G be a group and $B(G)$ be the space of all bounded complex-valued functions on G , equipped with the supremum norm $\|f\|_\infty$.

DEFINITION 2.1. A linear functional m on $B(G)$ is a *left (right) invariant mean* if:

- (α) $m(\bar{f}) = \overline{m(f)}$, for each $f \in B(G)$;
- (β) $\inf_x \{f(x)\} \leq m(f) \leq \sup_x \{f(x)\}$, for all real valued $f \in B(G)$;
- (γ) $m({}_x f) = m(f)$ ($m(f_x) = m(f)$), for all $x \in G$ and $f \in B(G)$, where ${}_x f(t) = f(xt)$ ($f_x(t) = f(tx)$).

The mean is two sided invariant if it is both left and right invariant.

Condition (β) is equivalent to $m(f) \geq 0$ if $f \geq 0$, and $m(1) = 1$, hence $\|m\| = 1$.

DEFINITION 2.2. A group G is *amenable* if it has a (two sided) invariant mean.

How to ascertain if a group G is amenable? The most important and used condition is due to Jaques Dixmier ([4]):

THEOREM 2.3. *The existence of an invariant mean on $B(G)$ is equivalent to the following: if $\{f_1, \dots, f_n\}$ are real valued functions in $B(G)$ and if $\{x_1, \dots, x_n\}$ are elements of G , then*

$$\sup_y \sum_{i=1}^n \{f_i(x_i y) - f_i(y)\} \geq 0.$$

As for Badora's class \mathcal{G} we have a condition which is not directly algebraic. In any case it is relatively easy to prove that Abelian groups and finite groups are amenable (see [9]).

If G is amenable and m is a left-invariant mean on $B(G)$, we may define a left-invariant finitely additive measure μ on the family $\mathcal{P}(G)$ of all subsets of G as follows: $\mu(E) = m(\chi_E)$, where χ_E is the characteristic function of $E \in \mathcal{P}(G)$.

We now prove that such a measure does not exist on the free group on two generators $F(a, b)$. Divide $F(a, b)$ into disjoint sets $\{H_i : i \in \mathbb{Z}\}$, where

$x \in H_i$ if and only if x expressed as a reduced word has the form

$$x = a^i b^{i_1} a^{i_2} \cdots, \quad i_1 \neq 0 \quad \text{if} \quad x \neq a^i.$$

Then, $\lambda_a : x \mapsto ax$ maps H_i to H_{i+1} , for all $i \in \mathbb{Z}$, whereas $\lambda_b : x \mapsto bx$ maps every set H_i , $i \neq 0$, into H_0 . If a left-invariant measure μ existed, then $\mu(H_i) = 0$ for all $i \in \mathbb{Z}$, and

$$\mu(H_0) \geq \mu\left(\bigcup_{i \neq 0} H_i\right), \quad \text{while} \quad \mu(H_0) + \mu\left(\bigcup_{i \neq 0} H_i\right) = \mu(G) = 1,$$

hence $\mu(H_0) \geq 1/2$, a contradiction.

Thus, the free group $F(a, b)$ is not amenable.

Since every subgroup of an amenable group is amenable, then if a group contains $F(a, b)$, it is not amenable.

A famous problem, the so-called Von Neumann–Day problem, asks if any non amenable group contains $F(a, b)$: the answer is negative and this have been proved in 1980 by Ol’shanskii ([14]).

Going back to stability, we prove here Székelyhidi’s theorem (see [5] and [22]).

THEOREM 2.4. *Let $f : G \rightarrow \mathbb{R}(\mathbb{C})$, G amenable group, and*

$$|f(xy) - f(x) - f(y)| \leq K \quad \text{for all} \quad x, y \in G.$$

Then there exists a unique homomorphism g such that

$$|f(x) - g(x)| \leq K \quad \text{for all} \quad x \in G.$$

PROOF. From $|f(xy) - f(x) - f(y)| \leq K$, we have that the function $y \mapsto f(xy) - f(y)$ is in $B(G)$ for each $x \in G$. Let m_y the left-invariant mean (on a function of y) and define

$$g(x) = m_y\{x f - f\}, \quad x \in G.$$

Then

$$\begin{aligned} g(xz) &= m_y\{xz f - f\} = m_y\{xz f - x f + x f - f\} \\ &= m_y\{xz f - x f\} + m_y\{x f - f\} \\ &= m_y\{z f - f\} + m_y\{x f - f\} = g(z) + g(x), \end{aligned}$$

so $g \in \text{Hom}(G, \mathbb{R})$. Moreover,

$$\begin{aligned} |g(x) - f(x)| &= |m_y\{x f - f\} - f(x)| = |m_y\{x f - f - f(x)\}| \\ &\leq \sup_{y \in G} |f(xy) - f(x) - f(y)| \leq K. \end{aligned} \quad \square$$

Following the ideas of Székelyhidi for the Cauchy equation, Dilian Yang in 2004 ([26]) was able to prove the following

THEOREM 2.5. *Let \mathcal{G} be an amenable group. Then the quadratic equation from G to \mathbb{R} is stable.*

A question immediately arises: does there exist any group where the Cauchy equation is not stable?

Since this group must be non amenable, the obvious idea is to look to $F(a, b)$. Indeed, consider the function $f: F(a, b) \rightarrow \mathbb{R}$ defined as follows:

$$f(x) = r(x) - s(x),$$

where $r(x)$ is the number of pairs ab , and $s(x)$ is the number of pairs $b^{-1}a^{-1}$ respectively in the reduced form of x . Anna Bahyrycz in [3] (see also [5]) proved in full details that

$$|f(xy) - f(x) - f(y)| \leq 1,$$

while for every homomorphism ϕ of G in \mathbb{R} , the difference $f - \phi$ is unbounded.

Also Yang considered this problem for the quadratic equation and proved that the function $f: F(a, b) \rightarrow \mathbb{R}$ given by $f(x) = r(x) + s(x)$ produces the searched example.

At this point we know that on $F(a, b)$ the additive equation is not stable and that there are non amenable groups which do not contain $F(a, b)$. So, we can ask whether for all groups containing $F(a, b)$ the additive equation is not stable; another weaker question may be the following: let $G \supset F(a, b)$, is it possible to extend the function f defined above on $F(a, b)$ to a function $\tilde{f}: G \rightarrow \mathbb{R}$ such that $|\tilde{f}(xy) - \tilde{f}(x) - \tilde{f}(y)|$ is bounded?

Frank Zorzitto and John Lawrence, in a private communication, proved that for both questions the answer is negative.

THEOREM 2.6. *There exists a group $G \supset F(a, b)$ such that any extension of the function f defined above on $F(a, b)$ has unbounded Cauchy difference.*

PROOF. Let G be the group whose presentation is

$$G = \langle a, b, c : c^{-1}(ab)^2c = (ab)^3 \rangle .$$

By the *Freiheitssatz* theorem (see [12, p. 252, Theorem 4.10]), the subgroup generated by a and b is free: let it be $F(a, b)$.

The function f on $F(a, b)$ defined above gives

$$f((ab)^n) = n, \quad f((ab)^{-n}) = -n, \quad n = 1, 2, \dots$$

Suppose there exists an extension $\tilde{f} : G \rightarrow \mathbb{R}$ of f such that

$$|\tilde{f}(xy) - \tilde{f}(x) - \tilde{f}(y)| \leq \delta,$$

for all $x, y \in G$ and some positive δ . Then, for $x, y, w, z \in G$ we have

$$\begin{aligned} |\tilde{f}(xywz) - \tilde{f}(x) - \tilde{f}(y) - \tilde{f}(w) - \tilde{f}(z)| &\leq |\tilde{f}(xywz) - \tilde{f}(xy) - \tilde{f}(wz)| \\ &\quad + |\tilde{f}(xy) + \tilde{f}(wz) - \tilde{f}(x) - \tilde{f}(y) - \tilde{f}(w) - \tilde{f}(z)| \\ &\leq |\tilde{f}(xywz) - \tilde{f}(xy) - \tilde{f}(wz)| + |\tilde{f}(xy) - \tilde{f}(x) - \tilde{f}(y)| \\ &\quad + |\tilde{f}(wz) - \tilde{f}(w) - \tilde{f}(z)| \leq 3\delta. \end{aligned}$$

In G for every positive integer n we have the following identities:

$$c^{-1}(ab)^{2n}c = (ab)^{3n}, \quad \text{or} \quad c^{-1}(ab)^{2n}c(ab)^{-3n} = e.$$

By combining this identity with the previous inequality we get:

$$|\tilde{f}(e) - \tilde{f}(c^{-1}) - \tilde{f}((ab)^{2n}) - \tilde{f}(c) - \tilde{f}((ab)^{-3n})| \leq \delta.$$

However, $\tilde{f}((ab)^{2n}) = f((ab)^{2n}) = 2n$ and $\tilde{f}((ab)^{-3n}) = f((ab)^{-3n}) = -3n$, hence

$$|\tilde{f}(e) - \tilde{f}(c^{-1}) - 2n - \tilde{f}(c) + 3n| = |\tilde{f}(e) - \tilde{f}(c^{-1}) - \tilde{f}(c) + n| \leq 3\delta$$

for every positive integer n : a contradiction. \square

THEOREM 2.7. *Any torsion free group H can be embedded into a group G such that the Cauchy equation is stable on G .*

PROOF. By using HNN-extensions ([12]) the group H can be embedded into a group G such that

$$\forall x \in G \exists y \in G : y^{-1}xy = x^2.$$

Let $f : G \rightarrow \mathbb{R}$ be such that

$$|f(xy) - f(x) - f(y)| \leq \delta$$

for all $x, y \in G$ and some nonnegative δ . If $\delta = 0$ we are done. Assume $\delta > 0$, we will prove that f has to be bounded, more precisely $|f(x)| < 2\delta$. Assume, on the contrary, that, for some $x \in G$, $|f(x)| \geq 2\delta$. From

$$|f(x^2) - 2f(x)| \leq \delta$$

it follows that

$$|2f(x)| - \delta \leq |f(x^2)|$$

so $|f(x^2)| \geq 3\delta$. From

$$|f(x) + f(x^2) - f(x^3)| \leq \delta$$

we get

$$|f(x)| + |f(x^2)| - \delta \leq |f(x^3)|,$$

hence $|f(x^3)| \geq 4\delta$. Repeating this procedure we obtain that

$$|f(x^n)| \geq (n+1)\delta, \quad n = 1, 2, \dots,$$

thus $f(x^n)$ is not bounded as n varies.

Now, take $y \in G$ such that $y^{-1}xy = x^2$, then for all positive integers n we have $y^{-1}x^ny = x^{2n}$. For each n ,

$$|f(x^{2n}) - 2f(x^n)| = |f(y^{-1}x^ny) - 2f(x^n)| \leq \delta.$$

But

$$|f(y^{-1}x^ny) - f(y^{-1}) - f(x^n) - f(y)| \leq 2\delta,$$

hence

$$|f(y^{-1}x^ny) - f(x^n)| \leq 2\delta + |f(y^{-1})| + |f(y)|.$$

Using the last two inequalities we obtain

$$|f(x^n)| \leq 3\delta + |f(y^{-1})| + |f(y)|, \quad n = 1, 2, \dots,$$

a contradiction. □

We have seen that amenability is an effective tool for proving stability for additive and quadratic equation. Clearly the possibility of using amenability is strictly connected with the structure of the functional equation involved. Other results using amenability are about polynomial equation ([21]), Drygas equation (see [7] and [25]) and, in a more sophisticated form, about Levi-Civita equation ([18, 19]).

As we have seen the amenability of the domain group implies the stability of the additive equation, while there are non amenable groups where the same equation is not stable and others where it is stable, thus stability is a property weaker than amenability.

Can we recover amenability by using stability in a certain stronger form? An answer is given by the following theorem ([5]), where by $\mathcal{C}f(x, y)$ we denote the Cauchy difference $f(xy) - f(x) - f(y)$:

THEOREM 2.8. *A group G is amenable if and only if for every n -tuple $f_1, f_2, \dots, f_n: G \rightarrow \mathbb{R}$, with*

$$|f_i(xy) - f_i(x) - f_i(y)| \leq K_i, \quad i = 1, \dots, n,$$

there exist n homomorphisms $\phi_i: G \rightarrow \mathbb{R}$ such that $f_i - \phi_i$, $i = 1, \dots, n$, are bounded and for all $x_1, x_2, \dots, x_n \in G$ the inequality

$$\inf_{y \in G} \sum_{i=1}^n \mathcal{C}f_i(x_i, y) \leq \sum_{i=1}^n \{\phi_i(x_i) - f_i(x_i)\} \leq \sup_{y \in G} \sum_{i=1}^n \mathcal{C}f_i(x_i, y).$$

PROOF. Suppose G amenable and let m be an invariant mean on $B(G)$. If $|f_i(xy) - f_i(x) - f_i(y)| \leq K_i$, then we set $\phi_i(x) = m\{x f_i - f_i\}$ and we know that $\phi_i \in \text{Hom}(G, \mathbb{R})$ and $\phi_i - f_i$ is bounded. Fix now $x_1, x_2, \dots, x_n \in G$, by the properties of m we have:

$$\inf_{y \in G} \sum_{i=1}^n \{f_i(x_i y) - f_i(y)\} \leq m\left\{ \sum_{i=1}^n \{x_i f_i - f_i\} \right\} \leq \sup_{y \in G} \sum_{i=1}^n \{f_i(x_i y) - f_i(y)\},$$

hence, by subtracting all $f_i(x_i)$, we obtain

$$\inf_{y \in G} \sum_{i=1}^n C f_i(x_i, y) \leq \sum_{i=1}^n \{\phi_i(x_i) - f_i(x_i)\} \leq \sup_{y \in G} \sum_{i=1}^n C f_i(x_i, y).$$

Conversely, let $f_1, f_2, \dots, f_n \in B(G)$, then obviously $|f_i(xy) - f_i(x) - f_i(y)| \leq K_i$ for each $i = 1, \dots, n$ and some nonnegative K_i 's, and the corresponding homomorphisms (existing by hypothesis) ϕ_i are equal to zero. Now, we show that the Dixmier inequality holds, i.e., that

$$\sup_{y \in G} \sum_{i=1}^n \{f_i(x_i y) - f_i(y)\} \geq 0.$$

If not, let

$$\sup_{y \in G} \sum_{i=1}^n \{f_i(x_i y) - f_i(y)\} = -\sigma < 0,$$

then

$$\sup_{y \in G} \sum_{i=1}^n \{f_i(x_i y) - f_i(x_i) - f_i(y)\} = -\sigma - \sum_{i=1}^n f_i(x_i)$$

and

$$-\sum_{i=1}^n f_i(x_i) \leq -\sigma - \sum_{i=1}^n f_i(x_i),$$

a contradiction. □

It is not at all immediate to use this last theorem for proving amenability, so an open problem of certain interest could be to elaborate some effective procedure to get the desired results.

3. Weak commutativity and stability

We come back to condition (1.4), i.e.,

$$\lim_{n \rightarrow \infty} 2^{-n} [f(x^{2^n} y^{2^n}) - f([xy]^{2^n})] = 0.$$

This condition is necessary and sufficient for stability of the Cauchy equation. This leads to the notion of the so-called *Tabor weak commutativity*.

DEFINITION 3.1. A group G is *Tabor weakly commutative* if

$$\forall x, y \in G \quad \exists n = n(x, y) \geq 2 : \quad x^{2^n} y^{2^n} = (xy)^{2^n}.$$

(Note that for $n = 1$ this is simply commutativity.)

Józef Tabor in 1984 ([23]) proved the following:

THEOREM 3.2. *Assume that the group G is Tabor weakly commutative, then the function g is additive.*

PROOF. By Theorem 1.4 and following remark, it is enough to construct a sequence $\{m(n)\}$ for which condition (1.5) is satisfied. To do this fix $x, y \in G$ and let $m_1 = n(x, y)$. Consider now the pair $(x^{2^{m_1}}, y^{2^{m_1}})$; by our hypothesis there exists $n(x^{2^{m_1}}, y^{2^{m_1}})$ such that

$$(x^{2^{m_1}} y^{2^{m_1}})^{2^{n(x^{2^{m_1}}, y^{2^{m_1}})}} = (x^{2^{m_1}})^{2^{n(x^{2^{m_1}}, y^{2^{m_1}})}} (y^{2^{m_1}})^{2^{n(x^{2^{m_1}}, y^{2^{m_1}})}}.$$

Since $(xy)^{2^{m_1}} = x^{2^{m_1}} y^{2^{m_1}}$, we obtain

$$(xy)^{2^{m_1+n(x^{2^{m_1}}, y^{2^{m_1}})}} = x^{2^{m_1+n(x^{2^{m_1}}, y^{2^{m_1}})}} y^{2^{m_1+n(x^{2^{m_1}}, y^{2^{m_1}})}}.$$

Setting $m_2 = m_1 + n(x^{2^{m_1}}, y^{2^{m_1}})$, we have

$$(xy)^{2^{m_2}} = x^{2^{m_2}} y^{2^{m_2}}.$$

By induction we construct the wanted sequence $\{m_n\}$. □

Which are the relations between amenability and Tabor weak commutativity? Z. Gajda and Z. Kominek in [8] proved that the group S of nonconstant affine maps of the plane, identified with $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}$ with the operation

$$(a, b) \cdot (c, d) = (ac, ad + b)$$

is not weakly commutative. Indeed, let $a > 1$ and consider the product $(1, a) \cdot (a, 0)$, we have

$$((1, a) \cdot (a, 0))^n = (a, a)^n = (a^n, a + a^2 + \cdots + a^n),$$

whereas

$$(1, a)^n \cdot (a, 0)^n = (1, na) \cdot (a^n, 0) = (a^n, na).$$

On the other hand, this group is amenable; to prove this it is enough to show that the subgroup of S given by $\{(1, b) : b \in \mathbb{R}\}$ is amenable and $S/\{(1, b) : b \in \mathbb{R}\}$ is amenable (see [9, Theorem 1.2.6]). Since $\{(1, b) : b \in \mathbb{R}\}$ is isomorphic to \mathbb{R} the first condition is satisfied. Let $(a, b) \in S$, then $(a, b)^{-1} = (a^{-1}, -ba^{-1})$;

$$\begin{aligned} (a, b)^{-1} \cdot (1, r) \cdot (a, b) &= (a^{-1}, -ba^{-1}) \cdot (1, r) \cdot (a, b) \\ &= (a^{-1}, a^{-1}r - a^{-1}b) \cdot (a, b) \\ &= (1, a^{-1}r + a^{-1}b - a^{-1}b) = (1, a^{-1}r), \end{aligned}$$

hence the subgroup $\{(1, b) : b \in \mathbb{R}\}$ is normal. By

$$(a, b) \cdot (1, r) = (a, ar + b), \quad (c, d) \cdot (1, r) = (c, cr + d),$$

we have

$$(a, ar + b) \cdot (c, cr + d) = (ac, acr + ad + ar + b)$$

and

$$(c, cr + d) \cdot (a, ar + b) = (ac, acr + cb + cr + d),$$

so these products belong to the same coset, i.e., $S/\{(1, b) : b \in \mathbb{R}\}$ is Abelian hence amenable.

Thus, we have shown that there exist amenable groups which are not weakly commutative. Conversely, let $B_r(n)$ be the so-called free Burnside group presented as follows:

$$B_r(n) = \langle a_1, \dots, a_r \mid \forall x : x^n = e \rangle,$$

(Another definition is $B_r(n) = F_r/F_r^n$, where F_r is the free group on r generators). It has been proved that for $r \geq 2$ and $n \geq 2^{48}$, n odd, the group $B_r(n)$ is infinite and non amenable (see [11, 13, 15, 16]). By a theorem proved in [24] by Toborg (see also [2]), since $B_r(n)$ is a torsion group and every element has odd order, then it is weakly commutative.

In order to introduce a different notion of weak commutativity we need some preliminary results.

Let $f: G \rightarrow \mathbb{R}$ be a function satisfying inequality (1.1). For $q \in \mathbb{N}$ we have the following

$$(3.1) \quad \left| \frac{f(x^q)}{q} - f(x) \right| \leq \frac{q-1}{q} K < K.$$

Indeed, it is true for $q = 1$ and assume valid for $q - 1$, then

$$\begin{aligned} |f(x^q) - qf(x)| &= |f(x^q) - f(x^{q-1}) - f(x) + f(x) + f(x^{q-1}) - qf(x)| \\ &\leq |f(x^q) - f(x^{q-1}) - f(x)| + |f(x^{q-1}) - (q-1)f(x)| \\ &\leq K + (q-2)K = (q-1)K. \end{aligned}$$

THEOREM 3.3. *Let $\{k_n\}$ be a sequence of integers greater than or equal to 2 and define $s_n := k_1 \cdot k_2 \cdots k_n$ and let f satisfy inequality (1.1). Then for each $x \in G$ the sequence $\{s_n^{-1}f(x^{s_n})\}$ is convergent.*

PROOF. We prove that $\{s_n^{-1}f(x^{s_n})\}$ is a Cauchy sequence. Fix $\varepsilon > 0$, $n > m$ and consider the difference

$$\frac{f(x^{s_n})}{s_n} - \frac{f(x^{s_m})}{s_m}.$$

By inequality (3.1), we have,

$$\left| \frac{f(x^{s_n})}{s_n} - \frac{f(x^{s_m})}{s_m} \right| = s_m^{-1} \left| \frac{s_m}{s_n} f(x^{s_n}) - f(x^{s_m}) \right| \leq s_m^{-1} K < \varepsilon$$

for $m > \frac{K}{\varepsilon}$.

□

Thanks to the previous theorem we can define

$$g(x) = \lim_{n \rightarrow \infty} \frac{f(x^{s_n})}{s_n},$$

moreover, by (3.1) we have

$$|g(x) - f(x)| \leq K.$$

In order to prove that g is additive, we assume the following property on the group G :

for each pair $x, y \in G$, $x \neq y$, there exists an integer $k = k(x, y) \geq 2$, such that

$$(3.2) \quad x^k y^k = (xy)^k,$$

we give to this property the name *Rätz weak commutativity* since it generalizes the condition presented by Jürg Rätz in [17].

Fix $x, y \in G$ and let k_1 be such that $x^{k_1} y^{k_1} = (xy)^{k_1}$, by (3.2) there exists $k_2 = k_2(x^{k_1}, y^{k_1})$ such that $(x^{k_1} y^{k_1})^{k_2} = x^{k_1 k_2} y^{k_1 k_2}$; by iterating this procedure we obtain a sequence $\{k_n\}$ such that

$$x^{k_1 k_2 \cdots k_n} y^{k_1 k_2 \cdots k_n} = (xy)^{k_1 k_2 \cdots k_n}, \quad n \in \mathbb{N}.$$

Thus, for $s_n = k_1 k_2 \cdots k_n$, we have

$$\begin{aligned} &= s_n^{-1} |f([xy]^{s_n}) - f(x^{s_n}) - f(y^{s_n})| \\ & \quad s_n^{-1} |f(x^{s_n} y^{s_n}) - f(x^{s_n}) - f(y^{s_n})| \leq s_n^{-1} K \end{aligned}$$

and taking the limit as $n \rightarrow \infty$, we have

$$g(xy) - g(x) - g(y) = 0$$

and, by uniqueness,

$$g(x) = \lim_{n \rightarrow \infty} \frac{f(x^{s_n})}{s_n} = \lim_{n \rightarrow \infty} \frac{f(x^{2^n})}{2^n}.$$

Note that Tabor weak commutativity is a special case of Rätz's one. The following obvious problem arises: does there exist any Rätz weakly commutative group which is not Tabor weakly commutative?

4. Commutator groups condition

We present another condition on the relevant group G which ensure that condition (1.4) is satisfied, so we have stability of the Cauchy equation.

THEOREM 4.1. *Condition (1.4) is satisfied if and only if for each pair $x, y \in G$, $x \neq y$, the function f satisfying (1.1) is bounded on the commutator group $G^1(x, y)$ of the subgroup $G(x, y)$ of G generated by x and y .*

PROOF. If condition (1.4) holds then the function g is additive. Take $u^{-1}v^{-1}uv$, $u, v \in G$, then

$$\begin{aligned} g(u^{-1}v^{-1}uv) &= g(u^{-1}) + g(v^{-1}) + g(u) + g(v) \\ &= -g(u) - g(v) + g(u) + g(v) = 0, \end{aligned}$$

so g is zero on $G^1(x, y)$ for each pair x, y , thus f is bounded on $G^1(x, y)$.

Fix now $x, y \in G$ and assume that f satisfies (1.1) and is bounded on $G^1(x, y)$, say $|f(\gamma)| \leq H$ for $\gamma \in G^1(x, y)$. For every $n \in \mathbb{Z}$ we have

$$u := y^{-n}x^{-n}(xy)^n = y^{-1}\gamma y, \quad \gamma \in G^1(x, y)$$

and $x^n y^n u = (xy)^n$. So we have

$$\begin{aligned} |f(x^n y^n) - f([xy]^n)| &\leq |f(x^n y^n) + f(u) - f([xy]^n)| + |f(u)| \\ &\leq K + |f(u) - f(y^{-1}) - f(\gamma y)| + |f(\gamma y) - f(\gamma) - f(y)| \\ &\quad + |f(y^{-1}) + f(y) - f(e)| + |f(e)| + |f(\gamma)| \leq 5K + |f(\gamma)| \leq 5K + H, \end{aligned}$$

thus, dividing by 2^n and taking the limit condition (1.4) follows. \square

The next theorem relates stability with properties of $G^1(x, y)$ (see [6]).

THEOREM 4.2. *Assume that for each pair $x, y \in G$, $x \neq y$, there exists and integer $N = N(x, y)$ such that each element of $G^1(x, y)$ is the product of at most $N(x, y)$ commutators. Then the equation of homomorphisms is stable.*

PROOF. We assume (1.1) and the theorem will be proved if we show that condition (1.4) holds. By the previous theorem it is enough to prove that f

is bounded on $G^1(x, y)$. Let $\gamma \in G^1(x, y)$, then $\gamma = c_1 c_2 \cdots c_N$, where the c_i 's are commutators of $G(x, y)$. Then

$$\begin{aligned} |f(\gamma)| &= |f(c_1 c_2 \cdots c_N)| \leq |f(c_1 c_2 \cdots c_N) - f(c_1) - f(c_2 \cdots c_N)| \\ &\quad + |f(c_2 \cdots c_N) - f(c_2) - f(c_3 \cdots c_N)| + \cdots \\ &\quad + |f(c_{N-1} c_N) - f(c_{N-1}) - f(c_N)| + |f(c_1)| + \cdots + |f(c_n)|, \end{aligned}$$

and, if $c_i = u^{-1} v^{-1} u v$,

$$\begin{aligned} |f(c_i)| &= |f(u^{-1} v^{-1} u v) + f(v u) - f(u v)| + |f(u v) - f(u) - f(v)| \\ &\quad + |f(u) + f(v) - f(v u)| \leq 3K; \end{aligned}$$

thus $|f(\gamma)| \leq 4NK$. □

Do we have in this way a new class of groups? The answer is positive. Consider the group $GL_2(\mathbb{R})$, it is well known that it is not amenable; now we show that it is not (Rätz and Tabor) weakly commutative. Indeed, take the matrices

$$x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix},$$

then it is easy to prove that

$$x^n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

and $y^2 = -I$, $y^3 = -y$ and $y^4 = I$. Hence we have for $n = 4k$, $x^n y^n = x^n$; for $n = 4k + 2$, $x^n y^n = -x^n$, for $n = 4k + 1$, $x^n y^n = x^n y$ and for $n = 4k + 3$, $x^n y^n = -x^n y$, and

$$x^n y = \begin{pmatrix} -1 - n & 2 + n \\ -1 & 1 \end{pmatrix}, \quad -x^n y = \begin{pmatrix} 1 + n & -2 - n \\ 1 & -1 \end{pmatrix},$$

whereas

$$xy = \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix}, \quad (xy)^2 = \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix}, \quad (xy)^3 = I.$$

Thus, for every $n \geq 2$ it is $x^n y^n \neq (xy)^n$ and $GL_2(\mathbb{R})$ is not weakly commutative.

On the other hand its commutator group is $SL_2(\mathbb{R})$ and each element of it is the product of at most 3 commutators. To see this, we start with two identities. If $a \neq 0$ and $ad - bc = 1$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix} \begin{pmatrix} 1 & ab \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix},$$

while, if $a = 0$,

$$\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -d/b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/b & 0 \\ 0 & b \end{pmatrix}.$$

Now, we have the following three identities:

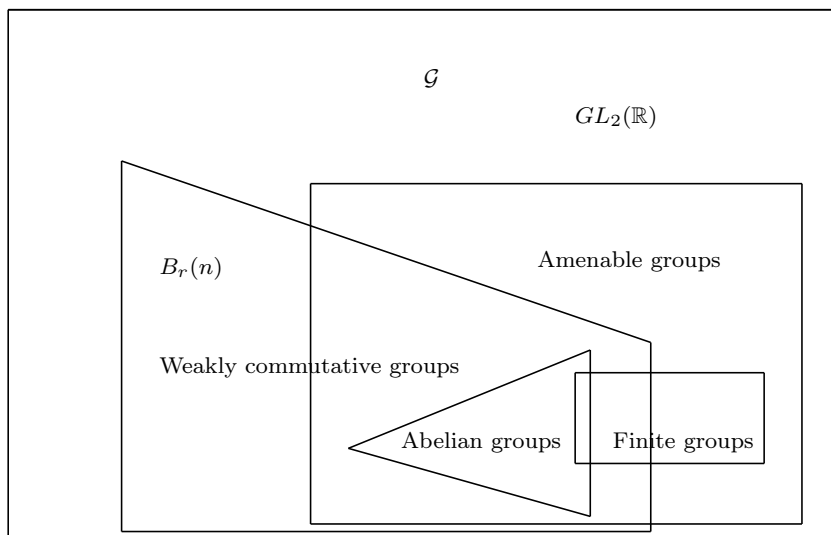
$$\begin{aligned} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{-1}, \\ \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix} &= \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1}, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}^{-1}. \end{aligned}$$

5. Conclusions and open problems

To finish, we can summarize the various relations among the classes of groups previously considered. We have Badora's class \mathcal{G} which is the set of groups where the Cauchy equation is stable; \mathcal{G} contains amenable groups and Rätz weakly commutative groups and these last two classes contain all Abelian groups. Finite groups are all amenable and some of them are weakly commutative (for instance the dihedral group D_4). The Burnside $B_r(n)$, for $r \geq 2$ and $n \geq 2^{48}$, is Tabor weakly commutative, but it is not amenable; finally the group $GL_2(\mathbb{R})$ belongs to \mathcal{G} but it is neither weak commutative nor amenable.

Thus, obvious open problems are the 'algebraic' characterization of the groups in \mathcal{G} which are neither amenable nor weakly commutative.

This paper is focused on groups and their properties, however the natural setting for the additive Cauchy equation are semigroups. Many of the results and considerations concerning groups can easily be translated to semigroups, like amenability and weak commutativity, while others are not (commutator



group etc.). It would be interesting to treat in more details the semigroup case.

As already noted, the theorem relating amenability to a sort of multi-stability is not very useful if we do not produce a way to apply it.

Last, it would be very interesting to produce a family of functional equations for which stability can be proved by using amenability and possibly some additional condition, like centrality used for Drygas equation.

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