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LEFT DERIVABLE MAPS AT NON-TRIVIAL IDEMPOTENTS ON NEST ALGEBRAS

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Abstract. Let $Alg\mathcal{N}$ be a nest algebra associated with the nest $\mathcal{N}$ on a (real or complex) Banach space $\mathbb{X}$. Suppose that there exists a non-trivial idempotent $P \in Alg\mathcal{N}$ with range $P(\mathbb{X}) \in \mathcal{N}$, and $\delta\colon Alg\mathcal{N} \to Alg\mathcal{N}$ is a continuous linear mapping (generalized) left derivable at P, i.e. $\delta(ab) = a\delta(b) + b\delta(a)$ ($\delta(ab) = a\delta(b) + b\delta(a) - ba\delta(I)$) for any $a, b \in Alg\mathcal{N}$ with $ab = P$, where I is the identity element of $Alg\mathcal{N}$. We show that δ is a (generalized) Jordan left derivation. Moreover, in a strongly operator topology we characterize continuous linear maps δ on some nest algebras $Alg\mathcal{N}$ with the property that $\delta(P) = 2P\delta(P)$ or $\delta(P) = 2P\delta(P) - P\delta(I)$ for every idempotent P in $Alg\mathcal{N}$.

1. Introduction

Throughout this paper, all algebras and vector spaces will be over $\mathbb{F}$, where $\mathbb{F}$ is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Let $\mathbb{A}$ be an algebra with unity 1, $\mathbb{M}$ be a left $\mathbb{A}$-module and $\delta\colon \mathbb{A} \to \mathbb{M}$ be a linear mapping. The mapping δ is said to be a *left derivation* (or a *generalized left derivation*) if $\delta(ab) = a\delta(b) + b\delta(a)$ (or $\delta(ab) = a\delta(b) + b\delta(a) - ba\delta(1)$) for all $a, b \in \mathbb{A}$. It is called a *Jordan left derivation* (or a *generalized Jordan left derivation*) if $\delta(a^2) = 2a\delta(a)$ (or $\delta(a^2) = 2a\delta(a) - a^2\delta(1)$) for any $a \in \mathbb{A}$. Obviously, any (generalized) left derivation is a (generalized) Jordan left derivation, but in general the converse is not true (see [15, Example 1.1]). The concepts of left derivation and Jordan left derivation were introduced by Brešar and Vukman

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in [4]. For results concerning left derivations and Jordan left derivations we refer the readers to [10] and the references therein.

In recent years, several authors studied the linear (additive) maps that behave like homomorphisms, derivations or left derivations when acting on special products (for instance, see [3, 7, 9, 6, 11, 12, 16] and the references therein). In this article we study the linear maps on nest algebras behaving like left derivations at idempotent-product elements.

Let $\mathbb{A}$ be an algebra with unity 1, $\mathbb{M}$ be a left $\mathbb{A}$-module and $\delta\colon \mathbb{A} \to \mathbb{M}$ be a linear mapping. We say that δ is *left derivable* (or *generalized left derivable*) at a given point $z \in \mathbb{A}$ if $\delta(ab) = a\delta(b) + b\delta(a)$ (or $\delta(ab) = a\delta(b) + b\delta(a) - ba\delta(1)$) for any $a, b \in \mathbb{A}$ with $ab = z$. In this paper, we characterize the continuous linear maps on nest algebras which are (generalized) left derivable at a non-trivial idempotent operator P. Moreover, in a strongly operator topology we describe continuous linear maps δ on some nest algebra $Alg\,\mathcal{N}$ with the property that $\delta(P) = 2P\delta(P)$ or $\delta(P) = 2P\delta(P) - P\delta(I)$ for every idempotent P in $Alg\,\mathcal{N}$, where I is the identity element of $Alg\,\mathcal{N}$.

The following are the notations and terminologies which are used throughout this article.

Let $\mathbb{X}$ be a Banach space. We denote by $\mathcal{B}(\mathbb{X})$ the algebra of all bounded linear operators on $\mathbb{X}$, and $\mathcal{F}(\mathbb{X})$ denotes the algebra of all finite rank operators in $\mathcal{B}(\mathbb{X})$. A *subspace lattice* $\mathcal{L}$ on a Banach space $\mathbb{X}$ is a collection of closed (under norm topology) subspaces of $\mathbb{X}$ which is closed under the formation of arbitrary intersection and closed linear span (denoted by $\vee$), and which includes $\{0\}$ and $\mathbb{X}$. For a subspace lattice $\mathcal{L}$, we define $Alg\,\mathcal{L}$ by

$$Alg\,\mathcal{L} = \{T \in \mathcal{B}(\mathbb{X}) \,|\, T(N) \subseteq N \text{ for all } N \in \mathcal{L}\}.$$

A totally ordered subspace lattice $\mathcal{N}$ on $\mathbb{X}$ is called a *nest* and $Alg\,\mathcal{N}$ is called a *nest algebra*. When $\mathcal{N} \neq \{\{0\}, \mathbb{X}\}$, we say that $\mathcal{N}$ is non-trivial. It is clear that if $\mathcal{N}$ is trivial, then $Alg\,\mathcal{N} = \mathcal{B}(\mathbb{X})$. Denote $Alg_{\mathcal{F}}\mathcal{N} := Alg\,\mathcal{N} \cap \mathcal{F}(\mathbb{X})$, the set of all finite rank operators in $Alg\,\mathcal{N}$ and for $N \in \mathcal{N}$, let $N_- = \vee\{M \in \mathcal{N} \,|\, M \subset N\}$. The identity element of a nest algebra will be denoted by I. An element P in a nest algebra is called a *non-trivial idempotent* if $P \neq 0, I$ and $P^2 = P$.

Let $\mathcal{N}$ be a non-trivial nest on a Banach space $\mathbb{X}$. If there exists a non-trivial idempotent $P \in Alg\,\mathcal{N}$ with range $P(\mathbb{X}) \in \mathcal{N}$, then we have $(I - P)(Alg\,\mathcal{N})P = \{0\}$ and hence

$$Alg\,\mathcal{N} = P(Alg\,\mathcal{N})P \dot{+} P(Alg\,\mathcal{N})(I - P) \dot{+} (I - P)(Alg\,\mathcal{N})(I - P)$$

as sum of linear spaces. This is so-called Peirce decomposition of $Alg\,\mathcal{N}$. The sets $P(Alg\,\mathcal{N})P$, $P(Alg\,\mathcal{N})(I - P)$ and $(I - P)(Alg\,\mathcal{N})(I - P)$ are closed in $Alg\mathcal{N}$. In fact, $P(Alg\mathcal{N})P$ and $(I-P)(Alg\mathcal{N})(I-P)$ are Banach subalgebras of $Alg\mathcal{N}$ whose unit elements are P and $I-P$, respectively and $P(Alg\mathcal{N})(I-P)$ is a Banach $(P(Alg\mathcal{N})P,(I-P)(Alg\mathcal{N})(I-P))$-bimodule. Also $P(Alg\mathcal{N})(I-P)$ is faithful as a left $P(Alg\mathcal{N})P$-module as well as a right $(I-P)(Alg\mathcal{N})(I-P)$-module. For more information on nest algebras, we refer to [5].

A subspace lattice $\mathcal{L}$ on a Hilbert space $\mathbb{H}$ is called a *commutative subspace lattice*, or a *CSL*, if the projections of $\mathbb{H}$ onto the subspaces of $\mathcal{L}$ commute with each other. If $\mathcal{L}$ is a CSL, then $Alg\,\mathcal{L}$ is called a *CSL algebra*. Each nest algebra on a Hilbert space is a CSL-algebra.

2. Main results

In order to prove our results we need the following result.

THEOREM 2.1 ([8]). *Let $\mathbb{A}$ be a Banach algebra with unity 1, $\mathbb{X}$ be a Banach space and let $\phi:\mathbb{A}\times\mathbb{A}\to\mathbb{X}$ be a continuous bilinear map with the property that*

$$a,b\in\mathbb{A},\ ab=1\Rightarrow\phi(a,b)=\phi(1,1).$$

Then

$$\phi(a,a)=\phi(a^2,1)$$

for all $a\in\mathbb{A}$.

PROPOSITION 2.2. *Let $\mathbb{A}$ be a Banach algebra with unity 1, and $\mathbb{M}$ be a unital Banach left $\mathbb{A}$-module. Let $\delta\colon\mathbb{A}\to\mathbb{M}$ be a continuous linear map. If δ is left derivable at 1, then δ is a Jordan left derivation.*

PROOF. Since $1\cdot 1=1$, it follows that $\delta(1)=2\delta(1)$ and therefore $\delta(1)=0$. Define a continuous bilinear map $\phi\colon\mathbb{A}\times\mathbb{A}\to\mathbb{M}$ by $\phi(a,b)=a\delta(b)+b\delta(a)$. Then $\phi(a,b)=\phi(1,1)$ for all $a,b\in\mathbb{A}$ with $ab=1$, since δ is left derivable at 1. By applying Theorem 2.1, we obtain $\phi(a,a)=\phi(a^2,1)$ for all $a\in\mathbb{A}$. So,

$$\delta(a^2)=2a\delta(a)\quad(a\in\mathbb{A}).\qquad\square$$

Corollary 2.3. *Let $\mathbb{A}$ be a Banach algebra with unity 1, and $\mathbb{M}$ be a unital Banach left $\mathbb{A}$-module. Let $x, y \in \mathbb{A}$ with $x+y=1$ and let $\delta\colon \mathbb{A} \to \mathbb{M}$ be a continuous linear map. If δ is left derivable at x and y, then δ is a Jordan left derivation.*

Proof. For $a, b \in \mathbb{A}$ with $ab=1$, we have $abx=x$ and $aby=y$. Since δ is left derivable at x and y, it follows that

$$\delta(x) = \delta(abx) = a\delta(bx) + bx\delta(a)$$

and

$$\delta(y) = \delta(aby) = a\delta(by) + by\delta(a).$$

Combining the two above equations, we get that

$$\delta(1) = \delta(x+y) = a\delta(bx) + bx\delta(a) + a\delta(by) + by\delta(a) = a\delta(b) + b\delta(a),$$

i.e. δ is left derivable at 1. It follows from Proposition 2.2 that δ is a Jordan left derivation. $\square$

Remark 2.4. If $\mathbb{A}$ is a CSL-algebra or a unital semisimple Banach algebra, then by [12] and [14] every continuous Jordan left derivation on $\mathbb{A}$ is zero. Hence it follows from Proposition 2.2 that every continuous linear map $\delta\colon \mathbb{A} \to \mathbb{A}$ which is left derivable at 1 is zero.

The following is our main result.

Theorem 2.5. *Let $\mathcal{N}$ be a nest on a Banach space $\mathbb{X}$ such that there exists non-trivial idempotent $P \in Alg\,\mathcal{N}$ with range $P(\mathbb{X}) \in \mathcal{N}$. If $\delta\colon Alg\,\mathcal{N} \to Alg\,\mathcal{N}$ is a continuous left derivable map at P, then δ is a Jordan left derivation.*

Proof. For a notational convenience, we denote $\mathbb{A} = Alg\,\mathcal{N}$, $\mathbb{A}_{11} = P\mathbb{A}P$, $\mathbb{A}_{12} = P\mathbb{A}(I-P)$ and $\mathbb{A}_{22} = (I-P)\mathbb{A}(I-P)$. As mentioned in the introduction $\mathbb{A} = \mathbb{A}_{11}\dot{+}\mathbb{A}_{12}\dot{+}\mathbb{A}_{22}$. Throughout the proof, a_{ij} and b_{ij} will denote arbitrary elements in $\mathbb{A}_{ij}$ for $1 \leq i, j \leq 2$.

First we show that $\delta(P) = 0$. Since $P^2 = P$, we have $2P\delta(P) = \delta(P)$. It follows from equation $2P\delta(P) = \delta(P)$ that $P\delta(P) = 0$ and it implies that $\delta(P) = 0$.

We complete the proof by verifying the following steps.

Step 1. $P\delta(a_{11}^2)P = 2a_{11}P\delta(a_{11})P$ and $P\delta(a_{11}^2)(I-P) = 2a_{11}P\delta(a_{11})(I-P)$.

For any a_{11}, b_{11} with $a_{11}b_{11} = P$, we have

$$a_{11}\delta(b_{11}) + b_{11}\delta(a_{11}) = \delta(P). \tag{2.1}$$

Multiplying this identity by P both from the left and from the right, we find

$$a_{11}P\delta(b_{11})P + b_{11}P\delta(a_{11})P = P\delta(P)P \quad (a_{11}b_{11} = P).$$

Define a continuous linear map $d\colon \mathbb{A}_{11} \to \mathbb{A}_{11}$ by $d(a_{11}) = P\delta(a_{11})P$. By above identity d is left derivable at P. Hence by Proposition 2.2, d is a Jordan left derivation, which implies

$$P\delta(a_{11}^2)P = 2a_{11}P\delta(a_{11})P \quad (a_{11} \in \mathbb{A}_{11}).$$

By multiplying (2.1) by P from the left and by $(I-P)$ from the right, we arrive at

$$a_{11}P\delta(b_{11})(I-P) + b_{11}P\delta(a_{11})(I-P) = P\delta(P)(I-P) \quad (a_{11}b_{11} = P).$$

Define a continuous linear map $D\colon \mathbb{A}_{11} \to \mathbb{A}_{12}$ by $D(a_{11}) = P\delta(a_{11})(I-P)$. It is easy to see that D is a left derivable at P. It follows from Proposition 2.2 that D is a Jordan left derivation. Thus,

$$P\delta(a_{11}^2)(I-P) = 2a_{11}P\delta(a_{11})(I-P) \quad (a_{11} \in \mathbb{A}_{11}).$$

Step 2. $P\delta(a_{22}) = 0$.

Since $(P + a_{22})P = P$, we have

$$(P + a_{22})\delta(P) + P\delta(P + a_{22}) = \delta(P).$$

From $\delta(P) = 0$ we get

$$P\delta(a_{22}) = 0.$$

Step 3. $P\delta(a_{12}) = 0$.

Applying δ to $(P + a_{12})P = P$, we get

$$(P + a_{12})\delta(P) + P\delta(P + a_{12}) = \delta(P).$$

Since $\delta(P) = 0$, it follows that

$$P\delta(a_{12}) = 0.$$

Step 4. $(I-P)\delta(a_{11})(I-P)=0$.

For any a_{11}, b_{11} with $b_{11}a_{11}=P$, we have $(I-P+b_{11})a_{11}=P$ and hence

$$(I-P+b_{11})\delta(a_{11})+a_{11}\delta(I-P+b_{11})=\delta(P).$$

Multiplying this identity by $I-P$ both from the left and from the right we arrive at

$$(I-P)\delta(a_{11})(I-P)=0.$$

Since any element in a Banach algebra with unit element is a sum of its invertible elements ([1]), by the linearity of δ and above identity we have

$$(I-P)\delta(a_{11})(I-P)=0$$

for all $a_{11}\in\mathbb{A}_{11}$.

Step 5. $(I-P)\delta(a_{12})(I-P)=0$.

Since $(P-a_{12})(I+a_{12})=P$, it follows that

$$(P-a_{12})\delta(I+a_{12})+(I+a_{12})\delta(P-a_{12})=\delta(P).$$

Multiplying this identity by $I-P$ both from the left and from the right and using the fact that $\delta(P)=0$, we find

$$(I-P)\delta(a_{12})(I-P)=0.$$

Step 6. $(I-P)\delta(a_{22})(I-P)=0$.

Applying δ to $(P+a_{12})(P-a_{12}a_{22}+a_{22})=P$, we see that

$$(P+a_{12})\delta(P-a_{12}a_{22}+a_{22})+(P-a_{12}a_{22}+a_{22})\delta(P+a_{12})=\delta(P).$$

Now, multiplying this identity from the left by P, from the right by $I-P$ and by Steps 2,3 and 5 and the fact that $\delta(P)=0$, we get $a_{12}(I-P)\delta(a_{22})(I-P)=0$. Since $a_{12}\in\mathbb{A}_{12}$ is arbitrary, we have $\mathbb{A}_{12}((I-P)\delta(a_{22})(I-P))=\{0\}$. From the fact that $\mathbb{A}_{12}$ is faithful as right $\mathbb{A}_{22}$-module, we arrive at

$$(I-P)\delta(a_{22})(I-P)=0.$$

Since $ab=PaPbP+PaPb(I-P)+Pa(I-P)b(I-P)+(I-P)a(I-P)b(I-P)$, for any $a,b\in\mathbb{A}$, by Steps 1–6, it follows that δ is a Jordan left derivation. □

Our next result characterizes the linear mappings on $Alg\mathcal{N}$ which are generalized left derivable at P.

THEOREM 2.6. *Let $\mathcal{N}$ be a nest on a Banach space $\mathbb{X}$ such that there exists a non-trivial idempotent $P \in Alg\mathcal{N}$ with range $P(\mathbb{X}) \in \mathcal{N}$. If $\delta\colon Alg\mathcal{N} \to Alg\mathcal{N}$ is a continuous generalized left derivable map at P, then δ is a generalized Jordan left derivation.*

PROOF. Define $\Delta\colon Alg\mathcal{N} \to Alg\mathcal{N}$ by $\Delta(a) = \delta(a) - a\delta(1)$. It is easy to see that Δ is a continuous left derivable map at P. By Theorem 2.5, Δ is a Jordan left derivation. Therefore

$$\begin{aligned}\delta(a^2) &= \Delta(a^2) + a^2\delta(1)\\ &= 2a\Delta(a) + a^2\delta(1)\\ &= 2a(\delta(a) - a\delta(1)) + a^2\delta(1)\\ &= 2a\delta(a) - a^2\delta(1)\end{aligned}$$

for all $a \in Alg\mathcal{N}$. So δ is a generalized Jordan left derivation. □

Since every continuous Jordan left derivation on a CSL algebra is zero ([12]), we have the following result.

COROLLARY 2.7. *Let $\mathcal{N}$ be a non-trivial nest on a Hilbert space $\mathbb{H}$. Let P be a non-trivial idempotent in $Alg\mathcal{N}$ with range $P(\mathbb{H}) \in \mathcal{N}$ and $\delta\colon Alg\mathcal{N} \to Alg\mathcal{N}$ be a continuous linear map.*

(i) *If δ is left derivable at P, then δ is zero.*

(ii) *If δ is generalized left derivable at P, then $\delta(a) = a\delta(1)$ for all $a \in Alg\mathcal{N}$.*

PROOF. (i) Since every continuous Jordan left derivation on a CSL algebra is zero ([12]), by Theorem 2.5, δ is zero.

(ii) By Theorem 2.6, δ is a generalized Jordan left derivation, so the mapping $\Delta\colon Alg\mathcal{N} \to Alg\mathcal{N}$ defined by $\Delta(a) = \delta(a) - a\delta(1)$ is a continuous Jordan left derivation. Therefore $\Delta = 0$ and hence $\delta(a) = a\delta(1)$ for all $a \in Alg\mathcal{N}$. □

Now, we characterize (generalized) left Jordan derivations which are continuous in the strongly operator topology, but in order to prove our result we must assume an additional (mild) condition concerning the nest $\mathcal{N}$. At present we have no counter-example, so it remains an open problem if this additional condition can be omitted.

The idea of the proof of Proposition 2.8 (i) comes from [2].

Proposition 2.8. *Let $\mathcal{N}$ be a nest on a Banach space $\mathbb{X}$, with each $N \in \mathcal{N}$ complemented in $\mathbb{X}$ whenever $N_- = N$. Let $\delta\colon Alg\mathcal{N} \to Alg\mathcal{N}$ be a continuous linear map in a strong operator topology.*

(i) *If $\delta(P) = 2P\delta(P)$ for every idempotent P in $Alg\mathcal{N}$, then $\delta = 0$.*

(ii) *If $\delta(P) = 2P\delta(P) - P\delta(I)$ for every idempotent P in $Alg\mathcal{N}$, then $\delta(a) = a\delta(I)$ for all $a \in Alg\mathcal{N}$.*

Proof. (i) For arbitrary idempotent operator $P \in Alg\mathcal{N}$, by hypothesis we have $\delta(P) = 2P\delta(P)$. It follows from equation $2P\delta(P) = \delta(P)$ that $P\delta(P) = 0$ and it implies that $\delta(P) = 0$.

Notice that $Alg_{\mathcal{F}}\mathcal{N}$ is contained in the linear span of the idempotents in $Alg\mathcal{N}$ (see [11]), which implies that $\delta(F) = 0$ for all finite rank operator F in $Alg\mathcal{N}$. Since δ is continuous under the strong operator topology and $\overline{Alg_{\mathcal{F}}\mathcal{N}}^{SOT} = Alg\mathcal{N}$ (see [13]), we find that $\delta(a) = 0$ for all $a \in Alg\mathcal{N}$.

(ii) Define $\Delta\colon Alg\mathcal{N} \to Alg\mathcal{N}$ by $\Delta(a) = \delta(a) - a\delta(I)$. It is easy to see that Δ is a continuous left map satisfying $\Delta(P) = 2P\Delta(P)$ for every idempotent P in $Alg\mathcal{N}$. So by (i) we have $\Delta = 0$ and hence $\delta(a) = a\delta(I)$ for all $a \in Alg\mathcal{N}$. □

It is obvious that the nests on Hilbert spaces, finite nests and the nests having order-type $\omega + 1$ or $1 + \omega^*$, where ω is the order-type of the natural numbers, satisfy the condition in Proposition 2.8 automatically.

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