

A NOTE ON MULTIPLICATIVE (GENERALIZED) (α, β) -DERIVATIONS IN PRIME RINGS

NADEEM UR REHMAN, RADWAN M. AL-OMARY,
 NAJAT MOHAMMED MUTHANA

Abstract. Let R be a prime ring with center $Z(R)$. A map $G: R \rightarrow R$ is called a multiplicative (generalized) (α, β) -derivation if $G(xy) = G(x)\alpha(y) + \beta(x)g(y)$ is fulfilled for all $x, y \in R$, where $g: R \rightarrow R$ is any map (not necessarily derivation) and $\alpha, \beta: R \rightarrow R$ are automorphisms. Suppose that G and H are two multiplicative (generalized) (α, β) -derivations associated with the mappings g and h , respectively, on R and α, β are automorphisms of R . The main objective of the present paper is to investigate the following algebraic identities: (i) $G(xy) + \alpha(xy) = 0$, (ii) $G(xy) + \alpha(yx) = 0$, (iii) $G(xy) + G(x)G(y) = 0$, (iv) $G(xy) = \alpha(y) \circ H(x)$ and (v) $G(xy) = [\alpha(y), H(x)]$ for all x, y in an appropriate subset of R .

1. Introduction

Throughout the present paper, R will denote an associative ring with centre $Z(R)$ and α, β will denote automorphisms on R . For given $x, y \in R$, the symbols $[x, y]$ and $x \circ y$ denote the commutator $xy - yx$ and anti-commutator $xy + yx$, respectively. For any pair $x, y \in R$ we shall write $[x, y]_{\alpha, \beta} = x\alpha(y) - \beta(y)x$. Given an integer $n \geq 2$, a ring R is said to be n -torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies either $a = 0$ or $b = 0$ and is semiprime if for $a \in R$, $aRa = (0)$ implies $a = 0$. An additive map δ from R to R

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is called a derivation of R if $\delta(xy) = \delta(x)y + x\delta(y)$ holds for all $x, y \in R$. Let $F: R \rightarrow R$ be a map associated with another map $\delta: R \rightarrow R$ such that $F(xy) = F(x)y + x\delta(y)$ holds for all $x, y \in R$. If F is additive and δ is a derivation of R , then F is said to be a generalized derivation of R – a concept introduced by Brešar ([4]). In [9], Hvala gave the algebraic study of generalized derivations of prime rings. We note that if R has the property that $Rx = (0)$ implies $x = 0$ and $\psi: R \rightarrow R$ is any function, and $\chi: R \rightarrow R$ is any additive map such that $\chi(xy) = \psi(x)y + x\psi(y)$ for all $x, y \in R$, then χ is uniquely determined by ψ and ψ must be a derivation by [4, Remark 1]. Obviously, every derivation is a generalized derivation of R . Thus, generalized derivations cover both the concept of derivations and left multiplier maps. Following [5], a multiplicative derivation of R is a map $G: R \rightarrow R$ which satisfies $G(xy) = G(x)y + xG(y)$ for all $x, y \in R$. Of course these maps need not be additive. To the best of our knowledge, the concept of multiplicative derivations appears for the first time in the work of Daif ([5]) and it was motivated by the work of Martindale ([10]). Further, the complete description of those maps was given by Goldmann and Šemrl in [8]. Such maps do indeed exist in the literature (viz. [5] and [8] where further references can be found). Daif and Tammam El-Sayiad ([6]) extended multiplicative generalized derivations as follows: a map $G: R \rightarrow R$ is called a multiplicative generalized derivation if there exists a derivation g such that $G(xy) = G(x)y + xg(y)$ for all $x, y \in R$. In this definition, if we consider that g is any map that is not necessarily a derivation or additive, then G is said to be multiplicative (generalized)-derivation which was introduced by Dhara and Ali ([7]). Thus, a map $G: R \rightarrow R$ (not necessarily additive) is said to be a multiplicative (generalized)-derivation if $G(xy) = G(x)y + xg(y)$ holds for all $x, y \in R$, where g is any map (not necessarily a derivation or an additive map). Hence, the concept of a multiplicative (generalized)-derivation covers the concept of a multiplicative derivation. Moreover, multiplicative (generalized)-derivation with $g = 0$ covers the notion of multiplicative centralizers (not necessarily additive). The examples of multiplicative (generalized)-derivations are multiplicative derivations and multiplicative centralizers. Let S be a nonempty subset of R . A mapping $f: R \rightarrow R$ is called centralizing on S if $[f(x), x] \in Z(R)$ for all $x \in S$ and is called commuting on S if $[f(x), x] = 0$ for all $x \in S$. In this direction, Posner ([11]) was the first who investigate commutativity of the ring. More precisely, he proved that: If R is a prime ring with a nonzero derivation δ on R such that δ is centralizing on R , then R is commutative.

Further, regarding commutativity in prime rings, Ashraf and Rehman ([3]), proved the following: let R be a prime ring and I a non-zero ideal of R . Suppose that δ is a non-zero derivation on R . If one of the following holds: (i) $\delta(xy) + xy \in Z(R)$; (ii) $\delta(xy) - xy \in Z(R)$ for all $x, y \in I$, then R must be commutative. Further, Ashraf et al. ([2]) extended their work, replacing the derivation δ with a generalized derivation F in a prime ring R . More

precisely, they proved the following: Let R be a prime ring and I a non-zero ideal of R . Suppose F is a generalized derivation associated with a nonzero derivation δ on R . If one of the following holds: (i) $F(xy) \pm xy \in Z(R)$; (ii) $F(xy) \pm yx \in Z(R)$; (iii) $F(x)F(y) \pm xy \in Z(R)$ for all $x, y \in I$, then R is commutative. Recently, Albas ([1]) studied the above mentioned identities in prime rings with central values.

Recently, Dhara and Ali ([7]) studied the following identities related to multiplicative (generalized)-derivations on semiprime rings: (i) $F(xy) \pm xy = 0$, (ii) $F(xy) \pm yx = 0$, (iii) $F(x)F(y) \pm xy \in Z(R)$, (iv) $F(x)F(y) \pm yx \in Z(R)$ for all x, y in some suitable subset of a semiprime ring R .

In the present paper, we generalize the concept of a multiplicative (generalized)-derivation to a multiplicative (generalized)- (α, β) -derivation. A mapping $G: R \rightarrow R$ (not necessarily additive) is called a multiplicative (generalized)- (α, β) -derivation of R , if $G(xy) = G(x)\alpha(y) + \beta(x)g(y)$ for all $x, y \in R$, where $g: R \rightarrow R$ is any map (not necessarily additive) and $\alpha, \beta: R \rightarrow R$ are automorphisms of R . One can find an example of a multiplicative generalized derivation, which is neither a derivation nor a generalized derivation.

EXAMPLE 1.1. Let

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$

Let us define $G, g, \alpha, \beta: R \rightarrow R$ by

$$G \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & bc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\alpha \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & -b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \quad \beta \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}.$$

Then it is straightforward to verify that G is not an additive map in R . Hence, G is a multiplicative (generalized)- (α, β) -derivation associated with the mapping g on R , but G is neither a generalized derivation nor a multiplicative (generalized)-derivation of R .

In the present paper, our aim is to investigate some identities with multiplicative (generalized)- $(\alpha; \beta)$ -derivations on some suitable subsets in prime rings.

2. Main Results

We begin our discussion with the following lemma.

LEMMA 2.1. *Let R be a prime ring and I be a nonzero left ideal of R . Let α, β be automorphisms of R . If $[x, y]_{\alpha, \beta} = 0$ for all $x, y \in I$, then R is commutative.*

PROOF. We have

$$(2.1) \quad [x, y]_{\alpha, \beta} = 0$$

for all $x, y \in I$. Replacing x by rx in (2.1), $r \in R$, we get

$$r[x, y]_{\alpha, \beta} + [r, \beta(y)]x = 0$$

for all $x, y \in I$ and $r \in R$. Application of (2.1) yields that $[r, \beta(y)]x = 0$ for all $x, y \in I$ and $r \in R$, that is, $[r, \beta(y)]RI = (0)$ for all $y \in I$ and $r \in R$. Thus, primeness of R forces that $[r, \beta(y)] = 0$ for all $y \in I$ and $r \in R$. Now, replace r by $\beta(t)$, $t \in R$, in the above expression, we find that $\beta([r, y]) = 0$, since β is automorphism, i.e., that $[r, y] = 0$. Again replacing y by sy for $s \in R$ in the last expression, we get $[r, s]y = 0$ that is, $[r, s]RI = (0)$. Hence, primeness of R gives that $[r, s] = 0$ for all $r, s \in R$, so that R is commutative. \square

THEOREM 2.1. *Let R be a prime ring and I be a nonzero left ideal of R . Suppose that G is a multiplicative (generalized)- (α, β) -derivation on R associated with the map g on R . If $G(xy) + \alpha(xy) = 0$ for all $x, y \in I$, then $G(x) = -\alpha(x)$ for all $x \in I$ and $\beta(I)g(I) = (0)$.*

PROOF. We have

$$(2.2) \quad G(xy) + \alpha(xy) = 0$$

for all $x, y \in I$. Replacing y by yz in (2.2), we get

$$(2.3) \quad G(xy)\alpha(z) + \beta(xy)g(z) + \alpha(xy)\alpha(z) = 0$$

for all $x, y, z \in I$. Using (2.2) in (2.3), we have

$$(2.4) \quad \beta(x)\beta(y)g(z) = 0$$

for all $x, y, z \in I$. Replacing y by ry in (2.4), $r \in R$, we get $\beta(x)\beta(r)\beta(y)g(z) = 0$. Now replacing r by $\beta^{-1}(g(z)r)$ we find that $\beta(x)g(z)R\beta(y)g(z) = (0)$ for all

$x, y, z \in I$. Thus, by primeness of R , we get $\beta(I)g(I) = (0)$. Thus, equation (2.2) implies that $G(x)\alpha(y) + \alpha(xy) = \{G(x) + \alpha(x)\}\alpha(y) = 0$. Replacing y by ry in the last expression and using primeness of R , we conclude that $G(x) = -\alpha(x)$. Thereby the proof is completed. \square

THEOREM 2.2. *Let R be a prime ring and I be a nonzero left ideal of R . Suppose that G is a multiplicative (generalized)- (α, β) -derivation on R associated with the map g on R . If $G(xy) + \alpha(yx) = 0$ for all $x, y \in I$, then R is commutative, $\beta(I)g(I) = (0)$ and $G(x) = -\alpha(x)$ for all $x \in I$.*

PROOF. We have the identity

$$(2.5) \quad G(xy) + \alpha(yx) = 0$$

for all $x, y \in I$. Replacing x by x^2 and y by xy , respectively, in (2.5) and then subtracting one from another, we obtain $\alpha(yx^2) = \alpha(xy^2)$ or $[x, y]x = 0$. Replacing y by ry in the last expression, we have $[x, r]yx = 0$, where $r \in R$. Since I is nonzero, so by primeness of R , we have $[x, r] = 0$. Substituting x by sx in the last expression, we obtain $[s, r]x = 0$, where $r, s \in R$. Primeness of R forces that R is commutative. Therefore $G(xy) + \alpha(yx) = 0$ becomes $G(xy) + \alpha(xy) = 0$. Thus, in view of Theorem 2.1, we have $G(x) = -\alpha(x)$ for all $x \in I$ and $\beta(I)g(I) = (0)$. This completes the proof. \square

THEOREM 2.3. *Let R be a prime ring and I be a nonzero left ideal of R . Suppose that G is a multiplicative (generalized)- (α, β) -derivation on R associated with the map g on R . If $G(xy) + G(x)G(y) = 0$ for all $x, y \in I$, then either $\alpha(I)[G(x), \alpha(x)] = (0)$ or $\beta(I)[G(x), \beta(x)] = (0)$ for all $x \in I$.*

PROOF. We have the identity

$$(2.6) \quad G(xy) + G(x)G(y) = 0$$

for all $x, y \in I$. Replacing y by yz in (2.6), we obtain

$$(2.7) \quad G(xy)\alpha(z) + \beta(xy)g(z) + G(x)G(y)\alpha(z) + G(x)\beta(y)g(z) = 0$$

for all $x, y, z \in I$. Using (2.6) in (2.7), we get

$$(2.8) \quad \beta(xy)g(z) + G(x)\beta(y)g(z) = 0$$

for all $x, y, z \in I$. Replacing x by xw in (2.8), we have

$$(2.9) \quad \beta(xwy)g(z) + G(x)\alpha(w)\beta(y)g(z) + \beta(x)g(w)\beta(y)g(z) = 0$$

for all $x, y, z, w \in I$. Again substituting y by wy in (2.8), we obtain

$$(2.10) \quad \beta(xwy)g(z) + G(x)\beta(w)\beta(y)g(z) = 0$$

for all $x, y, z, w \in I$. Subtracting (2.9) from (2.10), we

$$(2.11) \quad \{G(x)\alpha(w) + \beta(x)g(w) - G(x)\beta(w)\}\beta(y)g(z) = 0$$

for all $x, y, z, w \in I$. Replacing y by ry in (2.11), where $r \in R$, by primeness of R , we have $G(x)\alpha(w) + \beta(x)g(w) - G(x)\beta(w) = G(xw) - G(x)\beta(w) = 0$ or $\beta(y)g(z) = 0$. From (2.6), we have

$$(2.12) \quad G(xyz) = -G(xy)G(z) = -G(x)G(yz)$$

for all $x, y, z \in I$. Using $G(xy) - G(x)\beta(y) = 0$, equation (2.12) can be written as $G(x)\{\beta(y)G(z) - G(y)\beta(z)\} = 0$. Replacing x by xrw in the last expression, where $w \in I, r \in R$ and using primeness of R , we conclude that $\beta(w)[G(z), \beta(z)] = 0$. Now, the other case $\beta(x)g(y) = 0$ gives $G(xy) = G(x)\alpha(y)$ for all $x, y \in I$, then proceeding in the same way as we have done earlier for $G(xy) = G(x)\beta(y)$, we obtain $\alpha(x)[G(y), \alpha(y)] = 0$. Hence, we get the required result. \square

THEOREM 2.4. *Let R be a prime ring and I be a nonzero left ideal of R . Suppose that G and H are multiplicative (generalized)- (α, β) -derivations on R associated with the maps g and h on R , respectively. If $G(xy) = \alpha(y) \circ H(x)$ for all $x, y \in I$, then either R is commutative or $\alpha(I)[\alpha(I), H(I)] = (0)$.*

PROOF. We have the identity

$$(2.13) \quad G(xy) = \alpha(y) \circ H(x)$$

for all $x, y \in I$. Replacing y by yz in (2.13), we obtain

$$(2.14) \quad G(xy)\alpha(z) + \beta(xy)g(z) = (\alpha(y) \circ H(x))\alpha(z) + \alpha(y)[\alpha(z), H(x)]$$

for all $x, y, z \in I$. Using (2.13) in (2.14), we get

$$(2.15) \quad \beta(xy)g(z) = \alpha(y)[\alpha(z), H(x)]$$

for all $x, y, z \in I$. Replacing y by wy in (2.15), we have

$$(2.16) \quad \beta(xwy)g(z) = \alpha(wy)[\alpha(z), H(x)]$$

for all $x, y, z, w \in I$. Left multiply by $\alpha(w)$ to (2.15) and subtract it from (2.16), we obtain

$$(2.17) \quad \{\beta(x)\beta(w) - \alpha(w)\beta(x)\}\beta(y)g(z) = 0$$

for all $x, y, z, w \in I$. Replacing y by ry in (2.17), where $r \in R$ and using primeness of R , we get either $\beta(y)g(z) = 0$ or $\beta(x)\beta(w) - \alpha(w)\beta(x) = 0$. If $\beta(x)g(y) = 0$ holds for all $x, y \in I$, then from (2.15), we have $\alpha(I)[\alpha(I), H(I)] = (0)$. For the other case

$$(2.18) \quad \beta(x)\beta(y) - \alpha(y)\beta(x) = 0$$

for all $x, y \in I$. Replacing y by ry in (2.18), where $r \in R$, we get

$$(2.19) \quad \beta(x)\beta(ry) - \alpha(ry)\beta(x) = 0.$$

Left multiply by $\alpha(r)$ to (2.18) and subtract it from (2.19), we have $\{\beta(x)\beta(r) - \alpha(r)\beta(x)\}\beta(y) = 0$. Since I is nonzero, so primeness of R forces to write $\beta(x)\beta(r) - \alpha(r)\beta(x) = 0$. We can rewrite the last expression as $[\beta(x), r]_{\beta, \alpha} = 0$ for all $x \in I, r \in R$. Application of Lemma 2.1 yields that R is commutative. Thereby the proof is completed. \square

THEOREM 2.5. *Let R be a prime ring and I be a nonzero left ideal of R . Suppose that G and H are multiplicative (generalized)- (α, β) -derivations on R associated with the maps g and h on R , respectively. If $G(xy) = [\alpha(y), H(x)]$ for all $x, y \in I$, then either R is commutative or $\alpha(I)[\alpha(I), G(I)] = (0)$.*

PROOF. We have the identity

$$(2.20) \quad G(xy) = [\alpha(y), H(x)]$$

for all $x, y \in I$. Replacing y by yz in (2.20), we obtain

$$(2.21) \quad G(xy)\alpha(z) + \beta(xy)g(z) = [\alpha(y), H(x)]\alpha(z) + \alpha(y)[\alpha(z), H(x)]$$

for all $x, y, z \in I$. Using (2.20) in (2.21), we get

$$(2.22) \quad \beta(xy)g(z) = \alpha(y)[\alpha(z), H(x)]$$

for all $x, y, z \in I$. Note that the equation (2.22) is same as the equation (2.15) in Theorem 2.4; then proceeding in the same way as in Theorem 2.4, we get the required result. \square

3. Examples

In this section we construct some examples to show that the primeness condition of the ring in our results are essential.

EXAMPLE 3.1. Let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\} \quad \text{and} \quad I = \left\{ \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \mid b, c \in \mathbb{Z} \right\}.$$

Let us define $G, g, \alpha, \beta: R \rightarrow R$ by

$$\begin{aligned} G \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} &= \begin{pmatrix} 0 & -b \\ 0 & -c \end{pmatrix}, & g \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} &= \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}, \\ \alpha \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} &= \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}, & \beta \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} &= \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}. \end{aligned}$$

It is easy to verify that I is a left ideal on R , G is a multiplicative (generalized)- (α, β) -derivation associated with the map g , α and β are automorphisms on R and $G(xy) + G(x)G(y) = 0$ for all $x, y \in I$. Since

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\},$$

R is not a prime ring. We see that $\alpha(I)[G(x), \alpha(x)] \neq (0)$ and $\beta(I)[G(x), \beta(x)] \neq 0$ for all $x \in I$. Hence, the primeness hypothesis in Theorem 2.3 is crucial.

EXAMPLE 3.2. Let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\} \quad \text{and} \quad I = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}.$$

Let us define $G, g, \alpha, \beta, H, h: R \rightarrow R$ by

$$\begin{aligned} G \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} &= \begin{pmatrix} 0 & b \\ 0 & -c \end{pmatrix}, & g \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} &= \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \\ \alpha \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} &= \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}, & \beta \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} &= \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}, \\ H \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, & h \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} &= \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

It is easy to verify that I is a left ideal on R , G and H are multiplicative (generalized)- (α, β) -derivations associated with the maps g and h , respectively, α, β are automorphisms on R and $G(xy) = [\alpha(y), H(x)]$ for all $x, y \in I$. Since

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\},$$

R is not a prime ring. We see that R is not commutative and $\alpha(I)[\alpha(I), G(I)] \neq 0$. Hence, the primeness hypothesis in Theorem 2.5 is crucial.

EXAMPLE 3.3. Let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\} \quad \text{and} \quad I = \left\{ \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \mid b, c \in \mathbb{Z} \right\}.$$

Let us define $G, g, \alpha, \beta, H, h: R \rightarrow R$ by

$$\begin{aligned} G \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} &= \begin{pmatrix} a & -b \\ 0 & 0 \end{pmatrix}, & g \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} &= \begin{pmatrix} 0 & -2b \\ 0 & 0 \end{pmatrix}, \\ \alpha \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} &= \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, & \beta \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} &= \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}, \\ H \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} &= \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}, & h \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} &= \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

It is easy to verify that I is a left ideal on R , G and H are multiplicative (generalized)- (α, β) -derivations associated with the maps g and h , respectively, α, β are automorphisms on R and $G(xy) = \alpha(y) \circ H(x)$ for all $x, y \in I$. Since

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\},$$

R is not a prime ring. We see that R is not commutative and $\alpha(I)[\alpha(I), H(I)] \neq \{0\}$. Hence, the primeness hypothesis in Theorem 2.4 is crucial.

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NADEEM UR REHMAN
DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
ALIGARH U.P.
INDIA
e-mail: nu.rehman.mm@amu.ac.in

NAJAT MOHAMMED MUTHANA
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
KING ABDULAZIZ UNIVERSITY
JEDDAH
SAUDI ARABIA
e-mail: nmuthana@kau.edu.sa

RADWAN M. AL-OMARY
DEPARTMENT OF MATHEMATICS
IBB UNIVERSITY
IBB
YEMEN
e-mail: radwan959@yahoo.com