

## SOME GENERALIZATIONS OF NON-UNIQUE FIXED POINT THEOREMS OF ĆIRIĆ-TYPE FOR $(\Phi, \psi)$ -HYBRID CONTRACTIVE MAPPINGS

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**Abstract.** In this article, we establish some non-unique fixed point theorems of Ćirić's type for  $(\Phi, \psi)$ -hybrid contractive mappings by using a similar notion to that of the paper [M. Akram, A.A. Zafar and A.A. Siddiqui, *A general class of contractions: A-contractions*, Novi Sad J. Math. 38 (2008), no. 1, 25–33]. Our results generalize, extend and improve several ones in the literature.

### 1. Introduction

Let  $(X, d)$  be a complete metric space and  $T : X \rightarrow X$  a selfmapping of  $X$ . Suppose that  $F(T) = \{x \in X \mid Tx = x\}$  is the set of fixed points of  $T$ . The following definitions shall be required in the sequel:

The *orbit* of  $T$  at  $x$ , denoted  $O(x, T)$ , is defined by

$$O(x, T) = \{x, Tx, T^2x, \dots, T^n x, \dots\}.$$

In 1971, Ćirić ([11]) introduced the following two definitions to obtain some fixed point theorems.

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*Received: 17.11.2018. Accepted: 31.05.2019. Published online: 28.06.2019.*

(2010) Mathematics Subject Classification: 47H06, 47H10.

*Key words and phrases:* non-unique fixed point theorems of Ćirić, general class of contractions,  $(\Phi, \psi)$ -hybrid contractive mappings.

DEFINITION 1.1 ([11, 13]). A metric space  $(X, d)$  is said to be *T-orbitally complete* if  $T: X \rightarrow X$  is a selfmapping and if any Cauchy subsequence  $\{T^{n_i}x\}$  in orbit  $O(x, T)$ , with  $x \in X$ , converges in  $X$ .

DEFINITION 1.2 ([11, 13]). An operator  $T: X \rightarrow X$  is said to be *T-orbitally continuous* if  $T^{n_i}x \rightarrow x^* \implies T(T^{n_i}x) \rightarrow Tx^*$  as  $i \rightarrow \infty$ .

We give the following equivalent definition in the metric form:  
An operator  $T: X \rightarrow X$  is *orbitally continuous* if

$$\lim_{i \rightarrow \infty} d(T^{n_i}x, x^*) = 0 \implies \lim_{i \rightarrow \infty} d(T(T^{n_i}x), Tx^*) = 0.$$

In applications, it is possible to have nonlinear equations whose fixed points are not necessarily unique. Ćirić ([12]) established some results pertaining to this innovative notion of nonunique fixed points. Our purpose is to establish some nonunique fixed point theorems on a complete metric space for selfmappings by using Akram-Zafar-Siddiqui type contractive conditions. Our results generalize, extend and improve some previous results in the literature. In particular, our results generalize and improve some of the results of Ćirić ([12, 13]) and some recent results of the author (see [20]–[23], [26]). The classical Banach's fixed point theorem was established by Banach ([6]), using the following contractive condition: there exists  $c \in [0, 1)$  (fixed) such that  $\forall x, y \in X$ ,

$$(1.1) \quad d(Tx, Ty) \leq c d(x, y).$$

However, it is crucial to say that the mappings satisfying the contractive condition (1.1) are necessarily continuous. In order to have a wider class of contractive mappings than those satisfying (1.1), Kannan ([16]) generalized the Banach's fixed point theorem by employing the following contractive condition: there exists  $a \in [0, \frac{1}{2})$  such that

$$(1.2) \quad d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)], \quad \forall x, y \in X.$$

So, the mappings satisfying (1.2) need not be continuous and this is a very nice initiative of the author (see [16]). Several authors have generalized and extended Banach's fixed point theorem using similar notion as in (1.2). For such generalization and extension of Banach's fixed point theorem, interested readers may also consult Agarwal *et al.* ([4]), Chatterjea ([10]), Khamsi and Kirk ([18]), Zamfirescu ([31]), Zeidler ([32]) and a host of others in the literature.

However, it is noteworthy to say that several contractive conditions including Banach's contractive condition (1.1) have always been concerned with establishing the existence and uniqueness of the fixed point of the mapping.

Therefore, in order to include mappings whose fixed points may be not unique, Ćirić ([12]) introduced a new technique involving contractive conditions for such mappings, realizing the fact that there are also nonlinear equations with more than one fixed point as aforementioned. In particular, Ćirić ([12]) introduced, amongst others, the following two contractive conditions: For a mapping  $T: X \rightarrow X$ , there exists  $\lambda \in (0, 1)$  such that  $\forall x, y \in X$ ,

$$(1.3) \quad \min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(y, Tx)\} \\ \leq \lambda d(x, y), \quad \text{where } T \text{ is orbitally continuous;}$$

and also there exists  $\lambda \in (0, 1)$  such that  $\forall x, y \in X$ ,

$$(1.4) \quad \min\{d(Tx, Ty), \max\{d(x, Tx), d(y, Ty)\}\} \\ - \min\{d(x, Ty), d(y, Tx)\} \leq \lambda d(x, y).$$

Another contractivity condition worthy of note is the following:

DEFINITION 1.3 ([5]). A selfmap  $T: X \rightarrow X$  of a metric space  $(X, d)$  is said to be *A-contraction* if it satisfies the condition:

$$(A) \quad d(Tx, Ty) \leq \beta(d(x, y), d(x, Tx), d(y, Ty)), \quad \forall x, y \in X,$$

with some  $\beta \in A$ , where  $A$  is the set of all functions  $\beta: \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$  satisfying

- (i)  $\beta$  is continuous on the set  $\mathbb{R}_+^3$  (with respect to the Euclidean metric on  $\mathbb{R}^3$ );
- (ii)  $a \leq kb$  for some  $k \in [0, 1)$  whenever  $a \leq \beta(a, b, b)$ , or  $a \leq \beta(b, a, b)$ , or  $a \leq \beta(b, b, a)$ ,  $\forall a, b \in \mathbb{R}_+$ .

Akram *et al.* ([5]) employed the contractive condition (A) to prove that if  $X$  is a complete metric space, then the mapping  $T$  has a unique fixed point.

Olatinwo ([20]) generalized the results of Akram *et al.* ([5]) by employing the following more general contractive condition:

DEFINITION 1.4 ([20]). A selfmap  $T: X \rightarrow X$  of a metric space  $(X, d)$  is said to be a *generalized A-contraction* or  $G_A$ -contraction if it satisfies the condition:

$$d(Tx, Ty) \leq \alpha(d(x, y), d(x, Tx), d(y, Ty)), \\ [d(x, Tx)]^r [d(y, Ty)]^p d(x, Ty), d(y, Tx) [d(x, Tx)]^m,$$

$\forall x, y \in X, r, p, m \in \mathbb{R}_+$  with some  $\alpha \in G_A$ , where  $G_A$  is the set of all functions  $\alpha: \mathbb{R}_+^5 \rightarrow \mathbb{R}_+$  satisfying

- (i)  $\alpha$  is continuous on the set  $\mathbb{R}_+^5$  (with respect to the Euclidean metric on  $\mathbb{R}^5$ );
- (ii) if any of the conditions  $a \leq \alpha(b, b, a, c, c)$ , or,  $a \leq \alpha(b, b, a, b, b)$ , or,  $a \leq \alpha(a, b, b, b, b)$  holds for some  $a, b, c \in \mathbb{R}_+$ , then there exists  $k \in [0, 1)$  such that  $a \leq kb$ .

A further generalization of the results of Akram *et al.* ([5]) is contained in Olatinwo ([20]–[21]) and Olatinwo in [25] established some convergence theorems as well as some stability results using similar notion.

Motivated by the results of both Akram *et al.* ([5]) and Ćirić ([12]), in the present paper, we prove various and more general nonunique fixed point theorems by employing contractive conditions which are hybrids of those used in [5, 12], [20]–[23], [26]. Our results are indeed generalizations of the results of Ćirić ([12]) and those of several authors in the literature. For excellent study of mappings having nonunique fixed points, we refer to Achari ([1]–[3]), Ćirić ([12]–[13]), Ćirić *et al.* ([14]), Karapınar ([17]) and Pachpatte ([27]) as well as the articles of the author (see [22]–[24], [26]).

DEFINITION 1.5.

- (a) A function  $\psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is called a *comparison function* if it satisfies the following conditions:
  - (i)  $\psi$  is monotone increasing;
  - (ii)  $\lim_{n \rightarrow \infty} \psi^n(t) = 0, \forall t \geq 0$ , where  $\psi^n(t)$  denotes the  $n$ -th iterate of  $\psi$ .
- (b) A comparison function satisfying  $\sum_{n=0}^{\infty} \psi^n(t)$  converges for all  $t > 0$  is called a *(c)-comparison function*.

See Berinde ([7, 8]), Rus ([29]) and Rus *et al.* ([30]) for the definition and examples of comparison function.

REMARK 1.6. Every comparison function satisfies  $\psi(0) = 0$ .

To prove our results, we shall employ the following more general contractive conditions than those stated in (1.3) and (1.4):

Let  $(X, d)$  be a metric space.

- (a) For a mapping  $T: X \rightarrow X$ , there exist functions  $u, v, w: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $v(0) = 0 = w(0)$ , and functions  $\beta: \mathbb{R}_+^5 \rightarrow \mathbb{R}_+, \Phi, \psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that

$$(\Delta) \quad \Phi(M) \leq \beta(d(x, y), d(x, Tx), d(y, Ty),$$

$$u(d(x, Tx))v(d(y, Ty))d(x, Ty), d(y, Tx)u(d(x, Tx))),$$

$\forall x, y \in X$  such that  $M \geq 0$ , where

$$M = \min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - w(\min\{d(x, Ty), d(y, Tx)\}),$$

and the functions  $\beta, \Phi, \psi$  satisfy:

- (p<sub>1</sub>)  $\beta$  is continuous on the set  $\mathbb{R}_+^5$  (with respect to the Euclidean metric on  $\mathbb{R}^5$ );
  - (p<sub>2</sub>)  $\psi$  is a continuous (c)-comparison function,  $\Phi$  is an injective, continuous and subadditive monotone increasing function such that  $\Phi(0) = 0$  and  $\Phi(a) \leq \psi(\Phi(b))$  whenever  $\Phi(a) \leq \beta(b, b, a, 0, 0)$ ,  $\forall a, b \in \mathbb{R}_+$ .
- (b) For a mapping  $T: X \rightarrow X$ , there exist continuous functions  $u, v, w: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $v(0) = 0 = w(0)$ , and functions  $\beta: \mathbb{R}_+^5 \rightarrow \mathbb{R}_+$ ,  $\Phi, \psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that

$$(\Delta\star) \quad \Phi(N) \leq \beta(d(x, y), d(x, Tx), d(y, Ty), \\ u(d(x, Tx))v(d(y, Tx))d(x, Ty), d(y, Tx)u(d(x, Tx))),$$

$\forall x, y \in X$  such that  $N \geq 0$ , where

$$N = \min\{d(Tx, Ty), \max\{d(x, Tx), d(y, Ty)\}\} \\ - w(\min\{d(x, Ty), d(y, Tx)\}),$$

and the functions  $\beta, \Phi, \psi$  satisfy properties (p<sub>1</sub>), (p<sub>2</sub>).

REMARK 1.7. Each of the contractive conditions  $(\Delta)$  and  $(\Delta\star)$  can be reduced to several other ones in the literature. In particular, we have the following:

It is obvious that both contractive conditions (1.3) and (1.4) are special cases of contractive conditions  $(\Delta)$  and  $(\Delta\star)$  respectively if

$$\beta(t_1, t_2, t_3, t_4, t_5) = \lambda t_1, \quad \forall (t_1, t_2, t_3, t_4, t_5) \in \mathbb{R}_+^5, \lambda \in (0, 1).$$

## 2. Main results

THEOREM 2.1. *Let  $(X, d)$  be a complete metric space and let  $T: X \rightarrow X$  be a mapping satisfying the contractive condition  $(\Delta)$  with some functions  $u, v, w: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $v(0) = 0 = w(0)$ , and functions  $\beta: \mathbb{R}_+^5 \rightarrow \mathbb{R}_+$ ,  $\Phi, \psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying properties (p<sub>1</sub>) and (p<sub>2</sub>). If  $T$  is orbitally continuous, then  $T$  has a fixed point in  $X$ .*

PROOF. For  $x_0 \in X$ , let  $\{x_n\}_{n=0}^\infty$  be the Picard iteration associated to  $T$  defined by  $x_{n+1} = Tx_n$  ( $n = 0, 1, 2, \dots$ ). Note that  $x_n = Tx_{n-1} = T^n x_0$  ( $n = 0, 1, 2, \dots$ ). Let  $x = x_n$ ,  $y = x_{n+1}$ , then we have

$$\begin{aligned} M &= \min\{d(Tx_n, Tx_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})\} \\ &\quad - w(\min\{d(x_n, Tx_{n+1}), d(x_{n+1}, Tx_n)\}) \\ &= \min\{d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})\}. \end{aligned}$$

If  $d(x_q, x_{q+1}) = 0$  for some  $q \geq 0$ , then  $x_0$  is the limit point of  $\{T^n x_0\}$  and  $x_q$  is a fixed point of  $T$ .

Suppose that  $d(x_n, x_{n+1}) > 0$  ( $n = 0, 1, 2, \dots$ ). Using condition  $(\Delta)$  with  $x = x_n$ ,  $y = x_{n+1}$ , then we have

$$\begin{aligned} \Phi(M) &\leq \beta(d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), u(d(x_n, Tx_n))) \\ &\quad \times v(d(x_{n+1}, Tx_n), d(x_n, Tx_{n+1}), d(x_{n+1}, Tx_n), u(d(x_n, Tx_n))) \\ &= \beta(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), 0, 0) \end{aligned}$$

from which we obtain that

$$\begin{aligned} (2.1) \quad \Phi(\min\{d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})\}) \\ \leq \beta(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), 0, 0). \end{aligned}$$

We choose  $M = \min\{d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})\} = d(x_{n+1}, x_{n+2})$  and then apply property  $(p_2)$  of  $\beta$  so that from (2.1) we get

$$\Phi(d(x_{n+1}, x_{n+2})) \leq \psi(\Phi(d(x_n, x_{n+1}))),$$

that is,

$$(2.2) \quad \Phi(d(x_n, x_{n+1})) \leq \psi(\Phi(d(x_{n-1}, x_n))).$$

Using (2.2) inductively we obtain

$$\begin{aligned} \Phi(d(x_n, x_{n+1})) &\leq \psi(\Phi(d(x_{n-1}, x_n))) \\ &\leq \psi^2(\Phi(d(x_{n-2}, x_{n-1}))) \leq \dots \leq \psi^n(\Phi(d(x_0, x_1))), \end{aligned}$$

from which it follows that

$$(2.3) \quad \Phi(d(x_n, x_{n+1})) \leq \psi^n(\Phi(d(x_0, x_1))).$$

Applying the subadditivity of  $\Phi$  in the repeated application of the triangle inequality yields

$$(2.4) \quad \Phi(d(x_n, x_{n+q})) \leq \Phi(d(x_n, x_{n+1})) \\ + \Phi(d(x_{n+1}, x_{n+2})) + \cdots + \Phi(d(x_{n+q-1}, x_{n+q})).$$

Using (2.3) inductively in (2.4) gives

$$(2.5) \quad \Phi(d(x_n, x_{n+q})) \leq \psi^n(\Phi(d(x_0, x_1))) \\ + \psi^{n+1}(\Phi(d(x_0, x_1))) + \cdots + \psi^{n+q-1}(\Phi(d(x_0, x_1))) \\ = \sum_{k=n}^{n+q-1} \psi^k(\Phi(d(x_0, x_1))) \\ \leq \sum_{k=n}^{\infty} \psi^k(\Phi(d(x_0, x_1))), \quad n, q \in \mathbb{N}.$$

The right-hand side term in (9) tends to 0 as  $n \rightarrow \infty$ , thus leading to the fact that  $\Phi(d(x_n, x_{n+q})) \rightarrow 0$  as  $n \rightarrow \infty$  uniformly with respect to  $q$ . Therefore, by the continuity and injectivity of  $\Phi$  and the condition  $\Phi(0) = 0$ , we have that  $d(x_n, x_{n+q}) \rightarrow 0$  as  $n \rightarrow \infty$  uniformly with respect to  $q$ . That is,  $\{x_n\}_{n=0}^{\infty}$  is a Cauchy sequence in  $X$ . Since  $(X, d)$  is a complete metric space, there exists  $u \in X$  such that  $\lim_{n \rightarrow \infty} d(x_n, u) = 0$ , that is,  $\lim_{n \rightarrow \infty} x_n = u$ . Therefore, since  $x_n = T^n x_0$  and  $T$  is orbitally continuous, we have

$$0 = d(\lim_{n \rightarrow \infty} T(T^n x_0), Tu) = \lim_{n \rightarrow \infty} d(T(T^n x_0), Tu) = \lim_{n \rightarrow \infty} d(Tx_n, Tu) \\ = \lim_{n \rightarrow \infty} d(x_{n+1}, Tu) = d(u, Tu).$$

Thus,  $Tu = u$ , that is,  $u \in X$  is a fixed point of  $T$ . □

**THEOREM 2.2.** *Let  $(X, d)$  be a complete metric space and  $T: X \rightarrow X$  a mapping satisfying contractive condition  $(\Delta\star)$  with some continuous functions  $u, v, w: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $v(0) = 0 = w(0)$ , and functions  $\beta: \mathbb{R}_+^5 \rightarrow \mathbb{R}_+$ ,  $\Phi, \psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying properties  $(p_1)$ ,  $(p_2)$ . Then,  $T$  has a fixed point.*

**PROOF.** For  $x_0 \in X$ , let  $\{x_n\}_{n=0}^{\infty}$  defined by  $x_n = Tx_{n-1} = T^n x_0$  ( $n = 0, 1, 2, \dots$ ) be the Picard iteration associated with  $T$ . Note that

$x_n = Tx_{n-1} = T^n x_0$  ( $n = 0, 1, 2, \dots$ ). Let  $x = x_n$ ,  $y = x_{n+1}$ , then we have

$$\begin{aligned} N &= \min\{d(Tx_n, Tx_{n+1}), \max\{d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})\}\} \\ &\quad - w(\min\{d(x_n, Tx_{n+1}), d(x_{n+1}, Tx_n)\}) \\ &= \min\{d(x_{n+1}, x_{n+2}), \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}\} = d(x_{n+1}, x_{n+2}). \end{aligned}$$

If  $d(x_q, x_{q+1}) = 0$  for some  $q \geq 0$ , then  $x_0$  is the limit point of  $\{T^n x_0\}$  and  $x_q$  is a fixed point of  $T$ .

Assume that  $d(x_n, x_{n+1}) > 0$  ( $n = 0, 1, 2, \dots$ ). By using condition  $(\Delta\star)$  with  $x = x_n, y = x_{n+1}$ , then we obtain

$$\begin{aligned} \Phi(N) &\leq \beta(d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), u(d(x_n, Tx_n))) \\ &\quad \times v(d(x_{n+1}, Tx_n), d(x_n, Tx_{n+1}), d(x_{n+1}, Tx_n), u(d(x_n, Tx_n))) \\ &= \beta(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), 0, 0), \end{aligned}$$

from which we obtain that

$$(2.6) \quad \Phi(d(x_{n+1}, x_{n+2})) \leq \beta(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), 0, 0).$$

Then, applying property  $(p_2)$  of  $\beta$  again in (2.6) gives

$$\Phi(d(x_{n+1}, x_{n+2})) \leq \psi(\Phi(d(x_n, x_{n+1}))),$$

which leads to the form (2.5) again. It follows again that the sequence  $\{x_n\}_{n=0}^\infty$  is a Cauchy sequence in  $X$ . Since  $(X, d)$  is complete,  $\{x_n\}$  converges to some  $\nu \in X$ , that is,  $\lim_{n \rightarrow \infty} x_n = \nu$ .

Since  $N$  can be negative, we define now the function  $\Phi$  also for negative arguments so that  $\Phi(x) < 0$  for  $x < 0$  as follows: Let  $\Phi(x) := -\Phi(-x)$  for  $x < 0$ .

The continuity of  $\Phi$  is preserved.

Now, in the expression for  $N$ , let  $x = x_n$ ,  $y = \nu$  so that we have

$$\begin{aligned} N &= \min\{d(Tx_n, T\nu), \max\{d(x_n, Tx_n), d(\nu, T\nu)\}\} \\ &\quad - w(\min\{d(x_n, T\nu), d(\nu, Tx_n)\}) \\ &= \min\{d(x_{n+1}, T\nu), \max\{d(x_n, x_{n+1}), d(\nu, T\nu)\}\} \\ &\quad - w(\min\{d(x_n, T\nu), d(\nu, x_{n+1})\}), \end{aligned}$$



from which we have that

$$\begin{aligned}
 (2.7) \quad \lim_{n \rightarrow \infty} N &= \lim_{n \rightarrow \infty} [\min\{d(x_{n+1}, T\nu), \max\{d(x_n, x_{n+1}), d(\nu, T\nu)\}\} \\
 &\quad - w(\min\{d(x_n, T\nu), d(\nu, x_{n+1})\})] \\
 &= \min\{d(\nu, T\nu), \max\{d(\nu, \nu), d(\nu, T\nu)\}\} \\
 &\quad - w(\min\{d(\nu, T\nu), d(\nu, \nu)\}) \\
 &= \min\{d(\nu, T\nu), d(\nu, T\nu)\} = d(\nu, T\nu).
 \end{aligned}$$

By using condition  $(\Delta\star)$  again with  $x = x_n, y = \nu$ , we have that

$$\begin{aligned}
 (2.8) \quad \Phi(N) &\leq \beta(d(x_n, \nu), d(x_n, Tx_n), d(\nu, T\nu), \\
 &\quad u(d(x_n, Tx_n))v(d(\nu, Tx_n))d(x_n, T\nu), d(\nu, Tx_n)u(d(x_n, Tx_n))) \\
 &= \beta(d(x_n, \nu), d(x_n, x_{n+1}), d(\nu, T\nu), u(d(x_n, x_{n+1})) \\
 &\quad \times v(d(\nu, x_{n+1}))d(x_n, T\nu), d(\nu, x_{n+1})u(d(x_n, x_{n+1}))).
 \end{aligned}$$

As  $n \rightarrow \infty$  in (12), we get by the continuity of  $u, v, w, \Phi, \psi$  and the metric as well as by applying (11) in (12) that  $\Phi(d(\nu, T\nu)) \leq \beta(0, 0, d(\nu, T\nu), 0, 0)$ . Then, property  $(p_2)$  gives  $\Phi(d(\nu, T\nu)) \leq \psi(\Phi(0)) = 0$ , from which it follows by the condition on  $\Phi$  that  $\Phi(d(\nu, T\nu)) = 0$ , so that  $d(\nu, T\nu) = 0$ , that is,  $T\nu = \nu$ . □

REMARK 2.3. In order to establish that  $\{x_n\}$  is a Cauchy sequence in Theorem 2.1, the following observation is crucial: choosing

$$\min\{d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$$

is ignored as a reasonable recurrence inequality relation involving the comparison function  $\psi$  could not be obtained. More precisely, if  $\min\{d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$ , for some  $n \in \mathbb{N}$ , then putting  $y = d(x_n, x_{n+1})$  yields

$$0 < y \leq \psi^2(y),$$

whence, by monotonicity of  $\psi$ ,

$$0 < y \leq \psi^k(y), \quad \text{for } k \in \mathbb{N},$$

which contradicts condition (ii) of Definition 1.5.

**THEOREM 2.4.** *Let  $X$  be a nonempty set,  $d$  and  $\rho$  be two metrics on  $X$  and let  $T: X \rightarrow X$  be a mapping. Assume that:*

- (i) *there exists  $K > 0$  such that  $\rho(Tx, Ty) \leq Kd(x, y), \forall x, y \in X$ ;*
- (ii)  *$(X, \rho)$  is a complete metric space;*
- (iii)  *$T: (X, \rho) \rightarrow (X, \rho)$  is orbitally continuous;*
- (iv)  *$T: (X, d) \rightarrow (X, d)$  is a mapping satisfying  $(\Delta)$ .*

*Then,  $T: X \rightarrow X$  has a fixed point.*

**PROOF.** For  $x_0 \in X$ , let  $\{x_n\}_{n=0}^\infty$  defined by  $x_{n+1} = Tx_n, n = 0, 1, 2, \dots$ , be the Picard iteration associated with  $T$ . By condition (iv), we obtain as in Theorem 2.1 that, for  $p \in \mathbb{N}$ ,  $d(x_n, x_{n+p}) \rightarrow 0$  as  $n \rightarrow \infty$ . That is,  $\{x_n\}$  is a Cauchy sequence in  $(X, d)$ .

We now show that  $\{x_n\}$  is a Cauchy sequence in  $(X, \rho)$  as follows: By condition (i), we have, for  $p \in \mathbb{N}$ ,

$$\rho(x_n, x_{n+p}) = \rho(Tx_{n-1}, Tx_{n+p-1}) \leq Kd(x_{n-1}, x_{n+p-1}) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

that is,  $\rho(x_n, x_{n+p}) \rightarrow 0$  as  $n \rightarrow \infty$ . Thus,  $\{x_n\}$  is a Cauchy sequence in  $(X, \rho)$  too.

By condition (ii),  $(X, \rho)$  is a complete metric space which implies that there exists  $u \in X$  such that  $\lim_{n \rightarrow \infty} \rho(x_n, u) = 0$ , that is,  $\lim_{n \rightarrow \infty} x_n = u$ .

By condition (iii), since  $x_n = T^n x_0$  and  $T: (X, \rho) \rightarrow (X, \rho)$  is orbitally continuous, we have

$$\begin{aligned} 0 = \rho(\lim_{n \rightarrow \infty} T(T^n x_0), Tu) &= \lim_{n \rightarrow \infty} \rho(T(T^n x_0), Tu) \\ &= \lim_{n \rightarrow \infty} \rho(Tx_n, Tu) = \lim_{n \rightarrow \infty} \rho(x_{n+1}, Tu) = \rho(u, Tu). \end{aligned}$$

Therefore,  $\rho(u, Tu) = 0 \iff Tu = u$ . So,  $T$  has a fixed point  $u$ . □

**THEOREM 2.5.** *Let  $X$  be a nonempty set,  $d$  and  $\rho$  be two metrics on  $X$  and let  $T: X \rightarrow X$  be a mapping. Assume that:*

- (i) *there exists  $K > 0$  such that  $\rho(Tx, Ty) \leq Kd(x, y), \forall x, y \in X$ ;*
- (ii)  *$(X, \rho)$  is a complete metric space;*
- (iii)  *$T: (X, \rho) \rightarrow (X, \rho)$  is orbitally continuous;*
- (iv)  *$T: (X, d) \rightarrow (X, d)$  is a mapping satisfying  $(\Delta\star)$ .*

*Then,  $T: X \rightarrow X$  has a fixed point.*

**PROOF.** For  $x_0 \in X$ , let  $\{x_n\}_{n=0}^\infty$  defined by  $x_{n+1} = Tx_n, n = 0, 1, 2, \dots$ , be the Picard iteration associated with  $T$ . By condition (iv), we obtain as in Theorem 2.2 that  $\{x_n\}$  is a Cauchy sequence in  $(X, d)$ .

Further we proceed as in the proof of Theorem 2.4:

By condition (i), we have that  $\{x_n\}$  is a Cauchy sequence in  $(X, \rho)$  too.

By condition (ii),  $(X, \rho)$  is a complete metric space which implies that there exists  $u \in X$  such that  $\lim_{n \rightarrow \infty} \rho(x_n, u) = 0$ , that is,  $\lim_{n \rightarrow \infty} x_n = u$ .

Again, by condition (iii), since  $x_n = T^n x_0$  and  $T: (X, \rho) \rightarrow (X, \rho)$  is orbitally continuous, we obtain that  $\rho(u, Tu) = 0 \iff Tu = u$ . Hence,  $T$  has a fixed point  $u$ . □

EXAMPLE 2.6. Let  $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a Lebesgue-integrable mapping which is summable, nonnegative and such that  $\int_0^\epsilon \varphi(t) > 0$ , for each  $\epsilon > 0$ . Suppose  $\Phi, \beta$  are as defined in the Contractive conditions given in  $(\Delta)$  and  $(\Delta\star)$ . Then, we have the various cases below.

*Solution:* Let

$$M = \min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} \\ - w(\min\{d(x, Ty), d(y, Tx)\}) = d(Tx, Ty),$$

or,

$$N = \min\{d(Tx, Ty), \max\{d(x, Tx), d(y, Ty)\}\} \\ - w(\min\{d(x, Ty), d(y, Tx)\}) = d(Tx, Ty).$$

Suppose  $w(s) = 0, s \in \mathbb{R}_+$ , then we obtain the following cases from the Contractive conditions given in  $(\Delta)$  and  $(\Delta\star)$  :

Case (1): Putting  $\Phi(x) = \int_0^x \varphi(t)dt$  and

$$\beta(t_1, t_2, t_3, t_4, t_5) = k \int_0^{t_1} \varphi(t)dt,$$

where  $k \in [0, 1)$ , we obtain some kind of Branciari's contractive conditions of integral type (cf. [9, 28]):

$$\int_0^M \varphi(t)dt \leq k \int_0^{d(x,y)} \varphi(t)dt \quad \text{and} \quad \int_0^N \varphi(t)dt \leq k \int_0^{d(x,y)} \varphi(t)dt.$$

Case (2): Also, letting  $\Phi(x) = \int_0^x \varphi(t)dt$ ,

$$\beta(t_1, t_2, t_3, t_4, t_5) = k \int_0^{\max\{t_1, t_2, t_3, \frac{t_4+t_5}{2}\}} \varphi(t)dt,$$

where  $k \in [0, 1)$ , and  $u(t) = 1, t \in \mathbb{R}_+, v(t) = 1, t > 0$ , we obtain some kind of Rhoades' contractive condition of integral type (cf. [28]):

$$\int_0^M \varphi(t) dt \leq k \int_0^{h(x,y)} \varphi(t) dt,$$

where  $h(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}$ .

Case (3) Also, letting  $\Phi(x) = \int_0^x \varphi(t) dt$ ,

$$\beta(t_1, t_2, t_3, t_4, t_5) = k \int_0^{\max\{t_1, t_2, t_3, t_4, t_5\}} \varphi(t) dt,$$

where  $k \in [0, 1)$ , and  $u(t) = 1, t \in \mathbb{R}_+, v(t) = 1, t > 0$ , we obtain some other kind of Rhoades' contractive condition of integral type (cf. [28]):

$$\int_0^M \varphi(t) dt \leq k \int_0^{H(x,y)} \varphi(t) dt,$$

where  $H(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$ .

REMARK 2.7. In the article by Olatinwo [21], it was shown that the class of contractions employed in that paper (that is, the class of  $G_{\Phi, \psi}$ -contractions) is more general than the class of  $A$ -contractions introduced by the authors in [5].

REMARK 2.8. Our results generalize and extend several classical results in the literature, concerning unique and nonunique fixed points. In particular, both Theorem 2.1 and Theorem 2.2 are generalizations and extensions of the corresponding results of Ćirić ([12, 13]) and some results of Jaggi ([15]). Both Theorem 2.4 and Theorem 2.5 extend both Theorem 2.1 and Theorem 2.2 respectively as well as the corresponding results of Ćirić ([12, 13]) and some results of Jaggi ([15]). Both Theorem 2.4 and Theorem 2.5 also generalize the result of Maia ([19]). Indeed, the results of our present paper generalize the corresponding results by Olatinwo ([22, 23, 26]), but independent of the corresponding results of the author ([24]).

REMARK 2.9. We also employ this medium to announce that while proving the existence of the fixed point of  $T$ , the term " $d(T \lim_{n \rightarrow \infty} (T^n x_0), Tu)$ " that appeared was a typographical misprint in Theorem 2.1 and Theorem 2.3 of [22] as well as in Theorem 2.1 and Theorem 2.4 of [24]. Since  $T$  is orbitally continuous in those theorems (rather than being continuous), the misprint should change to " $d(\lim_{n \rightarrow \infty} T(T^n x_0), Tu)$ " (which is now correctly

written in the present article). Our interested readers can also see the correct term “ $d(\lim_{n \rightarrow \infty} T(T^n x_0), Tu)$ ” in the articles [23, 26] (which invariably becomes “ $\lim_{n \rightarrow \infty} d(T(T^n x_0), Tu)$ ” since metric is continuous).

**Acknowledgement.** This paper is dedicated to the late Professor Ljubomir Ćirić (one of my great mentors) who was born on Tuesday, 13<sup>th</sup> August, 1935 and whose demise occurred on Saturday, 23<sup>rd</sup> July, 2016. He was a pioneering expert of Fixed Point Theory in Serbia.

I also thank immensely the anonymous reviewers of this paper for perusing it, thereby making it publishable.

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