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REPRODUCING KERNELS OF WEIGHT SQUARE-SUMMABLE SEQUENCES HILBERT SPACES

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Abstract. In this paper we will introduce the concept of weighted reproducing kernel of $l^2(\mathbb{C})$ space, in similiar way as it is done in case of weighted reproducing kernel of Bergman space. We will give an explicit formula for it and prove that it depends analytically on weight. In addition, we will show some theorems about dependance of $l^2(\mathbb{C})$ space on weight.

1. Introduction

Concept of reproducing kernel is known in mathematics for a long time. G. Szegő and S. Bergman were one of the firsts who conducted research in this topic ([1], [5]). It is more general, however, and can be associated with general Hilbert space.

F.H. Szafraniec in [4] showed that $l^2(\mathbb{C},\mu)$ with a function $K(i,j) := \delta_{ij}/\mu(i)$ is an example of a reproducing kernel Hilbert space. He did not prove any properties of it however. In this paper we will show how the space $l^2(\mathbb{C},\mu)$ depends on weight μ and that the reproducing kernel of it is analytical in some sense.

Reproducing kernels play a significant role in mathematics. A. Odzijewicz proved that the calculation of Feynman path integral for some mechanical system is equivalent to finding reproducing kernel function of the Hilbert space of quantum states ([3]). He considered equation with weight of reproducing

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kernel in it. On the other hand, M. Drewnik and Z. Pasternak-Winiarski used reproducing kernels to recognize handwritten digits ([2]). In their paper they wanted to find the optimal weight (in some sense). In both these situations analyticity (or more general, differentiability) of reproducing kernel would give useful tools to solve the problems. That is why the aim of this paper is to show that dependance of the reproducing kernel of $l^2(\mathbb{C},\mu)$ on weight is analytic. We will start with the properties of $l^2(\mathbb{C},\mu)$ space, however.

For convenience, we will write $x(i)$ instead of x_i to denote i-th element of a sequence x belonging to our Hilbert space, leaving symbol x_i for i-th element of a sequence of elements of considered space.

2. Definition and properties

Let $l^2(\mathbb{C},\mu)$ be a set of all complex sequences $x = (x(1), x(2), \dots)$, square integrable in the sense

$$\| x \|_\mu^2 := \sum_{i\in\mathbb{N}} |x(i)|^2 \mu(i) < \infty \tag{2.1}$$

with an inner product given by

$$\langle x|y\rangle_\mu := \sum_{i\in\mathbb{N}} \overline{x(i)} y(i)\mu(i), \tag{2.2}$$

where $\mu\colon \mathbb{N} \to \mathbb{R}$ is a function, which we will name *weight*.

It is obvious that $\mu(i)$ must be greater than 0 for any natural i in order for (2.1) to have a sense of norm. On the other hand, if $\mu(i) > 0$ for any natural i, then (2.2) has a sense of an inner product.

For a given Hilbert space $\mathcal{H}$ of functions on a domain Ω for which evaluation functionals are continuous, we can define its reproducing kernel. It is a function $K\colon \Omega\times\Omega \to \mathbb{C}$, such that $\overline{K(z,\cdot)} \in \mathcal{H}$ and

$$\langle \overline{K(z,w)} | f(w)\rangle = f(z)$$

for any $f \in \mathcal{H}$ and $z \in \Omega$. By Riesz theorem, no Hilbert space can have more than one reproducing kernel. Moreover, if Hilbert space has a reproducing kernel, then it is given by

$$K(z,w) = \sum_{k\in I} \varphi_k(z)\overline{\varphi_k(w)},$$

where $\{\varphi_i\}$ is any complete orthonormal system of $\mathcal{H}$.

THEOREM 2.1. *Reproducing kernel of $l^2(\mathbb{C},\mu)$ is given by*

$$K_\mu(i,j) = \frac{1}{\mu(i)}\delta_{ij}.$$

PROOF. Let us take any element $x \in l^2(\mathbb{C},\mu)$. We have

$$\langle\overline{K_\mu(i,j)}|x(j)\rangle_\mu = \sum_{j\in\mathbb{N}} \frac{1}{\mu(j)}\delta_{ij}x(j)\mu(j) = \sum_{j\in\mathbb{N}}\delta_{ij}x(j) = x(i),$$

so K has a reproducing property. Of course $\overline{K_\mu(i,\cdot)} \in l^2(\mathbb{C},\mu)$ for any weight μ, so K_μ is a reproducing kernel of considered space. □

An interesting question to ask is how does space $l^2(\mathbb{C},\mu)$ change, as a set, with a change of a weight μ?

PROPOSITION 2.1. *Let μ_1, μ_2 be weights. Then $l^2(\mathbb{C},\mu_1+\mu_2) = l^2(\mathbb{C},\mu_1)\cap l^2(\mathbb{C},\mu_2)$ as sets.*

PROOF. Obviously,

$$\sum_{i\in\mathbb{N}}|x(i)|^2(\mu_1(i)+\mu_2(i)) = \sum_{i\in\mathbb{N}}|x(i)|^2\mu_1(i) + \sum_{i\in\mathbb{N}}|x(i)|^2\mu_2(i).$$

If $x \in l^2(\mathbb{C},\mu_1+\mu_2)$, then the sum on the left hand side is finite, so both sums on the right hand side must be also finite, because we sum non-negative numbers and in consequence, $x \in l^2(\mathbb{C},\mu_1)\cap l^2(\mathbb{C},\mu_2)$.

On the other hand, if $x \in l^2(\mathbb{C},\mu_1)\cap l^2(\mathbb{C},\mu_2)$, then both sums on the right hand side are finite and in consequence, the sum on the left hand side must be finite, which means that $x \in l^2(\mathbb{C},\mu_1+\mu_2)$. □

THEOREM 2.2. *Let μ_1, μ_2 be weights. Then the following conditions are equivalent:*
(i) *$l^2(\mathbb{C},\mu_1) = l^2(\mathbb{C},\mu_2)$ as sets;*
(ii) *there exist $m, M > 0$ such that*

$$m\mu_1(i) < \mu_2(i) < M\mu_1(i) \tag{2.3}$$

for any natural i.

Note that (2.3) means that

$$\frac{1}{M}\mu_2(i) < \mu_1(i) < \frac{1}{m}\mu_2(i).$$

PROOF. Let μ_1, μ_2 be weights which satisfy (2.3). Then

$$m \sum_{i\in\mathbb{N}} |x(i)|^2 \mu_1(i) \leqslant \sum_{i\in\mathbb{N}} |x(i)|^2 \mu_2(i) \leqslant M \sum_{i\in\mathbb{N}} |x(i)|^2 \mu_1(i).$$

If $x \in l^2(\mathbb{C}, \mu_1)$, then the sum on the right hand side is finite, so the sum in the middle must also be finite and $x \in l^2(\mathbb{C}, \mu_2)$. On the other hand, if $x \in l^2(\mathbb{C}, \mu_2)$, then the sum in the middle is finite, so the sum on the left hand side must also be finite and $x \in l^2(\mathbb{C}, \mu_1)$, so $l^2(\mathbb{C}, \mu_1) = l^2(\mathbb{C}, \mu_2)$ as sets.

Now let us assume that (ii) does not hold. It means that one of the inequalities in (2.3) is not true. Without loss of generality, we may assume that

$$\mu_2(i) < M\mu_1(i)$$

does not hold for any $M > 0$. Hence, for any $p \in \mathbb{N}$ there exists $i_p \in \mathbb{N}$, such that

$$p^2 \mu_1(i_p) \leq \mu_2(i_p),$$

which means that

$$\frac{\mu_1(i_p)}{\mu_2(i_p)} \leq \frac{1}{p^2} \tag{2.4}$$

for any $p \in \mathbb{N}$.

Now let us define an element x in the following way:

$$x(s) := \begin{cases} \frac{1}{\sqrt{\mu_2(s)}}, & s \in P, \\ 0, & s \notin P, \end{cases}$$

where $P := \{i_p\}$. As we can see,

$$\sum_{s\in\mathbb{N}} |x(s)|^2 \mu_1(s) = \sum_{s\in P} |x(s)|^2 \mu_1(s) = \sum_{s\in P} \frac{\mu_1(s)}{\mu_2(s)}.$$

But by (2.4) we have

$$\sum_{s\in P} \frac{\mu_1(s)}{\mu_2(s)} \leq \sum_{p\in\mathbb{N}} \frac{\mu_1(i_p)}{\mu_2(i_p)} \leq \sum_{p\in\mathbb{N}} \frac{1}{p^2} < \infty,$$

so $x \in l^2(\mathbb{C}, \mu_1)$.

However,

$$\sum_{s\in\mathbb{N}} |x(s)|^2\mu_2(s) = \sum_{s\in P} |x(s)|^2\mu_2(s) = \sum_{s\in P} \frac{\mu_2(s)}{\mu_2(s)} = \sum_{s\in P} 1 = \infty,$$

because P is infinite, so $x \notin l^2(\mathbb{C}, \mu_2)$ and in consequence $l^2(\mathbb{C}, \mu_1) \neq l^2(\mathbb{C}, \mu_2)$ as sets. □

Corollary 2.1. *The following conditions are equivalent:*
(i) *there are $m, M > 0$ such that*

$$m < \mu(i) < M$$

for any natural i,
(ii) $l^2(\mathbb{C}, \mu) = l^2(\mathbb{C}, 1)$ *as sets.*

Another good question is to ask how does Hilbert space structure (i.e. orthogonality) depends on μ.

Theorem 2.3. *The following conditions are equivalent:*
(i) *there exists $a > 0$ such that $\mu_1(i) = a\mu_2(i)$ for any natural i,*
(ii) *for any $x, y \in l^2(\mathbb{C}, \mu_1) \cap l^2(\mathbb{C}, \mu_2)$ we have $\langle x|y\rangle_{\mu_1} = 0 \Leftrightarrow \langle x|y\rangle_{\mu_2} = 0$.*

Proof. If (i) holds, then

$$\langle x|y\rangle_{\mu_1} = \sum_{i\in\mathbb{N}} \overline{x(i)}y(i)\mu_1(i) = a\sum_{i\in\mathbb{N}} \overline{x(i)}y(i)\mu_2(i) = a\langle x|y\rangle_{\mu_2},$$

so (ii) is true.

Now let us assume that (i) does not hold. It means that there exist natural j, k such that:

$$\alpha\mu_1(j) = \mu_2(j),$$
$$\beta\mu_1(k) = \mu_2(k)$$

and $\alpha \neq \beta$. Now let us define elements x and y in the following way:

$$x(i) := \delta_{ij} + \delta_{ik},$$
$$y(i) := \mu_1(k)\delta_{ij} - \mu_1(j)\delta_{ik}.$$

Of course $x, y \in l^2(\mathbb{C}, \mu_1) \cap l^2(\mathbb{C}, \mu_2)$, as all sequences with finite number of non-zero elements.

As we can see,

$$\langle x|y\rangle_{\mu_1} = \sum_{i\in\mathbb{N}} \overline{x(i)}y(i)\mu_1(i) = \mu_1(k)\mu_1(j) - \mu_1(j)\mu_1(k) = 0,$$

but

$$\begin{aligned}\langle x|y\rangle_{\mu_2} &= \sum_{i\in\mathbb{N}} \overline{x(i)}y(i)\mu_2(i) = \mu_1(k)\mu_2(j) - \mu_1(j)\mu_2(k) \\ &= (\alpha-\beta)\,\mu_1(k)\mu_1(j) \neq 0,\end{aligned}$$

so (ii) is not true. □

COROLLARY 2.2.
(i) *If $l^2(\mathbb{C},\mu_1) \neq l^2(\mathbb{C},\mu_2)$ as sets, then they have different Hilbert space structures, i.e. we can find two elements $x, y \in l^2(\mathbb{C},\mu_1)\cap l^2(\mathbb{C},\mu_2)$ such that $\langle x|y\rangle_{\mu_1} = 0$, but $\langle x|y\rangle_{\mu_2} \neq 0$.*
(ii) *If $l^2(\mathbb{C},\mu_1) = l^2(\mathbb{C},\mu_2)$ as sets, then they may have the same or different Hilbert space structures.*

Now let us think about a reproducing kernel K as a function which for a given i, j depends on weight, i.e. let us think about it as a function

$$K_{ij}\colon \mathcal{X} \ni \mu \mapsto K_{ij}(\mu) := K_\mu(i,j) \in \mathbb{C},$$

where $\mathcal{X}$ denotes the set of all possible (infinite) complex sequences μ such that $\mu(i) > 0$ for any natural i.

Let X be a linear space, $A \subset X$, $x \in A$ and Y a linear topological space. Assume that $v \in X$ is such that $x+hv \in A$ for $h \in \mathbb{R}$ from some neighbourhood of 0. Then we can define *Gâteaux derivative* of a function $f\colon A \to Y$ in a direction v at a point x in the following way:

$$\frac{\partial f(x)}{\partial v} := \lim_{h\to 0} \frac{f(x+hv)-f(x)}{h},$$

if the limit exists. We put $\frac{\partial^0 f(x)}{\partial v} := f(x)$.

THEOREM 2.4. *Let $\mu_k \in \mathcal{X}$ be defined in the following way:*

$$\mu_k(i) := \delta_{ik}.$$

Then for any natural k, K_{ij} is infinitely many times Gâteaux differentiable in a direction μ_k and it is true that

$$\frac{\partial^n K_{ij}(\mu)}{\partial(\mu_k)^n} = (-1)^n \frac{n!(\mu_k(i))^n}{(\mu(i))^{n+1}}\delta_{ij} \tag{2.5}$$

for $n \in \mathbb{N} \cup \{0\}$.

Proof. Formula (2.5) is correct for $n = 0$.

Now let us assume that (2.5) holds for $m \leq n$. Then we have

$$\begin{aligned}
\frac{\partial^{n+1} K_{ij}(\mu)}{\partial(\mu_k)^{n+1}} &= \lim_{h\to 0} \frac{(-1)^n n!(\mu_k(i))^n\delta_{ij}}{h}\left(\frac{1}{(\mu(i)+h\mu_k(i))^{n+1}} - \frac{1}{(\mu(i))^{n+1}}\right) \\
&= \lim_{h\to 0} \frac{(-1)^n n!(\mu_k(i))^n\delta_{ij}}{h}\left(\frac{(\mu(i))^{n+1} - (\mu(i)+h\mu_k(i))^{n+1}}{(\mu(i)+h\mu_k(i))^{n+1}(\mu(i))^{n+1}}\right) \\
&= \lim_{h\to 0} \frac{(-1)^n n!(\mu_k(i))^n\delta_{ij}}{h}\left(\frac{-\sum_{l=0}^{n}\frac{(n+1)!}{l!(n+1-l)!}(\mu(i))^l (h\mu_k(i))^{n+1-l}}{(\mu(i)+h\mu_k(i))^{n+1}(\mu(i))^{n+1}}\right) \\
&= \lim_{h\to 0} (-1)^n n!(\mu_k(i))^n\delta_{ij}\left(\frac{-\sum_{l=0}^{n}\frac{(n+1)!}{l!(n+1-l)!}(\mu(i))^l (h)^{n-l}(\mu_k(i))^{n+1-l}}{(\mu(i)+h\mu_k(i))^{n+1}(\mu(i))^{n+1}}\right) \\
&= (-1)^n n!(\mu_k(i))^n\delta_{ij}\left(\frac{-(n+1)(\mu(i))^n(\mu_k(i))}{(\mu(i))^{2n+2}}\right) \\
&= (-1)^{n+1}(n+1)!\frac{(\mu_k(i))^{n+1}}{(\mu(i))^{n+2}}\delta_{ij}.
\end{aligned}$$

□

Moreover K_{ij} depends analytically on weight in the following sense:

Theorem 2.5. *Let $\mu_0 \in \mathcal{X}$ be such that $c := \inf_{i\in\mathbb{N}} \mu_0(i) > 0$. Then*

$$K_{ij}(\mu) = \sum_{n=0}^{\infty} \frac{(-1)^n(\mu(i)-\mu_0(i))^n}{(\mu_0(i))^{n+1}}\delta_{ij} \tag{2.6}$$

for any $i, j \in \mathbb{N}$ and $\mu \in \mathcal{X}$ such that

$$\sup_{i\in\mathbb{N}} |\mu(i) - \mu_0(i)| < c.$$

PROOF. If $i \neq j$, then the right hand side of the equation is equal to 0 and $K_{ij}(\mu) = 0$, so in that case formula (2.6) is correct. Now let us assume that $i = j$. For any given i the right hand side of the formula is a real geometric series

$$\frac{1}{\mu_0(i)} \sum_{n=0}^{+\infty} \left(1 - \frac{\mu(i)}{\mu_0(i)}\right)^n$$

with a common ratio

$$q = 1 - \frac{\mu(i)}{\mu_0(i)}.$$

Therefore the series is convergent for $|q| < 1$, i.e. for $|\mu(i) - \mu_0(i)| < \mu_0(i)$ and its sum is equal to

$$\frac{1}{\mu_0(i)} \frac{1}{1-q} = \frac{1}{\mu(i)} = K_{ij}(\mu). \qquad \square$$

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