

ISOMORPHISM THEOREMS FOR COALGEBRAS

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Abstract. Let F be an endofunctor of a category \mathcal{C} . We prove isomorphism theorems for F -coalgebras under condition that the underlying category \mathcal{C} is exact; that is, regular with exact sequences. Also, F is not assumed to preserve pullbacks.

1. Introduction

The isomorphism theorems are three well known results in universal algebra. They belong to the folklore now. The situation in the coalgebra theory in contrast, is quite different. Given a *Set*-endofunctor which preserves (weak) pullbacks, the three isomorphism theorems for coalgebras hold due to Rutten [6]. An important observation is that the category *Set* has exact sequences; this means that every equivalence relation in *Set* is the kernel pair of its coequalizer. Further, for the category *Set* the first isomorphism theorem holds; that is, every image is isomorphic to a kernel pair factor. The reason is that *Set* has (regular epi)-mono factorizations. Consequently, the first isomorphism theorem for coalgebras holds for given any *Set*-endofunctor (see [3]). But this result depends on the axiom of choice. For instance, the axiom of choice is needed to show that every mono splits in the category *Set*.

In an arbitrary category, the presence of the axiom of choice is not always guaranteed. This note does not consider that hypothesis. In addition, the preservation of (weak) pullbacks is a very restrictive condition. Assuming

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such property, we get that every congruence relation is a bisimulation equivalence. In general though, a congruence relation needs not to be a bisimulation (see [1]).

The aim of this paper is to prove isomorphism theorems for coalgebras of an endofunctor, given an appropriate assumptions on the underlying category. Especially that the underlying category is exact, which means regular with exact sequences.

Let F be an endofunctor of an arbitrary category \mathcal{C} . We prove the three isomorphism theorems for F -coalgebras, provided that the category \mathcal{C} is exact. For the first and second isomorphism theorems we need the additional condition that F preserves monos. However, we need that F preserves pullbacks along monos for the third isomorphism theorem. As might be expected, congruences in this case are not the same as bisimulation equivalences. This is a weaker hypothesis than requiring F to preserve (weak) pullbacks.

2. Some basics

We recall the categorical concepts that will be most commonly used.

2.1. Pullbacks and their preservation

The *pullback* of morphisms $f: A \rightarrow C$ and $g: B \rightarrow C$ also called the *pullback of g along f* , is a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{p_1} & A \\ p_2 \downarrow & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$$

with the following property: if $u: D \rightarrow A$ and $v: D \rightarrow B$ are morphisms with $f \circ u = g \circ v$, then there is exactly one morphism $w: D \rightarrow P$ with $u = p_1 \circ w$ and $v = p_2 \circ w$. If the uniqueness requirement for w is dropped, we call (P, p_1, p_2) a *weak pullback*. Further, if f and g are monos, the pullback of f and g is called the *intersection* of f and g . The *kernel pair* of f denoted by $\ker(f)$, is the pullback of f and itself.

A functor F is said to preserve (weak) pullbacks, if it transforms every (weak) pullback into a (weak) pullback; i.e., for every (weak) pullback (P, p_1, p_2) of f and g we get (FP, Fp_1, Fp_2) is the (weak) pullback of Ff and

Fg . However, if at least one of f and g is a mono, we say that F preserves *pullbacks along monos*. Every functor F which preserves pullbacks along monos, also preserves monos (see [5]).

2.2. Relations

In a category with binary products, a *binary relation* from A to B is a subobject of $A \times B$. This is represented by a mono $m: R \rightarrow A \times B$ or equivalently, by a pair of arrows

$$\begin{array}{ccc} & & A \\ & r_1 \nearrow & \\ R & & \\ & r_2 \searrow & \\ & & B \end{array}$$

with the property that the induced arrow $\langle r_1, r_2 \rangle: R \rightarrow A \times B$ is a mono. A relation from A to A is called a *relation on A* .

Binary relations are ordered (as subobjects of $A \times A$) and can be composed. The relational composition is defined by applying pullbacks as follows: for a given binary relation R (represented by $r_1: R \rightarrow A$ and $r_2: R \rightarrow B$) in a finitely complete category with (regular epi)-mono factorizations, form the pullback of r_1 and r_2

$$\begin{array}{ccc} R \times_A R & \xrightarrow{t_1} & R \xrightarrow{r_1} A \\ t_2 \downarrow & & \downarrow r_2 \\ R & \xrightarrow{r_1} & A \\ r_2 \downarrow & & \\ A & & \end{array}$$

Factorize $\langle r_1 \circ t_1, r_2 \circ t_2 \rangle: R \times_A R \rightarrow A \times A$ as a regular epi followed by a mono, then the latter represents the composite $R \circ R$. R is said to be *transitive* if $R \circ R$ is smaller than R . The relation R is called *reflexive* if the diagonal map $\langle 1_A, 1_A \rangle: A \rightarrow A \times A$ factors through it and *symmetric* if there is an arrow $\tau: R \rightarrow R$ such that $r_1 \circ \tau = r_2$ and $r_2 \circ \tau = r_1$. We say that R is an *equivalence relation* if it is reflexive, symmetric and transitive.

2.3. Exact sequences

In a finitely complete category \mathcal{C} with coequalizers, the diagram

$$R \begin{array}{c} \xrightarrow{r_1} \\ \rightrightarrows \\ \xrightarrow{r_2} \end{array} A \xrightarrow{e} B$$

is called an *exact sequence* if R is the kernel pair of e and e is the coequalizer of the parallel pair (r_1, r_2) . We say that the category \mathcal{C} has exact sequences if every equivalence relation R in \mathcal{C} is the kernel pair of its coequalizer. The categories *Set* of sets and mappings, *Ab* of abelian groups and their homomorphisms, and *Top* of topological spaces and continuous mappings, have exact sequences.

A category \mathcal{C} will be called *regular* if every finite diagram has a limit, if every parallel pair of morphisms has a coequalizer and if regular epis are stable under pullbacks. In a regular category, every morphism can be written as $f = m \circ e$ where m is a mono and e a regular epi (see [2]). Thereafter, regular epis in a regular category are closed under composition. A regular category with exact sequences is called *exact*. Any topos is an exact category (see [4]).

3. Coalgebras

In the following, unless otherwise stated,

- \mathcal{C} is a category;
- F denotes an endofunctor of \mathcal{C} .

An *F-coalgebra* or a *coalgebra of type F* is a pair (A, a) consisting of an object A in \mathcal{C} together with a \mathcal{C} -morphism $a: A \rightarrow FA$. A is called the *carrier* or the *underlying object* and a the *coalgebra structure* of (A, a) . Given F -coalgebras (A, a) and (B, b) , the arrow $f: A \rightarrow B$ in \mathcal{C} is called an *F-homomorphism*, if it makes the following diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{a} & FA \\ f \downarrow & & \downarrow Ff \\ B & \xrightarrow{b} & FB \end{array}$$

This definition turns the class of coalgebras of type F and their homomorphisms into a category denoted \mathcal{C}_F .

A *subcoalgebra* of an F -coalgebra (A, a) is an object S in \mathcal{C} together with a coalgebra structure $s: S \rightarrow FS$ so that (S, s) is a subobject of (A, a) .

In the special case where the endofunctor F preserves monos, a subcoalgebra of (A, a) is a subobject S of A equipped with a coalgebra structure $s: S \rightarrow FS$ making the canonical arrow $S \hookrightarrow A$ an F -homomorphism.

By a *bisimulation* between F -coalgebras (A, a) and (B, b) we mean a binary relation R from A to B equipped with a coalgebra structure $r: R \rightarrow FR$ turning both projections $r_1: A \rightarrow FA$ and $r_2: R \rightarrow FB$ into F -homomorphisms:

$$\begin{array}{ccccc}
 A & \xleftarrow{r_1} & R & \xrightarrow{r_2} & B \\
 \downarrow a & & \downarrow r & & \downarrow b \\
 FA & \xleftarrow{Fr_1} & FR & \xrightarrow{Fr_2} & FB
 \end{array}$$

A *bisimulation on* (A, a) is a bisimulation between (A, a) and (A, a) . Any bisimulation on (A, a) which is an equivalence relation is called a *bisimulation equivalence*.

Let (A, a) and (B, b) be F -coalgebras. A binary relation K from A to B is a *precongruence* if for every cospan $(A \xrightarrow{i} Z \xleftarrow{j} B)$,

if $\begin{array}{ccc} & A & \\ & \nearrow K & \searrow i \\ & & Z \\ & \nwarrow & \nearrow j \\ & B & \end{array}$ commutes then so does $\begin{array}{ccccc} & A & \xrightarrow{a} & FA & \\ & \nearrow K & & \searrow Fi & \\ & & & & FZ \\ & \nwarrow & & & \nearrow Fj \\ & B & \xrightarrow{b} & FB & \end{array}$

A *congruence relation* is a precongruence which is an equivalence relation.

Every bisimulation equivalence is a congruence relation (see [7]). But the converse holds provided that F preserves weak pullbacks and \mathcal{C} has exact sequences.

4. Isomorphism theorems

Here, we prove the isomorphism theorems for coalgebras with any type of functor given. The underlying category needs not satisfy the axiom of choice.

DEFINITION 4.1. The *image* of a morphism $f: A \rightarrow C$ is a mono $m: im(f) \hookrightarrow C$ through which f factors: there exists a morphism $e: A \rightarrow im(f)$ such that $f = m \circ e$, and which is minimal in the sense that, for any

object B with a morphism $e' : A \rightarrow B$ and a mono $m' : B \rightarrow C$ such that $f = m' \circ e'$, there exists a unique morphism $u : im(f) \rightarrow B$ such that $m = m' \circ u$.

PROPOSITION 4.2 (First isomorphism theorem). *Suppose that the category \mathcal{C} is exact and the endofunctor F preserves monos. Then for every F -homomorphism $f : (A, a) \rightarrow (B, b)$, there is an isomorphism $A/ker(f) \cong im(f)$.*

PROOF. By hypothesis, the category \mathcal{C} is regular, and therefore it has (regular epi)-mono factorizations. Then every F -homomorphism $f : (A, a) \rightarrow (B, b)$ can be decomposed in \mathcal{C} as follows:

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 & \searrow e & \nearrow m \\
 & im(f) &
 \end{array}$$

with e a regular epi and m a mono. Also, f and e have the same kernel pair. Since the category \mathcal{C} has exact sequences, $ker(f)$ is the kernel pair of the coequalizer $\pi_{ker(f)} : A \rightarrow A/ker(f)$ of its projections. Hence $im(f)$ is isomorphic to $A/ker(f)$. But, $im(f)$ is equipped with a coalgebra structure turning e and m into F -homomorphisms; this is because the endofunctor F preserves monos. Consequently, the isomorphism $A/ker(f) \cong im(f)$ holds in the category \mathcal{C}_F . □

COROLLARY 4.3. *Under assumptions of Proposition 4.2, for any F -homomorphism $f : (A, a) \rightarrow (B, b)$, $ker(f)$ is a congruence relation on (A, a) .*

PROOF. Let $f : (A, a) \rightarrow (B, b)$ be an F -homomorphism. Then $A/ker(f)$ is equipped with a coalgebra structure turning $\pi_{ker(f)}$ into an F -homomorphism. Also, $ker(f)$ is a precongruence. Indeed, given a cospan $(A \xrightarrow{i} Z \xleftarrow{j} A)$ such that the following diagram commutes; u_1 and u_2 being the structural morphisms of $ker(f)$

$$\begin{array}{ccc}
 & A & \\
 u_1 \nearrow & & \searrow i \\
 ker(f) & & Z \\
 u_2 \searrow & & \nearrow j \\
 & A &
 \end{array}$$

By the universal property of coequalizers, there is a unique arrow $k : A/ker(f) \rightarrow Z$ such that $k \circ \pi_{ker(f)} = i$ and $k \circ \pi_{ker(f)} = j$. Thus,

$F(i) \circ a \circ u_1 = F(j) \circ a \circ u_2$; this to say that the following diagram commutes

$$\begin{array}{ccccc}
 & & A & \xrightarrow{a} & FA & \xrightarrow{Fi} & FZ \\
 & \nearrow^{u_1} & & & & & \\
 \ker(f) & & & & & & \\
 & \searrow_{u_2} & A & \xrightarrow{a} & FA & \xrightarrow{Fj} & FZ
 \end{array}$$

As in addition $\ker(f)$ is an equivalence relation, it is a congruence relation on (A, a) . □

Write A/K to denote the codomain of the coequalizer π_K of projections of an equivalence relation K on A .

PROPOSITION 4.4 (Second isomorphism theorem). *Assume the category \mathcal{C} is exact and the endofunctor F preserves monos. Let R and S be bisimulation equivalences on (A, a) such that R is smaller than S . There is a canonical arrow $\phi: A/R \rightarrow A/S$ such that $\phi \circ \pi_R = \pi_S$. Denote by S/R the kernel pair of ϕ : it is an equivalence relation on A/R and induces an isomorphism*

$$\phi': (A/R)/(S/R) \rightarrow A/S$$

such that $\phi = \phi' \circ \pi_{S/R}$.

PROOF. The existence of the isomorphism ϕ' arises from Proposition 4.2. □

LEMMA 4.5. *Suppose that the category \mathcal{C} is regular. Let (A, a) be an F -coalgebra, (B, b) a subcoalgebra of (A, a) represented by the arrow $m: B \rightarrow A$ and R a bisimulation equivalence on (A, a) with projections r_1 and r_2 . Form the pullback of r_2 and m*

$$\begin{array}{ccc}
 R \times_A B & \xrightarrow{u} & B \\
 v \downarrow & & \downarrow m \\
 R & \xrightarrow{r_2} & A
 \end{array}$$

The image B^R of the composite morphism $R \times_A B \xrightarrow{u} B \xrightarrow{m} A$ is a subcoalgebra of (A, a) , provided that the endofunctor F preserves pullbacks along monos.

PROOF. The endofunctor F preserves monos since it preserves pullbacks along monos. Hence m is a mono. For the same reason, $R \times_A B$ is equipped with a coalgebra structure turning v into an F -homomorphism. So, the image B^R of the composite morphism $R \times_A B \xrightarrow{v} R \xrightarrow{r_1} A$ is a subcoalgebra of (A, a) . \square

COROLLARY 4.6. *The assumptions as in Lemma 4.5 are used. So, $B \times B$ is a subobject of $A \times A$. The intersection Q of R and $B \times B$ is a bisimulation equivalence on (B, b) provided that the category \mathcal{C} is exact.*

PROOF. Denote by a_1 and a_2 the structural morphisms of the product of A and A . From universality of the product, there is a unique arrow $\bar{m}: B \times B \rightarrow A \times A$ such that $a_1 \circ \bar{m} = m \circ b_1$ and $a_2 \circ \bar{m} = m \circ b_2$; b_1 and b_2 being the structural morphisms of the product of B and B . Since m is a mono due to F preserving pullbacks along monos, \bar{m} is a mono. Hence $B \times B$ is a subobject of $A \times A$. Let the mono $w: R \rightarrow A \times A$ represent R . The intersection Q of R and $B \times B$ is by definition the pullback

$$\begin{array}{ccc} Q & \xrightarrow{t} & R \\ s \downarrow & & \downarrow w \\ B \times B & \xrightarrow{\bar{m}} & A \times A \end{array}$$

of \bar{m} and w . We claim that Q together with arrows $q_i = b_i \circ s: Q \rightarrow B$, $i = 1, 2$; is a bisimulation equivalence on (B, b) . First of all, Q is a binary relation on B since monos are stable under pullbacks. According to the fact that the endofunctor F preserves pullbacks along monos, Q is a bisimulation on (B, b) . Consider the diagonal map $\langle 1_A, 1_A \rangle: A \rightarrow A \times A$. By the fact that R is a reflexive relation on A , there is an arrow $h_A: A \rightarrow R$ such that $w \circ h_A = \langle 1_A, 1_A \rangle$. Given the diagonal map $\langle 1_B, 1_B \rangle: B \rightarrow B \times B$, we have that $a_i \circ w \circ h_A \circ m = a_i \circ \langle 1_A, 1_A \rangle \circ m = 1_A \circ m = m \circ 1_B = m \circ b_i \circ \langle 1_B, 1_B \rangle = a_i \circ \bar{m} \circ \langle 1_B, 1_B \rangle$; $i = 1, 2$. The equality $w \circ (h_A \circ m) = \bar{m} \circ \langle 1_B, 1_B \rangle$ holds as the pair (a_1, a_2) is a mono source. From universality of the pullback, there is a unique factorization $h_B: B \rightarrow Q$ such that $t \circ h_B = h_A \circ m$ and $s \circ h_B = \langle 1_B, 1_B \rangle$. So, the diagonal map $\langle 1_B, 1_B \rangle$ factors through s , whence Q is reflexive. Since R is a symmetric relation on A , there is a an arrow $\tau: R \rightarrow R$ such that $r_1 \circ \tau = r_2$ and $r_2 \circ \tau = r_1$. Let $\sigma: B \times B \rightarrow A \times A$ be the arrow such that $b_1 \circ \sigma = b_2$ and $b_2 \circ \sigma = b_1$. Then $a_1 \circ w \circ \tau \circ t = r_1 \circ \tau \circ t = r_2 \circ t = a_2 \circ w \circ t = a_2 \circ \bar{m} \circ s = m \circ b_2 \circ s = m \circ b_1 \circ \sigma \circ s = a_1 \circ \bar{m} \circ \sigma \circ s$. Similarly, $a_2 \circ w \circ \tau \circ t = a_2 \circ \bar{m} \circ \sigma \circ s$. Hence $w \circ \tau \circ t = \bar{m} \circ \sigma \circ s$ because the pair (a_1, a_2) is a mono source. As a result, there is a unique arrow $\rho: Q \rightarrow Q$ such that $t \circ \rho = \tau \circ t$ and $s \circ \rho = \sigma \circ s$. It follows that $q_1 \circ \rho = b_1 \circ s \circ \rho = b_1 \circ \sigma \circ s = b_2 \circ s = q_2$ and

$q_2 \circ \rho = b_2 \circ s \circ \rho = b_2 \circ \sigma \circ s = b_1 \circ s = q_1$. This proves that Q is a symmetric relation on B . Form the pullback of q_1 and q_2

$$\begin{array}{ccc}
 Q \times_B Q & \xrightarrow{z_1} & Q \xrightarrow{q_1} B \\
 z_2 \downarrow & & \downarrow q_2 \\
 Q & \xrightarrow{q_1} & B \\
 q_2 \downarrow & & \\
 B & &
 \end{array}$$

Then $a_1 \circ w \circ t \circ z_2 = a_1 \circ \bar{m} \circ s \circ z_2 = m \circ b_1 \circ s \circ z_2 = m \circ q_1 \circ z_2 = m \circ q_2 \circ z_1 = m \circ b_2 \circ s \circ z_1 = a_2 \circ \bar{m} \circ s \circ z_1 = a_2 \circ w \circ t \circ z_1$. Let π_R be the coequalizer of r_1 and r_2 . The equality $\pi_R \circ r_1 = \pi_R \circ r_2$ implies that $\pi_R \circ a_1 \circ w \circ t \circ z_1 = \pi_R \circ a_2 \circ w \circ t \circ z_1 = \pi_R \circ a_1 \circ w \circ t \circ z_2 = \pi_R \circ a_2 \circ w \circ t \circ z_2$. Besides, R is the kernel pair of π_R , since the category \mathcal{C} has exact sequences. There is therefore a unique arrow $\varphi: Q \times_B Q \rightarrow R$ such that $a_1 \circ w \circ \varphi = a_1 \circ w \circ t \circ z_1$ and $a_2 \circ w \circ \varphi = a_2 \circ w \circ t \circ z_2$. Let $\psi: Q \times_B Q \rightarrow B \times B$ be the unique arrow such that $b_1 \circ \psi = q_1 \circ z_1$ and $b_2 \circ \psi = q_2 \circ z_2$. Then $a_1 \circ \bar{m} \circ \psi = m \circ b_1 \circ \psi = m \circ q_1 \circ z_1 = m \circ b_1 \circ s \circ z_1 = a_1 \circ \bar{m} \circ s \circ z_1 = a_1 \circ w \circ t \circ z_1 = a_1 \circ w \circ \varphi$. One proves in the same way that $a_2 \circ \bar{m} \circ \psi = a_2 \circ w \circ \varphi$. Thus, $\bar{m} \circ \psi = w \circ \varphi$ due to the fact that the pair (a_1, a_2) is a mono source. By the universal property of pullbacks, there is a unique arrow $j: Q \times_B Q \rightarrow Q$ such that $s \circ j = \psi$ and $t \circ j = \varphi$. Consider the following commutative diagram

$$\begin{array}{ccc}
 Q \times_B Q & \longrightarrow & Q \circ Q \\
 j \downarrow & \dashrightarrow \gamma & \downarrow \\
 Q & \xrightarrow{s} & B \times B
 \end{array}$$

There is a unique arrow $\gamma: Q \circ Q \rightarrow Q$ making both triangles commute. Hence $Q \circ Q$ is a subobject of Q ; that is, Q is a transitive relation on A . Consequently, Q is an equivalence relation on A . □

There is a one-to-one correspondence between subcoalgebras of a regular quotient of (A, a) and regular quotients of subcoalgebras of (A, a) , as the third isomorphism theorem states.

PROPOSITION 4.7 (Third isomorphism theorem). *Suppose that the category \mathcal{C} is exact and the endofunctor F preserves pullbacks along monos. Let (A, a) be an F -coalgebra, (B, b) a subcoalgebra of (A, a) represented by a mono $m: B \rightarrow A$ and R a bisimulation equivalence on (A, a) with projections r_1 and r_2 . Denote by U the intersection $R \cap (B^R \times B^R)$. The following holds:*

$$B/Q \cong B^R/U$$

Q comes from 4.6.

PROOF. Since the category \mathcal{C} is exact, it has exact sequences. Then R is the kernel pair of π_R . Consider the composite morphism $\varepsilon: B \xrightarrow{m} A \xrightarrow{\pi_R} A_R$. From universality of the pullback, there is a unique arrow $g: \ker(\varepsilon) \rightarrow R$ such that $r_1 \circ g = a_1 \circ w \circ g = m \circ p_1$ and $r_2 \circ g = a_2 \circ w \circ g = m \circ p_2$; p_1 and p_2 being the structural morphisms of $\ker(\varepsilon)$. Let $p: \ker(\varepsilon) \rightarrow B \times B$ be the unique arrow such that $b_1 \circ p = p_1$ and $b_2 \circ p = p_2$. Then $a_1 \circ \bar{m} \circ p = m \circ b_1 \circ p = m \circ p_1 = a_1 \circ w \circ g$ and $a_2 \circ \bar{m} \circ p = m \circ b_2 \circ p = m \circ p_2 = a_2 \circ w \circ g$. It follows that $\bar{m} \circ p = w \circ g$ as the pair (a_1, a_2) is a mono source. Consequently, there is a unique arrow $\psi: \ker(\varepsilon) \rightarrow Q$ such that $t \circ \psi = g$ and $s \circ \psi = p$. Also, $\varepsilon \circ q_1 = \varepsilon \circ q_2$ as $\pi_R \circ a_1 \circ w \circ t = \pi_R \circ a_2 \circ w \circ t$ because $\pi_R \circ r_1 = \pi_R \circ r_2$. For this reason, there is a unique arrow $\varphi: Q \rightarrow \ker(\varepsilon)$ such that $p_1 \circ \varphi = q_1$ and $p_2 \circ \varphi = q_2$. It is straightforward to check that $\varphi \circ \psi = 1_{\ker(\varepsilon)}$ and $\psi \circ \varphi = 1_Q$; this shows that $\ker(\varepsilon)$ and Q are isomorphic.

We claim that $\text{im}(\varepsilon)$ is isomorphic to B^R/U . Indeed, let ε' be the composite morphism $\pi_R \circ i$, where $i: B^R \rightarrow A$ is the subobject witness. Denote by s_1 and s_2 the structural morphisms of the product of B^R and itself. There is a unique arrow $h: \ker(\varepsilon') \rightarrow B^R \times B^R$ such that $s_1 \circ h = t_1$ and $s_2 \circ h = t_2$; t_1 and t_2 being the structural morphisms of $\ker(\varepsilon')$. In addition, there is a unique arrow $i \times i: B^R \times B^R \rightarrow A \times A$ such that $a_1 \circ (i \times i) = i \circ s_1$ and $a_2 \circ (i \times i) = i \circ s_2$. Hence, we have $\pi_R \circ (i \circ t_1) = (\pi_R \circ i) \circ t_1 = \varepsilon' \circ t_1 = \varepsilon' \circ t_2 = (\pi_R \circ i) \circ t_2 = \pi_R \circ (i \circ t_2)$. By the universal property of pullbacks, there is a unique arrow $k: \ker(\varepsilon') \rightarrow R$ such that $a_1 \circ w \circ k = i \circ t_1 = i \circ (s_1 \circ h) = (i \circ s_1) \circ h = (a_1 \circ (i \times i)) \circ h = a_1 \circ ((i \times i) \circ h)$ and $a_2 \circ w \circ k = i \circ t_2 = i \circ (s_2 \circ h) = (i \circ s_2) \circ h = (a_2 \circ (i \times i)) \circ h = a_2 \circ ((i \times i) \circ h)$. That is, $w \circ k = (i \times i) \circ h$ as the pair (a_1, a_2) is a mono source

$$\begin{array}{ccc}
 \ker(\varepsilon') & & \\
 \begin{array}{l} \nearrow \tau \\ \searrow k \end{array} & \begin{array}{l} \xrightarrow{h} \\ \xrightarrow{c} \\ \downarrow d \end{array} & \begin{array}{l} B^R \times B^R \\ \\ A \times A \end{array} \\
 U & \xrightarrow{c} & B^R \times B^R \\
 \downarrow d & & \downarrow i \times i \\
 R & \xrightarrow{w} & A \times A
 \end{array}$$

There is therefore a unique arrow $\sigma: \ker(\varepsilon') \rightarrow U$ such that $c \circ \sigma = h$ and $d \circ \sigma = k$; c and d being the structural morphisms of U . However, we have also $\varepsilon' \circ (s_1 \circ c) = (\pi_R \circ i) \circ (s_1 \circ c) = \pi_R \circ (i \circ s_1) \circ c = \pi_R \circ (a_1 \circ (i \times i)) \circ c = (\pi_R \circ a_1) \circ ((i \times i) \circ c) = (\pi_R \circ a_1) \circ (w \circ d) = \pi_R \circ (a_1 \circ w) \circ d = (\pi_R \circ r_1) \circ d = (\pi_R \circ r_2) \circ d = \pi_R \circ (a_2 \circ w) \circ d = (\pi_R \circ a_2) \circ (w \circ d) = (\pi_R \circ a_2) \circ ((i \times i) \circ c) = \pi_R \circ (a_2 \circ (i \times i)) \circ c = \pi_R \circ (i \circ s_2) \circ c = (\pi_R \circ i) \circ (s_2 \circ c) = \varepsilon' \circ (s_2 \circ c)$. So there is a unique arrow $\tau: U \rightarrow \ker(\varepsilon')$ such that $t_1 \circ \tau = s_1 \circ c$ and $t_2 \circ \tau = s_2 \circ c$ or equivalently, $s_1 \circ (h \circ \tau) = s_1 \circ c$ and $s_2 \circ (h \circ \tau) = s_2 \circ c$; i.e., $h \circ \tau = c$ because the pair (s_1, s_2) is a mono source. This implies that $c \circ (\sigma \circ \tau) = (c \circ \sigma) \circ \tau = h \circ \tau = c$.

But, c is a mono and monos are stable under pullbacks. Thus, $\sigma \circ \tau = 1_U$. Also, $t_1 \circ (\tau \circ \sigma) = (t_1 \circ \tau) \circ \sigma = (s_1 \circ c) \circ \sigma = s_1 \circ (c \circ \sigma) = s_1 \circ h = t_1$ and $t_2 \circ (\tau \circ \sigma) = (t_2 \circ \tau) \circ \sigma = (s_2 \circ c) \circ \sigma = s_2 \circ (c \circ \sigma) = s_2 \circ h = t_2$. Then $\tau \circ \sigma = 1_{\ker(\varepsilon')}$ as the pair (t_1, t_2) is a mono source. Consequently, $\ker(\varepsilon')$ and U are isomorphic.

Besides, regular epis in the category \mathcal{C} are stable under composition as it is exact. Let $e: R \times_A B \rightarrow B^R$ denote the regular epi such that $r_1 \circ v = i \circ e$. Then $\text{im}(\varepsilon')$ is isomorphic to $\text{im}(\varepsilon'')$, where $\varepsilon'' = \pi_R \circ r_2 \circ v$; this follows from the fact that $\varepsilon'' = \pi_R \circ r_2 \circ v = \pi_R \circ r_1 \circ v = \pi_R \circ i \circ e = \varepsilon' \circ e$. Also, r_2 is a retraction because R is a reflexive relation. Thereafter, u is a retraction and retractions are stable under pullbacks. Since $\varepsilon'' = \pi_R \circ r_2 \circ v = \pi_R \circ r_1 \circ v = \pi_R \circ m \circ u = \varepsilon \circ u$, one deduces that $\text{im}(\varepsilon'')$ is isomorphic to $\text{im}(\varepsilon)$. Under Proposition 4.2, the isomorphism $B/Q \cong B^R/U$ holds. \square

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