Annales Mathematicae Silesianae **33** (2019), 159–167
DOI: 10.2478/amsil-2018-0011

APPROXIMATE ANALYTICAL SOLUTIONS TO CONFORMABLE MODIFIED BURGERS EQUATION USING HOMOTOPY ANALYSIS METHOD

Ali Kurt, Orkun Tasbozan

Abstract. In this paper the authors aspire to obtain the approximate analytical solution of Modified Burgers Equation with newly defined conformable derivative by employing homotopy analysis method (HAM).

1. Introduction

Differential calculus as a subfield of calculus is related to the rate of the changes in quantities. And the process of finding a derivative is called differentiation. Generally we all accustomed to derivatives of integer order. But – as is often the case – such integer order derivatives do not suffice for describing the real world phenomena. Because of the complexity of the problems arising in natural world, scientists needed a new version of derivative of not necessarily integer order. For this reason they plunged into a quest for them in explaining real world phenomena. So the adventure of the fractional order derivative started. In 1695 L'Hospital raised the question "What is the meaning of $\frac{d^n y}{dx^n}$ if $n = \frac{1}{2}$". Leibniz replied "This is an apparent paradox from which, one day, useful consequences will be drawn". From then on many versions of fractional derivative have been proposed, for instance Riemann–Liouville derivative, Caputo derivative, Grünwald–Letnikov derivative etc. ([12, 7, 15]). Nowadays

Received: 07.02.2018. Accepted: 25.08.2018. Published online: 06.12.2018.
(2010) Mathematics Subject Classification: 35R11, 34K28, 74H15.
Key words and phrases: modified Burgers equation, conformable derivative, homotopy analysis method.

a new, more applicable, more understandable derivative called "conformable derivative" has been introduced ([6]).

Definition 1. Let $f\colon [0,\infty) \to \mathbb{R}$ be a function and $\alpha \in (0,1)$. The α^{th} order *conformable fractional derivative* of f is defined by

$$T_\alpha(f)(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

for all $t > 0$. Let f is α-differentiable in some $(0,a), a > 0$ and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then define $f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t)$ and the *conformable fractional integral* of a function f starting from $a \geq 0$ is defined as

$$I_\alpha^a(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

As it is seen this new derivative looks like the limit definition of standard derivative and satisfies all the requirements of the standard derivative such as common formula of the derivative of the product of two function, formula of the derivative of the quotient of two functions, chain rule etc.

To examine the applicability and usefulness of the conformable derivative, scientists made huge number of scientific articles on it ([5, 17]). For example T. Abdeljawad ([1]) presented conformable versions of the chain rule, exponential functions, Gronwalls inequality, integration by parts, Taylor power series expansions and Laplace transform. Conformable derivative has been used by A. Kurt et al. ([8]) in obtaining the analytical and approximate solution of Burgers equation. A. Atangana et al. ([2]) introduced the new properties of conformable derivative. M. Eslami ([4]) obtained the exact travelling wave solutions for fractional coupled nonlinear Schrödinger equations. Hence, there are many open problems to be considered in this new area.

Some properties of this new definition are given in references [6, 1].

In this paper Homotopy Analysis Method (HAM) is applied to modified Burgers equation to obtain the approximate analytical solution. HAM includes an auxiliary parameter $\hbar$ which is used for adjusting and arranging the convergence region of the solution.

Modified Burgers equation which is one of the important type of Burgers equation arises in various scientific areas such as, plasma physics, solid-state physics, optical fibers, biology, fluid dynamics, chemical kinetics etc. ([14]). As a consequence of this importance, scientists paid great attention to obtain the exact or numerical solutions of modified Burgers equation.

2. Homotopy analysis method

In order to show the fundamentals of HAM which is an effective and powerful mathematical method for finding the approximate analytical solution of nonlinear partial differential equations ([16, 13]) let's give a brief description of the method. Consider the following differential equation

$$\mathcal{N}\left[u(x,t)\right] = 0$$

where $\mathcal{N}$ is a nonlinear operator, x and t show independent variables and $u(x,t)$ is an unknown function. By means of HAM, we can construct the zero-order deformation equation ([9, 10])

$$(1-p)\mathcal{L}\left[\phi(x,t;p) - u_0(x,t)\right] = p\hbar\mathcal{N}\left[\phi(x,t;p)\right] \tag{2.1}$$

where $p \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, $\mathcal{L}$ is an auxiliary linear operator, $u_0(x,t)$ is an initial guess of $u(x,t)$, $\phi(x,t;p)$ is an unknown function, succesively. In this way, it would be conceivable to choose auxiliary parameters and operators in HAM. When p is chosen as $p = 0$ and $p = 1$ it is obvious that equation (2.1) becomes

$$\phi(x,t;0) = u_0(x,t), \quad \phi(x,t;1) = u(x,t)$$

respectively. Thus, as long as p differs from 0 to 1, the solutions $\phi(x,t;p)$ changes from the initial value $u_0(x,t)$ to the solution $u(x,t)$. If we expand $\phi(x,t;p)$ to Taylor series with respect to the embedding parameter p, we get

$$\phi(x,t;p) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)p^m,$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;p)}{\partial p^m}\bigg|_{p=0}. \tag{2.2}$$

Then choosing auxiliary linear operator, the initial guess and the auxiliary parameter $\hbar$ in a proper way, the series expressed above converges at $p = 1$, and

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t),$$

which must be one of the solutions of the genuine nonlinear equation, as expressed by Liao ([10, 11]). With reference to (2.2), the governing equation can be reduced from the zero-order deformation equation (2.1). Define the vector

$$\overrightarrow{\mathbf{u}_n} = \{u_0(x,t), u_1(x,t), ..., u_n(x,t)\}.$$

Differentiating equation (2.1) m-times with respect to p, then setting $p = 0$ and dividing by $m!$, we obtain the m

$$\mathcal{L}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = \hbar R_m\left(\mathbf{u}_{m-1}\right), \tag{2.3}$$

where

$$R_m\left(\mathbf{u}_{m-1}\right) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}\left[\phi(x,t;p)\right]}{\partial p^{m-1}}\bigg|_{p=0}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

It should be emphasized that $u_m(x,t)$ for $m \geq 1$ is governed by linear equation (2.3) with the boundary condition that comes from the problem. Hence, it can be easily solved by using symbolic computation software such as Mathematica.

To show the effectiveness, convenience and applicability of the methods on nonlinear CPDEs, we employ the method to the conformable Modified Burgers Equation given in the next section.

3. Application of the HAM

Consider the Modified Burgers Equation ([3])

$$\frac{\partial^\alpha u}{\partial t^\alpha} + u^2\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0, \tag{3.1}$$

where $\alpha \in (0,1)$, the derivative means conformable derivative, with initial condition

$$u(x,0) = -\frac{\sqrt{3}}{\sqrt{1-\cosh(2x)-\sinh(2x)}} \tag{3.2}$$

obtained by giving the values $r = 0$, $\nu = 1$, $c = 1$ to the analytical solution in article [3].

To solve the equation (3.1) with initial condition (3.2) by aid of homotopy analysis method, let us consider the linear operator defined as follows

$$\mathcal{L}\left[\phi(x,t;p)\right] = D_t^{\alpha}\phi(x,t;p)$$

with the property

$$\mathcal{L}\left[s\right] = 0$$

where s is constant. From the equation (3.1) and for the given special value of $\nu = 1$ the nonlinear operator can be defined as follows

$$\mathcal{N}\left[\phi(x,t;p)\right] = \frac{\partial^{\alpha}\phi(x,t;p)}{\partial t^{\alpha}} + \phi(x,t;p)^2\frac{\partial\phi(x,t;p)}{\partial x} - \frac{\partial^2\phi(x,t;p)}{\partial x^2}.$$

From the properties of this new definition given in [6], nonlinear operator can be written as follows

$$\mathcal{N}\left[\phi(x,t;p)\right] = t^{1-\alpha}\frac{\partial\phi(x,t;p)}{\partial t} + \phi^2(x,t;p)\frac{\partial\phi(x,t;p)}{\partial x} - \frac{\partial^2\phi(x,t;p)}{\partial x^2}.$$

Thus, the zero-order deformation equation is set up as

$$(1-p)\mathcal{L}\left[\phi(x,t;p) - u_0(x,t)\right] = p\hbar\mathcal{N}\left[\phi(x,t;p)\right]. \tag{3.3}$$

By choosing $p = 0$ and $p = 1$ we get

$$\phi(x,t;0) = u_0(x,t) = u(x,0), \quad \phi(x,t;1) = u(x,t).$$

Therefore, since the embedding parameter p increases from 0 to 1, the solution $\phi(x,t;p)$ differs from the initial value $u_0(x,t)$ from the solution $u(x,t)$. By expanding $\phi(x,t;p)$ in Taylor series dependent to the embedding parameter p:

$$\phi(x,t;p) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)p^m$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m\phi(x,t;p)}{\partial p^m}\bigg|_{p=0}.$$

Supposing that the auxiliary linear operator, the initial guess and the auxiliary parameter $\hbar$ are properly chosen, the above series converges at $p = 1$, and

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t),$$

which must be one of the solutions of the original nonlinear equations, as proven by Liao ([10]). Then if we differentiate equation (3.3) m times with respect to the embedding parameter p, we have the mth-order deformation equation

$$\mathcal{L}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = \hbar R_m\left(\mathbf{u}_{m-1}\right) \tag{3.4}$$

where

$$\begin{aligned} R_m\left(\mathbf{u}_{m-1}\right) = t^{1-\alpha}\frac{\partial u_{m-1}(x,t)}{\partial t} &- \frac{\partial^2 u_{m-1}(x,t)}{\partial x^2} \\ &+ \sum_{n=0}^{m-1}\left(\sum_{k=0}^{n} u_k(x,t)u_{n-k}(x,t)\right)\frac{\partial u_{m-1-n}(x,t)}{\partial x} \end{aligned}$$

and

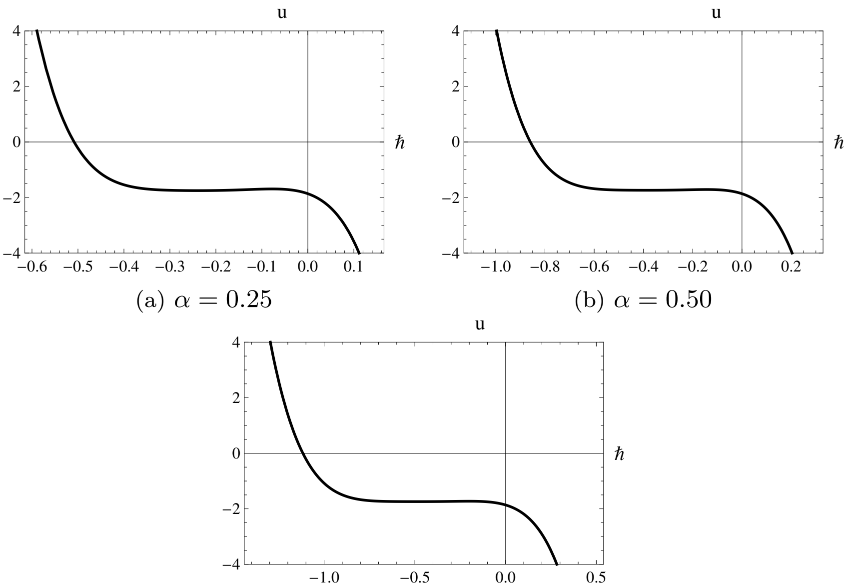
$$\chi_m = \begin{cases} 0, m \leq 1, \\ 1, m > 1. \end{cases}$$

The solutions of the mth-order deformation equation (3.4) for $m \geq 1$ result in

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar\mathcal{L}^{-1}\left[R_m\left(\mathbf{u}_{m-1}\right)\right]. \tag{3.5}$$

By using (3.5) with initial condition given by (3.2) we obtain respectively

$$\begin{aligned} u_0(x,t) &= -\frac{\sqrt{3}}{\sqrt{1-\cosh(2x)-\sinh(2x)}}, \\ u_1(x,t) &= -\frac{\sqrt{3}e^{2x}ht^{\alpha}}{\left(1-e^{2x}\right)^{3/2}\alpha}, \\ u_2(x,t) &= -\frac{\sqrt{3}e^{2x}ht^{\alpha}}{\left(1-e^{2x}\right)^{3/2}\alpha} - \frac{\sqrt{3}e^{3x}h^2t^{\alpha}\left(3t^{\alpha}\cosh(x)-\left(t^{\alpha}+4\alpha\right)\sinh(x)\right)}{2\left(1-e^{2x}\right)^{5/2}\alpha^2}, \\ &\ \vdots \end{aligned}$$



(a) $\alpha = 0.25$ (b) $\alpha = 0.50$

(c) $\alpha = 0.75$

Figure 1. The $\hbar$-curves of 5th-order approximate solutions obtained by the HAM for different values of α

So, the series solutions found out by HAM can be written in the form

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \dots . \tag{3.6}$$

The series solutions of $u(x,t)$ can be obtained by using (3.6).

The auxiliary parameter $\hbar$, which is involved in our HAM solution series, provides us with a simple way to adjust and control the convergence of the solution series. To obtain a suitable range for $\hbar$, we consider the so-called $\hbar$-curves for different values of α, shown in Figure 1, to choose an appropriate value of $\hbar$ which guarantee that the solution series is convergent, as pointed by Liao ([10]), by finding the valid region of $\hbar$ which corresponds to the line segments nearly parallel to the horizontal axis.

To show the applicability of the method, the HAM solutions given by (3.6) of the conformable Modified Burgers Equation are compared with its exact solution from [3]

$$u(x,t) = -\frac{\sqrt{3}}{\sqrt{1-\cosh\left(2x-2\frac{t^\alpha}{\alpha}\right)-\sinh\left(2x-2\frac{t^\alpha}{\alpha}\right)}}$$

given in Tables 1–3 for $x=-2$, different values of α and $\hbar$.

Table 1. Absolute errors and numerical solutions of problem for $x = -2$, $\alpha = 0.25$, $\hbar = -0.25$

t	Numerical	Exact	Error
0.05	−1.73107	−1.73223	0.00116
0.10	−1.73155	−1.73213	0.00058
0.15	−1.73208	−1.73209	0.00001
0.20	−1.73255	−1.73208	0.00048
0.25	−1.73296	−1.73207	0.00090

Table 2. Absolute errors and numerical solutions of problem for $x = -2$, $\alpha = 0.50$, $\hbar = -0.30$

t	Numerical	Exact	Error
0.05	−1.73883	−1.73655	0.00229
0.10	−1.73639	−1.73471	0.00168
0.15	−1.73496	−1.73383	0.00113
0.20	−1.73401	−1.73332	0.00070
0.25	−1.73336	−1.73299	0.00037

Table 3. Absolute errors and numerical solutions of problem for $x = -2$, $\alpha = 0.75$, $\hbar = -0.50$

t	Numerical	Exact	Error
0.05	−1.73262	−1.74201	0.00939
0.10	−1.73206	−1.73924	0.00718
0.15	−1.73164	−1.73746	0.00582
0.20	−1.73130	−1.73621	0.00491
0.25	−1.73103	−1.73531	0.00428

4. Conclusion

In this paper, HAM is used to obtain the approximate analytical solution of modified Burgers equation where derivatives are denoted by conformable derivative. It is known that modified Burgers equation is one of the attractive nonlinear equations because of its applications in number theory, gas dynamics, heat conduction, elasticity theory, turbulence theory, shock wave theory, fluid mechanics, termaviscous fluids, hydrodynamic waves, and elastic waves. Consequently, it is seen that HAM is an effective, efficient and suitable method for obtaining approximate analytical solution of conformable fractional partial differential equations.

References

[1] Abdeljawad T., *On conformable fractional calculus*, J. Comput. Appl. Math. **279** (2015), 57–66.

[2] Atangana A., Baleanu D., Alsaedi A., *New properties of conformable derivative*, Open Math. **13** (2015), 889–898.

[3] Çenesiz Y., Baleanu D., Kurt A., Tasbozan O., *New exact solutions of Burgers' type equations with conformable derivative*, Waves Random Complex Media **27** (2017), no. 1, 103–116.

[4] Eslami M., *Exact traveling wave solutions to the fractional coupled nonlinear Schrodinger equations*, Appl. Math. Comput. **285** (2016), 141–148.

[5] He S., Sun K., Mei X., Yan B., Xu S., *Numerical analysis of a fractional-order chaotic system based on conformable fractional-order derivative*, Eur. Phys. J. Plus **132** (2017), no. 1, Art. 36, 11 pp.

[6] Khalil R., Al Horani M., Yousef A., Sababheh M., *A new definition of fractional derivative*, J. Comput. Appl. Math. **264** (2014), 65–70.

[7] Kilbas A.A., Srivastava H.M., Trujillo J.J., *Theory and Applications of Fractional Differential Equations*, Elsevier Science, San Diego, 2006.

[8] Kurt A., Çenesiz Y., Tasbozan O., *On the solution of Burgers' equation with the new fractional derivative*, Open Phys. **13** (2015), 355–360.

[9] Liao S.J., *The proposed homotopy analysis technique for the solution of nonlinear problems*, Ph.D. Thesis, Shanghai Jiao Tong University, Shanghai, 1992.

[10] Liao S.J., *Beyond Perturbation. Introduction to the Homotopy Analysis Method*, Chapman & Hall/CRC, Boca Raton, 2004.

[11] Liao S.J., *Notes on the homotopy analysis method: some definitions and theorems*, Commun. Nonlinear Sci. Numer. Simul. **14** (2009), no. 4, 983–997.

[12] Miller K.S., Ross B., *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, 1993.

[13] Nicolescu B.N., Macarie T., Petrescu T.C., *Application of homotopy analysis method for solving equation of vehicle's move in the linear case*, Applied Mechanics and Materials **822** (2016), 3–11.

[14] Oruç Ö., Bulut F., Esen A., *A Haar wavelet-finite difference hybrid method for the numerical solution of the modified Burgers' equation*, J. Math. Chem. **53** (2015), no. 7, 1592–1607.

[15] Podlubny I., *Fractional Differential Equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Academic Press, San Diego, 1999.

[16] Tasbozan O., Kurt A., *Approximate analytical solution of ZK-BBM equation*, Sohag J. Math. **2** (2015), no. 2, 57–60.

[17] Ünal E., Gökdoğan A., *Solution of conformable fractional ordinary differential equations via differential transform method*, Optik **128** (2017), 264–273.

Department of Mathematics
Faculty of Science and Art
Mustafa Kemal University
Hatay, 31000
Turkey
e-mail: alikurt@mku.edu.tr
e-mail: otasbozan@mku.edu.tr