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APPLICATIONS OF STOCHASTIC SEMIGROUPS TO QUEUEING MODELS

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Abstract. Non-markovian queueing systems can be extended to piecewise-deterministic Markov processes by appending supplementary variables to the system. Then their analysis leads to an infinite system of partial differential equations with an infinite number of variables and non-local boundary conditions. We show how one can study such systems by using the theory of stochastic semigroups.

1. Introduction

A basic model of a queueing system ([2]) is a single-server system in which customers arrive for a service and the times taken to serve the customers (service times) are independent and identically distributed (i.i.d.) random variables. On arrival each customer must wait until a server is free, giving priority to earlier arrivals, and the times between arrivals of two consecutive customers (inter-arrival times) are also i.i.d. The service times and the inter-arrival times are assumed to be independent. Let $N(t)$ be the number of customers in the system at time t, those being served at time t and those waiting to be served. This process is not Markov, unless both the inter-arrival and the service times are exponentially distributed. Then the queue is called an M/M/1 queue, where the code means memoryless inter-arrival

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times/memoryless service times/single server system. If customers arrive according to a Poisson process in which case the inter-arrival times are exponentially distributed then the queue is called M/G/1, where now G indicates that the service-times distribution is general. The earliest investigation to analyze this queueing system was by Cox ([8]), who introduced the supplementary variable: the time that the customer in service has been at the service point, and constructed the forward Kolmogorov equations (see (2.2)–(2.5)) for the resulting Markov process. For the history of the supplementary variables technique and its use in queueing models we refer the reader to [7] and [12], where the references to treatments of the M/G/1 queue may also be found.

We focus here on applications of the theory of stochastic semigroups ([5, 21]) to the particular example of the M/G/1 queue. We rewrite the forward Kolmogorov equations in the form of an abstract Cauchy problem on the space of integrable functions and prove its well-posedness using semigroup methods. We then study the existence of stationary solutions and show that the time dependent solutions converge to the stationary solution if and only if the traffic intensity is strictly smaller then 1. In all previous studies of the M/G/1 queue with the tools of functional analysis (see [12]) it was assumed that the hazard rate function μ as in (2.1) associated with the distribution of service time is bounded and that $\inf_x \mu(x) > 0$. In particular, these assumptions are not satisfied for queues with uniformly, gamma, Weibull or Pareto distributed service times. In our approach we do not need these assumptions. We prove well-posedness using a generation result from [14] that is an extension of Greiner's theorem ([11]) by allowing unbounded perturbations of boundary conditions in the setting of an AL-space. Convergence to a stationary solution is obtained by using the theory of partially integral stochastic semigroups ([20]). For applications of methods of operator semigroups to concrete problems from the theory of queues we refer to [15, 12, 13, 25], where the spectral theory was used to get the existence of stationary solutions. Our approach with stochastic semigroups can be used in virtually all queueing models which can be studied by supplementary variables, see [17] for a recent review of such models.

2. The Markov process for the M/G/1 queue

We consider the M/G/1 queue, in which customers arrive according to a Poisson process with rate α and the service times of customers are independently distributed random variables with probability density function $b(x)$. Define the state of the process to be $(0, 0)$ if there are no customers present and to be (s, n) if the customer being currently served is at the service point for a time s and there are n customers in the system. We write $X(t) = (x(t), N(t))$, where $N(t)$ is the system size and $x(t)$ is the time already spent in service by time t of a customer being served. A customer will leave the service point in the next Δt time units with probability $\mu(x(t))\Delta(t) + o(\Delta t)$, where μ is the hazard rate function

$$\mu(x) = \frac{b(x)}{\int_x^\infty b(r)dr}, \quad x > 0, \tag{2.1}$$

and b is the probability density of the distribution of the service time.

The process $\{X(t)\}_{t\geq 0}$ has values in the set

$$E = \{(0,0)\} \cup [0,\infty) \times \mathbb{N}$$

and it is an example of a piecewise deterministic Markov process ([9, 21]). It evolves deterministically according to the differential equation

$$X'(t) = g(X(t)),$$

where $g(x,n) = (1,0)$ for $x \geq 0$, $n \geq 1$, $g(0,0) = (0,0)$, and it changes its values in a random way only at random times $t_k, k \geq 0$, where $0 = t_0 < t_1 < \ldots < t_k < \ldots$. Here the random times (t_k) denote departure and arrival times of customers. We have

$$\mathbb{P}(t_1 > t | X(0) = (x,n)) = \exp\left\{-\int_0^t \varphi(x+r,n)dr\right\},$$

where we set

$$\varphi(x,n) = \begin{cases} \alpha + \mu(x) & \text{for } x \geq 0, n \geq 1, \\ \alpha, & \text{for } x \geq 0, n = 0, \end{cases}$$

and $\mu(0) = 0$. Once the jump occurs at time t_k and $X(t_k^-) = (x,n)$, where $X(t_k^-) = \lim_{t\uparrow t_k} X(t)$, then we choose $X_k = X(t_k)$ according to the following transition probability

$$\mathbb{P}(X_k \in B | X(t_k^-) = (x,n)) = \mathcal{P}((x,n), B), \quad (x,n) \in E,$$

where $\mathcal{P}((0,0), B) = 1_B(0,1)$ and

$$\mathcal{P}((x,n), B) = \frac{\alpha}{\alpha+\mu(x)} 1_B(x, n+1) + \frac{\mu(x)}{\alpha+\mu(x)} 1_B(0, n-1)$$

for $x \geq 0, n \geq 1$, and any Borel subset B of E.

Let $p_0(t)$ be the probability that there are no customers at time t present in the system and let $p_n(t,x)$ be the probability density function of $X(t)$ when $N(t) = n$ at time t. The M/G/1 queue can be described by the following system of partial differential equations ([8])

$$\frac{\partial p_0(t)}{\partial t} = -\alpha p_0(t) + \int_0^\infty \mu(x) p_1(t,x)dx, \tag{2.2}$$

$$\frac{\partial p_1(t,x)}{\partial t} = -\frac{\partial p_1(t,x)}{\partial x} - (\alpha + \mu(x))p_1(t,x), \tag{2.3}$$

$$\frac{\partial p_n(t,x)}{\partial t} = -\frac{\partial p_n(t,x)}{\partial x} - (\alpha + \mu(x))p_n(t,x) + \alpha p_{n-1}(t,x), \tag{2.4}$$

for $n \geq 2, x > 0$, supplemented by boundary conditions

$$p_1(t,0) = \int_0^\infty p_2(t,x)\mu(x)dx + \alpha p_0(t), \tag{2.5}$$

$$p_n(t,0) = \int_0^\infty p_{n+1}(t,x)\mu(x)dx, \quad n \geq 2, \tag{2.6}$$

and the initial conditions

$$p_0(0) = f(0,0), \quad p_n(0,x) = f(x,n), \quad x > 0, n \geq 1, \tag{2.7}$$

which are nonnegative and such that

$$f(0,0) + \sum_{n=1}^\infty \int_0^\infty f(x,n)dx = 1.$$

Using the theory of stochastic semigroups we prove in Section 4 the following:

THEOREM 2.1. *The system* (2.2)–(2.7) *has a unique nonnegative time-dependent solution and*

$$p_0(t) + \sum_{n=1}^\infty \int_0^\infty p_n(t,x)dx = 1, \quad t \geq 0.$$

We also study the long time behavior of the solutions of (2.2)–(2.7). Let N_b be the number of arrivals during the service time of one customer. Conditioning on the duration of the service time, we see that the distribution of N_b is

$$a_k := \mathbb{P}(N_b = k) = \int_0^\infty e^{-\alpha x}\frac{(\alpha x)^k}{k!} b(x)dx, \quad k \geq 0, \tag{2.8}$$

and its generating function is given by

$$G_a(z) = \sum_{n=0}^{\infty} a_n z^n = \int_0^{\infty} e^{-\alpha(1-z)x} b(x)dx = \hat{b}(\alpha(1-z)), \tag{2.9}$$

where $\hat{b}$ denotes the Laplace transform of the service time distribution with density b. We define the traffic intensity ρ, i.e., the mean number of arrivals within the average service time, by

$$\rho = \alpha \int_0^{\infty} xb(x)dx.$$

Note that $G_a'(1) = \rho$. If $\rho < 1$ then we define a probability distribution $q = (q_n)_{n\geq 0}$ with the help of its generating function of the form

$$G_q(z) = \sum_{n=0}^{\infty} q_n z^n = \frac{(1-\rho)z(1-z)}{G_a(z) - z}, \quad z \in (0,1). \tag{2.10}$$

The proof of the next result is also given in Section 4.

THEOREM 2.2. *Suppose that the traffic intensity satisfies* $\rho < 1$ *and the distribution* $(q_n)_{n\geq 0}$ *is defined by* (2.10). *Then there exists a stationary solution* p^* *of* (2.2)–(2.7), *it is given by* $p_0^* = 1 - \rho$,

$$p_n^*(x) = \alpha \sum_{k=0}^{n-1} e^{-\alpha x} \frac{(\alpha x)^k}{k!} q_{n-k} \int_x^{\infty} b(z)dz, \quad x \geq 0, n \geq 1, \tag{2.11}$$

and the unique time-dependent solution of (2.2)–(2.7) *converges to* p^**:*

$$\lim_{t\to\infty} (|p_0(t) - p_0^*| + \sum_{n=1}^{\infty} \int_0^{\infty} |p_n(t,x) - p_n^*(x)|dx) = 0.$$

Conversely, if the unique time-dependent solution of (2.2)–(2.7) *is convergent then* $\rho < 1$.

Now suppose that $\rho < 1$. If the process is at the stationary distribution, a departing customer leaves n customers in the system with probability

$$v_n = \int_0^{\infty} p_n^*(x)dx, \quad n = 1, 2, \ldots, \quad v_0 = 1 - \rho.$$

Making use of (2.11) we show that the generating function of the sequence $(v_n)_{n\geq 0}$ is of the form

$$G_v(z) = \frac{(1-\rho)(1-z)G_a(z)}{G_a(z)-z}, \quad z \in (0,1). \tag{2.12}$$

This is known as the Pollaczek–Khinchin formula (see e.g. [22]).

For this purpose, we set

$$r_k = \int_0^\infty e^{-\alpha x}\frac{(\alpha x)^k}{k!}\alpha \int_x^\infty b(z)dzdx, \quad k \geq 0,$$

and, by (2.11), we see that

$$v_n = \sum_{k=0}^{n-1} r_k q_{n-k}, \quad n \geq 1. \tag{2.13}$$

To calculate r_k we write $\alpha = (\alpha + \mu(x)) - \mu(x)$ and use (2.1) to arrive at

$$r_k = \int_0^\infty e^{-\alpha x}\frac{(\alpha x)^k}{k!}(\alpha+\mu(x))\int_x^\infty b(z)dzdx - \int_0^\infty e^{-\alpha x}\frac{(\alpha x)^k}{k!}b(x)dx.$$

It follows from (2.1) that

$$e^{-\alpha x}\int_x^\infty b(z)dz = e^{-\alpha x - \int_0^x \mu(z)dz}.$$

This together with integration by parts implies that

$$r_k = r_{k-1} - a_k, \quad k \geq 1, \quad r_0 = 1 - a_0,$$

where $(a_k)_{k\geq 0}$ is defined by (2.8). Consequently,

$$r_k = 1 - \sum_{l=0}^{k} a_l = \sum_{l=k+1}^{\infty} a_l, \quad k \geq 0,$$

and the generating function of the sequence $(r_k)_{k\geq 0}$ is of the form

$$G_r(z) = \frac{1-G_a(z)}{1-z}, \quad z \in (0,1). \tag{2.14}$$

Applying (2.13) we deduce that

$$G_v(z) = 1 - \rho + \sum_{n=1}^{\infty} v_n z^n = 1 - \rho + G_r(z)G_q(z), \quad z \in (0,1).$$

Simple computations using (2.10) and (2.14) give (2.12).

3. Stochastic operators and semigroups

In this section we collect preliminary material concerning stochastic semigroups. Let $(E,\mathcal{E},m)$ be a σ-finite measure space and $L^1 = L^1(E,\mathcal{E},m)$ be the space of integrable functions. We denote by $D(m) \subset L^1$ the set of all *densities* on E, i.e.

$$D(m) = \{f \in L^1_+ : \|f\| = 1\}, \quad \text{where } L^1_+ = \{f \in L^1 : f \geq 0\},$$

and $\|\cdot\|$ is the norm in L^1. A linear operator $P\colon L^1 \to L^1$ such that $P(D(m)) \subseteq D(m)$ is called *stochastic* or *Markov* ([18]). It is called *substochastic* if $Pf \geq 0$ and $\|Pf\| \leq \|f\|$ for all $f \in L^1_+$. If $\mathcal{D}$ is a linear subspace of L^1 then a linear operator $P\colon \mathcal{D} \to L^1$ is called *positive*, if $Pf \geq 0$ for $f \in \mathcal{D} \cap L^1_+$. A positive and everywhere defined operator is bounded and its norm is determined through its values on L^1_+.

A family of stochastic (substochastic, positive) operators $\{P(t)\}_{t\geq 0}$ on L^1 which is a C_0-*semigroup*, i.e.,

(1) $P(0) = I$ (the identity operator);
(2) $P(t+s) = P(t)P(s)$ for every $s,t \geq 0$;
(3) for each $f \in L^1$ the mapping $t \mapsto P(t)f$ is continuous: for each $s \geq 0$

$$\lim_{t\to s} \|P(t)f - P(s)f\| = 0;$$

is called a *stochastic (substochastic, positive) semigroup*. The infinitesimal *generator* of $\{P(t)\}_{t\geq 0}$ is by definition the operator A with domain $\mathcal{D}(A) \subset L^1$ defined as

$$\mathcal{D}(A) = \{f \in L^1 : \lim_{t\downarrow 0} \frac{1}{t}(P(t)f - f) \text{ exists}\},$$

$$Af = \lim_{t\downarrow 0} \frac{1}{t}(P(t)f - f), \quad f \in \mathcal{D}(A).$$

Let $(A,\mathcal{D}(A))$ be a linear operator. If for some real λ the operator $\lambda - A := \lambda I - A$ is one-to-one, onto, and $(\lambda - A)^{-1}$ is a bounded linear operator, then λ is said to belong to the resolvent set $\rho(A)$ and $R(\lambda, A) := (\lambda - A)^{-1}$ is called the resolvent of A at λ. If A is the generator of the substochastic semigroup $\{P(t)\}_{t\geq 0}$ then $(0,\infty) \subset \rho(A)$ and we have the integral representation

$$R(\lambda, A)f = \int_0^\infty e^{-\lambda s} P(s)f\, ds \quad \text{for} \quad f \in L^1.$$

The operator $\lambda R(\lambda, A)$ is substochastic and $R(\mu, A)f \leq R(\lambda, A)f$ for $\mu > \lambda > 0$, $f \in L^1_+$.

Following Arendt ([1]), a linear operator A is said to be *resolvent positive* if there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subseteq \rho(A)$ and $R(\lambda, A) \geq 0$ for all $\lambda > \omega$. Generators of substochastic semigroups are resolvent positive and we have the following result being a consequence of the fundamental Hille–Yosida theorem (see e.g. [4, 21]). If the operator $(A, \mathcal{D}(A))$ is resolvent positive, $\mathcal{D}(A)$ is dense in L^1, and

$$\int_E Af\, dm \leq 0 \quad \text{for all nonnegative } f \in \mathcal{D}(A),$$

then $(A, \mathcal{D}(A))$ generates a substochastic semigroup on L^1.

We now provide a result from [14] concerning the existence of positive semigroups generated by operators defined on non-trivial domains and providing positive and integrable solutions of initial-boundary value problems. It is a generalization of Greiner's theorem ([11]) on perturbation of the domain of a generator. For a recent review of different approaches used in the study of initial-boundary value problems see [6].

We assume that there is a second L^1 space denoted by $L^1_\partial = L^1(E_\partial, \mathcal{E}_\partial, m_\partial)$, where $(E_\partial, \mathcal{E}_\partial, m_\partial)$ is a σ-finite measure space. Let $\mathcal{D}$ be a linear subspace of L^1. We assume that $A\colon \mathcal{D} \to L^1$ and $\Psi_0, \Psi\colon \mathcal{D} \to L^1_\partial$ are linear operators satisfying the following conditions:

(i) for each $\lambda > 0$, the operator $\Psi_0\colon \mathcal{D} \to L^1_\partial$ restricted to the nullspace $\mathcal{N}(\lambda I - A) = \{f \in \mathcal{D} : \lambda f - Af = 0\}$ of the operator $(\lambda I - A, \mathcal{D})$ has a positive right inverse, i.e., there exists a positive operator $\Psi(\lambda)\colon L^1_\partial \to \mathcal{N}(\lambda I - A)$ such that $\Psi_0 \Psi(\lambda) f_\partial = f_\partial$ for $f_\partial \in L^1_\partial$;

(ii) the operator $\Psi\colon \mathcal{D} \to L^1_\partial$ is positive and $\|\Psi\Psi(\lambda)\| < 1$ for all $\lambda > 0$;

(iii) the operator $A_0 = A$ with $\mathcal{D}(A_0) = \{f \in \mathcal{D} : \Psi_0 f = 0\}$ is resolvent positive and $\mathcal{D}(A_0)$ is dense in L^1;

(iv) for all nonnegative $f \in \mathcal{D}$

$$\int_E Af\, dm - \int_{E_\partial} \Psi_0 f\, dm_\partial \leq 0. \tag{3.1}$$

THEOREM 3.1 ([14]). *Suppose that conditions* (i)–(iv) *hold. Then the operator* $(A_\Psi, \mathcal{D}(A_\Psi))$ *defined by*

$$A_\Psi f = Af, \quad f \in \mathcal{D}(A_\Psi) = \{f \in \mathcal{D} : \Psi_0(f) = \Psi(f)\}, \tag{3.2}$$

is the generator of a positive semigroup on L^1. *Moreover, the resolvent of* A_Ψ *at* $\lambda > \omega$ *is given by*

$$R(\lambda, A_\Psi) f = (I + \Psi(\lambda)(I - \Psi\Psi(\lambda))^{-1}\Psi) R(\lambda, A_0) f, \quad f \in L^1. \tag{3.3}$$

We also need the following corollary to the Kato perturbation theorem ([16, 10, 24, 3]).

Theorem 3.2. *Assume conditions* (i)–(iv). *Let* $(A_\Psi, \mathcal{D}(A_\Psi))$ *be as in* (3.2) *and let* $B\colon \mathcal{D}(A_\Psi) \to L^1$ *be a positive operator satisfying*

$$\int_E (A_\Psi f + Bf)\, dm = 0 \quad \textit{for all } f \in \mathcal{D}(A_\Psi) \cap L^1_+. \tag{3.4}$$

If for some $\lambda > 0$

$$\lim_{n\to\infty} \|(BR(\lambda, A_\Psi))^n\| = 0 \tag{3.5}$$

then the operator $(A_\Psi + B, \mathcal{D}(A_\Psi))$ *is the generator of a stochastic semigroup* $\{P(t)\}_{t\geq 0}$ *on* L^1.

Remark 3.3. Condition (3.4) in Theorem 3.2 implies that the operator $(A_\Psi, \mathcal{D}(A_\Psi))$ satisfies

$$\int_E A_\Psi f\, dm \leq 0 \quad \text{for all } f \in \mathcal{D}(A_\Psi) \cap L^1_+,$$

since $Bf \geq 0$ for $f \geq 0$. Hence, the operator $(A_\Psi, \mathcal{D}(A_\Psi))$ generates a substochastic semigroup. From condition (3.4) it also follows that the operator $BR(\lambda, A_\Psi)$ is substochastic (see e.g. [23, Corollary 3.2]).

It should be noted that condition (3.5) equivalently states that the spectral radius of the bounded operator $BR(\lambda, A_\Psi)$ is strictly smaller than 1.

We now combine the generation results with the approach from [23, 5] to study the existence of invariant densities for the semigroup $\{P(t)\}_{t\geq 0}$ of Theorem 3.2. Consider the substochastic operator $K\colon L^1 \to L^1$ defined by ([23])

$$Kf = \lim_{\lambda\to 0} BR(\lambda, A_\Psi)f, \quad f \in L^1, \tag{3.6}$$

where the limit exists in L^1 and pointwise (see [23, Theorem 3.6]). It follows from [5, Theorem 3.3] that if K has an invariant density f_*, i.e., $Kf_* = f_*$, and if $\overline{f}_*$ defined by

$$\overline{f}_* = \sup_{\lambda>0} R(\lambda, A_\Psi)f_* = \lim_{\lambda\to 0} R(\lambda, A_\Psi)f_* \tag{3.7}$$

belongs to L^1, then $\overline{f}_*$ is invariant for the semigroup $\{P(t)\}_{t\geq 0}$, i.e., $P(t)\overline{f}_* = \overline{f}_*$, $t \geq 0$. The limit in (3.7) is taken pointwise and in general it does not have to belong to L^1.

Note that given any $\lambda, \mu \in \rho(A_0)$ we have ([11, Lemma 1.3]) $\Psi(\lambda) = \Psi(\mu) + (\mu - \lambda)R(\lambda, A_0)\Psi(\mu)$. Thus $\Psi(\lambda) \geq \Psi(\mu)$ for $0 < \lambda \leq \mu$. Consequently, for any nonnegative $f_\partial \in L^1_\partial$ we can define the pointwise limit

$$\Psi(0)f_\partial = \lim_{\lambda \to 0} \Psi(\lambda) f_\partial. \tag{3.8}$$

THEOREM 3.4. *Assume conditions* (i)–(iv) *and* (3.4)–(3.5). *Suppose that* $0 \in \rho(A_0)$ *and that* $\Psi(0)$ *as in* (3.8) *extends to a bounded linear operator* $\Psi(0)\colon L^1_\partial \to L^1$ *with* $\|\Psi\Psi(0)\| < 1$. *Then the semigroup* $\{P(t)\}_{t\geq 0}$ *has an invariant density if and only if the operator* K *defined as in* (3.6) *has an invariant density. Moreover, the equation* $Kf_* = f_*$ *can be rewritten as*

$$f_* = BR(0, A_0)f_* + B\Psi(0)f_\partial, \quad f_\partial = \Psi R(0, A_0)f_* + \Psi\Psi(0)f_\partial, \tag{3.9}$$

and $\overline{f}_*$ *given by*

$$\overline{f}_* = R(0, A_0)f_* + \Psi(0)f_\partial \tag{3.10}$$

is invariant for the semigroup $\{P(t)\}_{t\geq 0}$.

PROOF. Since $0 \in \rho(A_0)$, the operator $R(0, A_0)$ is bounded and

$$\lim_{\lambda \to 0} R(\lambda, A_0)f = R(0, A_0)f, \quad f \in L^1.$$

The operator $\Psi(0)$ is bounded and $I - \Psi\Psi(0)$ is invertible. Thus, $R_0 f \in L^1$ for all $f \in L^1$, where (see (3.3))

$$R_0 f := \lim_{\lambda \to 0} R(\lambda, A_\Psi)f = R(0, A_0)f + \Psi(0)(I - \Psi\Psi(0))^{-1}\Psi R(0, A_0)f.$$

This implies that

$$\lim_{\lambda \to 0} \lambda R(\lambda, A_\Psi)f = 0,$$

which together with [23, Theorem 3.6] shows that the operator K is stochastic.

Now if $Kf_* = f_*$ for $f_* \in D(m)$ then $\overline{f}_* = R_0 f_* \in L^1$. Thus $\overline{f}_*/\|\overline{f}_*\|$ is an invariant density for the semigroup $\{P(t)\}_{t\geq 0}$, by [5, Theorem 3.3]. To prove the converse assume that $\tilde{f}_*$ is an invariant density for the semigroup $\{P(t)\}_{t\geq 0}$. Since $\tilde{f}_* \in \mathcal{D}(A_\Psi)$ and $(B, \mathcal{D}(A_\Psi))$ is a positive operator, we have $B(\tilde{f}_*) \in L^1$ and $B(\tilde{f}_*) \geq 0$. Hence, $B(\tilde{f}_*)/\|B(\tilde{f}_*)\|$ is an invariant density for K, by the same argument as in [5, Corollary 3.11]. $\square$

We conclude this section with a general result from [20] concerning possible asymptotic behavior of stochastic semigroups. A stochastic semigroup $\{P(t)\}_{t\ge 0}$ is called *asymptotically stable* if it has an invariant density f_* such that

$$\lim_{t\to\infty} \|P(t)f - f_*\| = 0 \quad \text{for all } f \in D(m)$$

and *partially integral* if, for some $t > 0$, the operator $P(t)$ is *partially integral*, i.e., there exists a measurable function $q\colon E \times E \to [0,\infty)$ such that

$$\int_E \int_E q(x,y)\, m(dx)\, m(dy) > 0 \quad \text{and} \quad P(t)f(x) \ge \int_E q(x,y) f(y)\, m(dy)$$

for (m-a.e.) $x \in E$ and for every density f.

THEOREM 3.5 ([20, Theorem 2]). *Let $\{P(t)\}_{t\ge 0}$ be a partially integral stochastic semigroup. Assume that the semigroup $\{P(t)\}_{t\ge 0}$ has only one invariant density f_*. If $f_* > 0$ a.e. then the semigroup $\{P(t)\}_{t\ge 0}$ is asymptotically stable.*

4. Stochastic semigroup for the M/G/1 queue

We consider the Borel sets $E = \{(0,0)\} \cup (0,\infty) \times \mathbb{N}$ and $E_\partial = \{0\} \times \{1,2,\ldots\}$ with Borel σ-algebras and measures

$$m = \delta_{(0,0)} + \sum_{n\in\mathbb{N}} \mathrm{Leb} \times \delta_n, \quad m_\partial = \sum_{n\in\mathbb{N}} \delta_0 \times \delta_n,$$

where Leb is the Lebesgue measure on $\mathbb{R}_+$. We take

$$\begin{aligned} Af(0,0) &= -\alpha f(0,0), \\ Af(x,n) &= -\frac{\partial}{\partial x} f(x,n) - (\alpha + \mu(x)) f(x,n), \quad x > 0, n \ge 1, \end{aligned}$$

on its maximal domain $\mathcal{D}$, which consists of all $f \in L^1$ such that $Af \in L^1$,

$$\sum_{n=1}^{\infty} \int_0^\infty \mu(x)|f(x,n)|dx < \infty$$

and the function $x \mapsto f(x,n)$ is absolutely continuous on $(0,\infty)$ for each $n \ge 1$.

We define operators $\Psi_0, \Psi\colon \mathcal{D} \to L^1_\partial$ by

$$\Psi_0 f(0,n) = \lim_{x\to 0^+} f(x,n), \quad n \geq 1, f \in \mathcal{D},$$

and

$$\Psi f(0,1) = \alpha f(0,0) + \int_0^\infty \mu(x) f(x,2)\,dx, \tag{4.1}$$

$$\Psi f(0,n) = \int_0^\infty \mu(x) f(x,n+1)\,dx, \quad n \geq 2, f \in \mathcal{D}. \tag{4.2}$$

Finally, let $B\colon \mathcal{D} \to L^1$ be given by

$$Bf(0,0) = \int_0^\infty \mu(x) f(x,1)\,dx, \quad Bf(x,1) = 0, \tag{4.3}$$

$$Bf(x,n) = \alpha f(x,n-1), \quad x > 0, n \geq 2, f \in \mathcal{D}. \tag{4.4}$$

Now using Theorem 3.2 we prove the following generation theorem which implies Theorem 2.1.

THEOREM 4.1. *The operator $(A_\Psi + B, \mathcal{D}(A_\Psi))$ is the generator of a stochastic semigroup $\{P(t)\}_{t\geq 0}$ on L^1.*

Before we prove Theorem 4.1 we state a couple of lemmas.

LEMMA 4.2. *The resolvent of the operator $A_0 f = Af$, $f \in \mathcal{D}(A_0) = \{f \in \mathcal{D}\colon \Psi_0 f = 0\}$, at $\lambda \geq 0$ is of the form*

$$R(\lambda, A_0) f(0,0) = \frac{1}{\lambda+\alpha} f(0,0),$$
$$R(\lambda, A_0) f(x,n) = e^{-\Lambda_\lambda(x)} \int_0^x e^{\Lambda_\lambda(y)} f(y,n)\,dy, \quad x > 0, n \geq 1,$$

where

$$\Lambda_\lambda(x) = (\lambda+\alpha)x + \int_0^x \mu(z)\,dz. \tag{4.5}$$

PROOF. The idea of the proof comes from [19]. Let $f \in L^1$ and $\lambda \geq 0$. We define $R_\lambda f(0,0) = f(0,0)/(\lambda+\alpha)$ and

$$R_\lambda f(x,n) = e^{-\Lambda_\lambda(x)} \int_0^x e^{\Lambda_\lambda(y)} f(y,n)\,dy, \quad x > 0, n \geq 1.$$

We first show that $R_\lambda f \in \mathcal{D}$. If $0 \le y \le x$ then we have

$$\int_0^y \mu(z)dz \le \int_0^x \mu(z)dz.$$

This and Fubini's theorem give

$$\begin{aligned}\int_0^\infty e^{-\Lambda_\lambda(x)} \int_0^x e^{\Lambda_\lambda(y)}|f(y,n)|dydx &= \int_0^\infty \int_y^\infty e^{-\Lambda_\lambda(x)}dx\, e^{\Lambda_\lambda(y)}|f(y,n)|dy \\ &\le \int_0^\infty \int_y^\infty e^{-(\lambda+\alpha)(x-y)}dx|f(y,n)|dy\end{aligned}$$

for all $n \ge 1$, implying that $\|R_\lambda f\| \le \|f\|/(\lambda+\alpha)$. Hence R_λ is a bounded linear operator. Now, observe that

$$\int_y^\infty \mu(x)e^{-\Lambda_\lambda(x)}dx = \int_y^\infty b(x)e^{-(\lambda+\alpha)x}dx \le e^{-\Lambda_\lambda(y)}, \quad y \ge 0.$$

Thus

$$\sum_{n=1}^\infty \int_0^\infty \mu(x)|R_\lambda f(x,n)|dx \le \|f\| < \infty.$$

Since the function $x \mapsto e^{-\Lambda_\lambda(x)}$ is absolutely continuous and $y \mapsto e^{\Lambda_\lambda(y)}f(y,n)$ is integrable on intervals $(0,x)$, $x>0$, we see that $R_\lambda f$ is absolutely continuous. For any absolutely continuous function $f(\cdot,n)$ we have

$$\lambda f(x,n) - Af(x,n) = e^{-\Lambda_\lambda(x)}\frac{\partial}{\partial x}\left(e^{\Lambda_\lambda(x)}f(x,n)\right). \tag{4.6}$$

Consequently, $(\lambda I - A)R_\lambda f = f$ and $R_\lambda f \in \mathcal{D}$. Since

$$|R_\lambda f(x,n)| \le \int_0^x |f(y,n)|dy, \quad x>0, n \ge 1,$$

we see that $\Psi_0(R_\lambda f)(0,n) = 0$ for all n. Hence $R_\lambda f \in \mathcal{D}(A_0)$. It is easy to check that

$$R_\lambda(\lambda f - A_0 f) = f, \quad f \in \mathcal{D}(A_0).$$

We conclude that $R_\lambda = R(\lambda, A_0)$ as required. □

Lemma 4.3. *Let $\lambda \geq 0$. The right inverse $\Psi(\lambda)$ of Ψ_0 restricted to the nullspace of the operator $\lambda I - A$ is given by $\Psi(\lambda) f_\partial(0,0) = 0$ and*

$$\Psi(\lambda) f_\partial(x,n) = f_\partial(0,n) e^{-\Lambda_\lambda(x)}, \quad n \geq 1, x > 0, f_\partial \in L^1_\partial, \tag{4.7}$$

where Λ_λ is defined by (4.5). If Ψ is as in (4.1)–(4.2) then

$$\|\Psi\Psi(\lambda)\| \leq c_\lambda < 1, \quad \text{where } c_\lambda = \int_0^\infty \mu(x) e^{-\Lambda_\lambda(x)} dx. \tag{4.8}$$

Moreover,

$$\begin{aligned}(I - \Psi\Psi(\lambda))^{-1} & \Psi R(\lambda, A_0) f(0,1) \\ & \leq \frac{\alpha}{\lambda+\alpha} f(0,0) + \sum_{k=0}^{\infty} c_\lambda^k \int_0^\infty f(x,k+2)\, dx \leq \|f\|\end{aligned} \tag{4.9}$$

for any nonnegative $f \in L^1$.

Proof. The first claim follows from formula (4.6). Since $\Psi(\lambda) f(0,0) = 0$, we get

$$\Psi\Psi(\lambda) f_\partial(0,n) = \int_0^\infty \mu(x) f_\partial(0,n+1) e^{-\Lambda_\lambda(x)} dx, \quad n \geq 1. \tag{4.10}$$

This gives the first inequality in (4.8). From (2.1) and (4.5) we have

$$\int_0^\infty \mu(x) e^{-\Lambda_\lambda(x)} dx = \int_0^\infty b(x) e^{-(\lambda+\alpha)x} dx.$$

Since b is a probability density function, the last integral is strictly smaller than 1, completing the proof of (4.8).

For each $f_\partial \in L^1_\partial$ and $n \geq 1$ we have $\Psi\Psi(\lambda) f_\partial(0,n) = c_\lambda f_\partial(0,n+1)$, by (4.10), implying that

$$(I - \Psi\Psi(\lambda))^{-1} f_\partial(0,n) = \sum_{k=0}^{\infty} (\Psi\Psi(\lambda))^k f_\partial(0,n) = \sum_{k=0}^{\infty} c_\lambda^k f_\partial(0,k+n).$$

Let $f \in L^1$ be nonnegative. Then

$$\Psi R(\lambda, A_0) f(0,1) = \frac{\alpha}{\lambda+\alpha} f(0,0) + \int_0^\infty \mu(x) R(\lambda, A_0) f(x,2)\, dx$$

and

$$\Psi R(\lambda, A_0) f(0,n) = \int_0^\infty \mu(x) R(\lambda, A_0) f(x,n+1)\, dx, \quad n \geq 2.$$

Since

$$\int_0^\infty \mu(x)R(\lambda, A_0)f(x,n+1)dx \le \int_0^\infty f(x,n+1)dx$$

for any $n \ge 2$, we get (4.9). □

Now we are ready to prove the generation theorem using Theorem 3.2.

Proof of Theorem 4.1. It follows from Lemma 4.3 that conditions (i) and (ii) hold. By Lemma 4.2, the operator $(A_0, \mathcal{D}(A_0))$ is resolvent positive. Since the space of Lipschitz continuous functions with compact support is contained in $\mathcal{D}(A_0)$ and is dense in L^1, the domain of the operatorA_0 is dense in L^1. For nonnegative $f \in \mathcal{D}$ we have

$$\begin{aligned}\int_E Af\,dm &= Af(0,0) + \sum_{n=1}^\infty \int_0^\infty Af(x,n)dx \\ &= -\alpha f(0,0) + \sum_{n=1}^\infty \left(\Psi_0 f(0,n) - \int_0^\infty (\alpha+\mu(x))f(x,n)dx\right) \\ &= -\alpha f(0,0) - \sum_{n=1}^\infty \int_0^\infty (\alpha+\mu(x))f(x,n)dx + \int_{E_\partial} \Psi_0 f\,dm_\partial.\end{aligned} \tag{4.11}$$

This implies that (3.1) holds for all nonnegative $f \in \mathcal{D}$. Thus, we conclude that the operator A satisfies conditions (iii)–(iv).

It remains to check conditions (3.4) and (3.5). Since

$$\int_E Bf\,dm = \int_0^\infty \mu(x)f(x,1)dx + \sum_{n=2}^\infty \int_0^\infty \alpha f(x,n-1)dx$$

and (4.11) holds for nonnegative $f \in \mathcal{D}(A_\Psi) \subset \mathcal{D}$, we conclude that

$$\int_E (A_\Psi f + Bf)dm = -\alpha f(0,0) - \sum_{n=2}^\infty \int_0^\infty \mu(x)f(x,n)dx + \int_{E_\partial} \Psi_0 f = 0$$

and so (3.4) holds. The operator $BR(\lambda, A_\Psi)$ is substochastic (see Remark 3.3). Thus $0 \le \|(BR(\lambda, A_\Psi))^{n+1}\| \le \|(BR(\lambda, A_\Psi))^n\|$ for all n. Consequently, the limit in (3.5) exists. We will now show that the limit in (3.5) is equal to zero. Since $Bf(\cdot,1) = 0$ for $f \in \mathcal{D}$, we see that for any nonnegative $f \in \mathcal{D}$

$$\|BR(\lambda, A_\Psi)Bf\| = BR(\lambda, A_\Psi)Bf(0,0) + \sum_{n=2}^\infty \int_0^\infty BR(\lambda, A_\Psi)Bf(x,n)dx$$

and $R(\lambda, A_0)Bf(\cdot,1) = 0$. It follows from (3.3) and (4.7) that

$$\begin{aligned} BR(\lambda, A_\Psi)Bf(0,0) &= \int_0^\infty \mu(x)R(\lambda, A_\Psi)Bf(x,1)dx \\ &= c_\lambda(I - \Psi\Psi(\lambda))^{-1}\Psi R(\lambda, A_0)Bf(0,1). \end{aligned}$$

This together with (4.9) and (4.4) implies that

$$\begin{aligned} BR(\lambda, A_\Psi)Bf(0,0) &\le c_\lambda \frac{\alpha}{\lambda+\alpha} Bf(0,0) + c_\lambda \sum_{k=0}^\infty c_\lambda^k \int_0^\infty Bf(x,k+2)dx \\ &\le c_\lambda \frac{\alpha}{\lambda+\alpha} Bf(0,0) + \alpha c_\lambda \|f\|. \end{aligned}$$

We have

$$\begin{aligned} \sum_{n=2}^\infty \int_0^\infty BR(\lambda, A_\Psi)Bf(x,n)dx &= \alpha \sum_{n=1}^\infty \int_0^\infty R(\lambda, A_\Psi)Bf(x,n)dx \\ &= \alpha \left(\|R(\lambda, A_\Psi)Bf\| - R(\lambda, A_\Psi)Bf(0,0)\right). \end{aligned}$$

Since $\Psi(\lambda)f_\partial(0,0) = 0$ for any $f_\partial \in L^1_\partial$, we get

$$R(\lambda, A_\Psi)Bf(0,0) = R(\lambda, A_0)Bf(0,0) = \frac{1}{\lambda+\alpha}Bf(0,0).$$

We have $\lambda\|R(\lambda, A_\Psi)\| \le 1$, and so

$$\|R(\lambda, A_\Psi)Bf\| \le \frac{1}{\lambda}\|Bf\| \le \frac{1}{\lambda}Bf(0,0) + \frac{\alpha}{\lambda}\|f\|,$$

where we used (4.4). Consequently,

$$\|BR(\lambda, A_\Psi)Bf\| \le \frac{\alpha}{\lambda}Bf(0,0) + \frac{\alpha^2}{\lambda}\|f\| + \alpha\|f\|$$

for any nonnegative $f \in \mathcal{D}$. Now, if we take a nonnegative $f \in L^1$ then $R(\lambda, A_\Psi)f$ is nonnegative and $R(\lambda, A_\Psi)f \in \mathcal{D}$ with $\|R(\lambda, A_\Psi)f\| \le \|f\|/\lambda$. This together with (4.9) implies that

$$\begin{aligned} \|(BR(\lambda, A_\Psi))^2 f\| &\le \frac{\alpha}{\lambda}BR(\lambda, A_\Psi)f(0,0) + \left(\frac{\alpha^2}{\lambda} + \alpha\right)\|R(\lambda, A_\Psi)f\| \\ &\le \left(\frac{2\alpha}{\lambda} + \frac{\alpha^2}{\lambda^2}\right)\|f\| \end{aligned}$$

and shows that $\|(BR(\lambda,A))^2\| \le \alpha(\alpha+2\lambda)/\lambda^2$. Thus for sufficiently large λ the norm of the iterates of $BR(\lambda,A)$ converges to zero. □

In the proof of Theorem 2.2 we need a couple of lemmas.

LEMMA 4.4. *If the traffic intensity satisfies $\rho < 1$ then the stochastic semigroup $\{P(t)\}_{t\ge 0}$ from Theorem 4.1 has a unique invariant density f^*, $f^* > 0$ a.e., and $f^*(\cdot,n) = p_n^*$ for $n \ge 0$, where p_n^* is as in* (2.11). *Conversely, if $\{P(t)\}_{t\ge 0}$ has an invariant density then $\rho < 1$.*

PROOF. We have

$$R(0,A_0)f(x,n) = e^{-\Lambda_0(x)}\int_0^x e^{\Lambda_0(y)}f(y,n)dy, \quad x>0, n\ge 1,$$

by Lemma 4.2, and

$$\Psi(0)f_\partial(x,n) = e^{-\Lambda_0(x)}f_\partial(0,n), \quad n \ge 1,$$

by Lemma 4.3. The operators $R(0,A_0)$ and $\Psi(0)$ are bounded. Moreover, $\|\Psi\Psi(0)\| < 1$, by (4.8). It follows from Theorem 3.4 that $\{P(t)\}_{t\ge 0}$ has an invariant density if and only if the operator K defined as in (3.6) has an invariant density. To find an invariant density f_* for K we make use of (3.9). We have

$$f_*(x,n) = BR(0,A_0)f_*(x,n) + B\Psi(0)f_\partial(x,n), \quad (x,n)\in E,$$

$$f_\partial(0,n) = \Psi R(0,A_0)f_*(0,n) + \Psi\Psi(0)f_\partial(0,n), \quad n\ge 1.$$

Since $Bf(\cdot,1) = 0$ for all $f \in \mathcal{D}$, we see that $f_*(\cdot,1) = 0$. This and (4.3) show that $BR(0,A_0)f_*(0,0) = 0$. Thus

$$\begin{aligned} f_*(0,0) &= \int_0^\infty \mu(x)\Psi(0)f_\partial(x,1)dx \\ &= \int_0^\infty \mu(x)e^{-\Lambda_0(x)}dx f_\partial(0,1) = a_0 f_\partial(0,1). \end{aligned}$$

We also have $BR(0,A_0)f_*(\cdot,2) = 0$ and $\Psi\Psi(0)f_\partial(0,n) = a_0 f_\partial(0,n+1)$, $n \ge 1$. Hence,

$$f_*(x,2) = B\Psi(0)f_\partial(x,2) = \alpha\Psi(0)f_\partial(x,1) = \alpha e^{-\Lambda_0(x)}f_\partial(0,1), \quad x>0,$$

and

$$f_\partial(0,1) = f_*(0,0) + a_1 f_\partial(0,1) + a_0 f_\partial(0,2). \tag{4.12}$$

It is easily seen that for $n \geq 2$ we have

$$f_*(x,n) = \alpha e^{-\Lambda_0(x)} \sum_{k=0}^{n-2} \frac{(\alpha x)^k}{k!} f_\partial(0, n-1-k), \quad x > 0,$$

and

$$f_\partial(0,n) = \sum_{k=0}^{n} a_k f_\partial(0, n+1-k). \tag{4.13}$$

To find the sequence $q_n = f_\partial(0,n)$, $n \geq 1$, we use the generating function approach. Multiplying (4.13) by z^n and (4.12) by z, we calculate that

$$\begin{aligned} G_q(z) = \sum_{n=1}^{\infty} f_\partial(0,n) z^n &= f_*(0,0)z + \sum_{n=1}^{\infty} \sum_{k=0}^{n} a_k f_\partial(0, n+1-k) z^n \\ &= f_*(0,0)z + G_a(z) \frac{G_q(z)}{z} - f_*(0,0), \end{aligned}$$

where G_a is as in (2.9). Hence,

$$G_q(z) = \frac{f_*(0,0)(1-z)z}{G_a(z) - z}, \quad z \in (0,1).$$

Note that

$$\sum_{n=1}^{\infty} \int_0^{\infty} f_*(x,n)dx = \rho \sum_{n=1}^{\infty} f_\partial(0,n)$$

and

$$\sum_{n=1}^{\infty} f_\partial(0,n) = \lim_{z \to 1^-} G_q(z) = \frac{f_*(0,0)}{1-\rho}.$$

Consequently, $\|f_*\| = 1$ if and only if

$$f_*(0,0) + \rho \frac{f_*(0,0)}{1-\rho} = 1.$$

Thus, f_* is an invariant density for the operator K if and only if $f_*(0,0) = 1 - \rho > 0$.

It follows from (3.10) that $\overline{f}_* = R(0, A_0)f_* + \Psi(0)f_\partial$ is invariant for the semigroup $\{P(t)\}_{t\geq 0}$. We have $\overline{f}_*(0,0) = f_*(0,0)/\alpha$ and

$$\begin{aligned}\overline{f}_*(x,n) &= e^{-\Lambda_0(x)}\left(\int_0^x e^{\Lambda_0(y)} f_*(y,n)dy + f_\partial(0,n)\right)\\ &= e^{-\Lambda_0(x)}\sum_{k=0}^{n-1}\frac{(\alpha x)^k}{k!}f_\partial(0,n-k), \quad x>0, n\geq 1.\end{aligned}$$

Note that

$$\sum_{n=1}^{\infty}\int_0^\infty \overline{f}_*(x,n)dx = \frac{\rho}{\alpha}\sum_{n=1}^{\infty} f_\partial(0,n).$$

If we take $f^* = \alpha\overline{f}_*$ then $f_\partial(0,1)>0$ and f^* is a strictly positive invariant density for the semigroup $\{P(t)\}_{t\geq 0}$. □

Lemma 4.5. *The stochastic semigroup $\{P(t)\}_{t\geq 0}$ from Theorem 4.1 is partially integral.*

Proof. We show that for each $t>0$ there exists a nonnegative measurable function $k_t(y)$ such that

$$P(t)f(0,0) \geq \int_0^\infty k_t(y)f(y,1)dy$$

for all densities f and that there exists $t>0$ such that

$$\int_0^\infty k_t(y)dy > 0.$$

To this end, recall that the stochastic semigroup $\{P(t)\}_{t\geq 0}$ obtained with the help of Theorem 3.2 is given by the Dyson–Phillips expansion ([3, 4])

$$P(t)f = \sum_{n=0}^{\infty} S_n(t)f, \quad f\in L^1,$$

where $\{S_0(t)\}_{t\geq 0}$ is the substochastic semigroup generated by the operator $(A_\Psi, \mathcal{D}(A_\Psi))$ and

$$S_{n+1}(t)f = \int_0^t S_n(t-s)BS_0(s)f\,ds, \quad f\in\mathcal{D}(A_\Psi), n\geq 0.$$

It follows from (3.3) that $R(\lambda, A_\Psi) - R(\lambda, A_0)$ is a positive operator. Thus the semigroup $\{S_0(t)\}_{t\geq 0}$ dominates the semigroup $\{S(t)\}_{t\geq 0}$ generated by the operator $(A_0, \mathcal{D}(A_0))$, and so we obtain $S_0(t)f \geq S(t)f$ for all nonnegative f. It is easy to check that (see [19])

$$S(t)f(x,n) = f(x-t,n)1_{(0,\infty)}(x-t)e^{-\int_{x-t}^{x}\mu(z)dz}, \quad x>0, n\geq 1,$$

and $S(t)f(0,0) = e^{-\alpha t}f(0,0)$, $t>0$, $f \in L^1$. Since B is a positive operator, we see that

$$P(t)f \geq S_2(t)f \geq \int_0^t S(t-s)BS(s)f\,ds$$

for all nonnegative $f \in \mathcal{D}$. Consequently, we get

$$\begin{aligned} P(t)f(0,0) &\geq \int_0^t e^{-\alpha(t-s)} \int_s^\infty \mu(x)f(x-s,1)e^{-\int_{x-s}^{x}\mu(z)dz} dxds \\ &\geq \int_0^\infty k_t(y)f(y,1)dy, \end{aligned}$$

where

$$k_t(y) = \int_0^t e^{-\alpha(t-s)}e^{-\int_y^{y+s}\mu(z)dz}ds, \quad y>0,$$

is nontrivial for sufficiently large t. Now taking $q(x,n,y,k)=0$ for $n,k\geq 0$, $x,y\geq 0$, and $q(0,0,y,1) = k_t(y)$, we conclude that the semigroup $\{P(t)\}_{t\geq 0}$ is partially integral. □

Proof of Theorem 2.2. Suppose first that $\rho < 1$. It follows from Lemma 4.4 that there is an invariant density for the semigroup $\{P(t)\}_{t\geq 0}$ of the given form and that it is unique and strictly positive a.e. This together with Lemma 4.5 implies that all assumptions of Theorem 3.5 hold. Thus the semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable. Suppose now that $\{P(t)\}_{t\geq 0}$ is asymptotically stable. Then it has an invariant density. Thus $\rho < 1$, by Lemma 4.4. □

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