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Report of Meeting

The Nineteenth Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities Zakopane (Poland), January 30–February 2, 2019

The Nineteenth Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities was held in Hotel Geovita in Zakopane, Poland, from January 30 to February 2, 2019. The meeting was organized by the Institute of Mathematics of the University of Silesia.

16 participants came from the University of Debrecen (Hungary), 13 from the University of Silesia in Katowice (Poland) and one from the Pedagogical University of Cracow (Poland).

Professor Maciej Sablik opened the Seminar and welcomed the participants to Zakopane.

The scientific talks presented at the Seminar focused on the following topics: equations in a single variable and in several variables, iteration theory, equations on abstract algebraic structures, regularity properties of the solutions of certain functional equations, functional inequalities, Hyers–Ulam stability, functional equations and inequalities involving mean values, generalized convexity. Interesting discussions were generated by the talks.

There was also a Problem Session and a festive dinner.

The closing address was given by Professor Zsolt Páles. His invitation to the Twentieth Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities in January 2020 in Hungary was gratefully accepted.

Summaries of the talks in alphabetical order of the authors follow in section 1, problems and remarks in chronological order in section 2, and the list of participants in the final section.

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1. Abstracts of talks

Roman Badora: *The Ulam stability problem for the equation $f(x\star g(y)) = f(x)f(y)$*

The functional equation $f(x \star g(y)) = f(x)f(y)$ is a joint generalization of the exponential equation $f(x+y) = f(x)f(y)$ and the equation $f(xf(y)) = f(x)f(y)$. Its origin is applied to turbulent fluid motion in the averaging theory. This equation is connected with some linear operators, that is, the Reynolds operator (M.-L. Dubreil-Jacotin (1953), Y. Matras (1969)), the averaging operator, the multiplicatively symmetric operator (J. Aczél and J. Dhombres (1989)) and it was studied by M.-L. Dubreil-Jacotin, Z. Daróczy, Y. Matras, J. Dhombres, C.F.K. Jung, V. Boonyasombat, G. Barbançon, J.R. Jung, N. Brillouët-Belluot, J. Brzdęk.

The stability problem for the equation $f(x \star g(y)) = f(x)f(y)$ was considered by A. Najdecki (2007) and J. Chung (2014). We discuss the Ulam stability problem for this equation for vector-valued mappings.

Karol Baron: *Weak limit of iterates of some random–valued functions and its application*

Given a probability space $(\Omega, \mathcal{A}, P)$, a complete and separable metric space X with the σ–algebra $\mathcal{B}$ of all its Borel subsets, and a $\mathcal{B}\otimes\mathcal{A}$–measurable and contractive in mean $f\colon X\times\Omega \to X$ we consider iterates $(f^n(x,\cdot))_{n\in\mathbb{N}}$ of f, defined on $X\times\Omega^{\mathbb{N}}$ by $f^0(x,\omega) = x$ and $f^n(x,\omega) = f\left(f^{n-1}(x,\omega),\omega_n\right)$ for $n\in\mathbb{N}$ (cf. [4, Sec. 1.4]), and obtained in [1] the weak limit π^f of this sequence. Basing on some properties of π^f established in [3] and on the main result of [2], given a Lipschitz F mapping X into a separable Banach space Y we characterize solvability of the equation

$$\varphi(x) = \int_\Omega \varphi\left(f(x,\omega)\right) P(d\omega) + F(x)$$

in the class of Lipschitz functions $\varphi\colon X\to Y$ by

$$\int_X F(z)\pi^f(dz) = 0.$$

Some examples illustrate that question.

References

[1] K. Baron, *On the convergence in law of iterates of random-valued functions*, Aust. J. Math. Anal. Appl. **6** (2009), no. 1, Art. 3, 9 pp.

[2] K. Baron and J. Morawiec, *Lipschitzian solutions to linear iterative equations revisited*, Aequationes Math. **91** (2017), 161–167.

[3] R. Kapica, *The geometric rate of convergence of random iteration in the Hutchinson distance*, Aequationes Math. **93** (2019), 149–160.

[4] M. Kuczma, B. Choczewski and R. Ger, *Iterative functional equations*, Encyclopedia of Mathematics and its Applications **32**, Cambridge University Press, Cambridge, 1990.

Mihály Bessenyei: *Fractals for minimalists* (Joint work with Evelin Pénzes)

The aim of the talk is to present an elementary way to fractals which completely avoids advanced analysis and uses purely naive set theory. The method relies on fixed point methods, where the role of the Banach Contraction Principle is replaced by a slightly improved version of the Knaster–Tarski Fixed Point Theorem.

Zoltán Boros: *Monomially linked multiadditive functions*

Applying a classical theorem [2, Theorem 1] and following arguments similar to those in a recent paper by Masaaki Amou [1], we establish the following statement.

Theorem. *If $n \in \mathbb{N}$, $1 < k_j \in \mathbb{N}$ $(j = 1, 2, \ldots, n)$, $C \in \mathbb{R}$, and the multiadditive function $F\colon \mathbb{R}^n \to \mathbb{R}$ satisfies the functional equation*

$$(1) \qquad F\left(x_1^{k_1}, x_2^{k_2}, \ldots, x_n^{k_n}\right) = C x_1^{k_1-1} x_2^{k_2-1} \ldots x_n^{k_n-1} F\left(x_1, x_2, \ldots, x_n\right)$$

for all $x_j \in \mathbb{R}$ $(j = 1, 2, \ldots, n)$, then $F = \sum_{I \in \mathcal{K}} F_I$, where the function $F_I\colon \mathbb{R}^n \to \mathbb{R}$ is a derivation or a linear function with respect to its jth variable whenever j does or does not belong to the index set I, respectively. Here $\mathcal{K}$ consists of all subsets I of $\{1, 2, \ldots, n\}$ that satisfy the equation $\prod_{j \in I} k_j = C$. It is supposed that $\prod_{j \in \emptyset} k_j = 1$ (i.e., in case $C = 1$ we obtain that F is multilinear). In case $\mathcal{K} = \emptyset$ we have $F = 0$.

We note that Amou proved such a statement for $k_j = -1$ $(j = 1, 2, \ldots, n)$, when equation (1) is restricted to nonzero variables.

References

[1] M. Amou, *Multiadditive functions satisfying certain functional equations*, Aequationes Math. **93** (2019), 345–350.

[2] A. Nishiyama and S. Horinouchi, *On a system of functional equations*, Aequationes Math. **1** (1968), 1–5.

Pál Burai: *Convexity generated by certain circulant matrices* (Joint work with Judit Makó and Patrícia Szokol)

In this presentation we examine the properties of Schur-convex functions, that are induced by certain circulant matrices. More precisely, let $t = (t_1, \dots, t_n)$ be a probability vector, and

$$T = \mathrm{circ(t_1, \dots, t_n)} = \begin{pmatrix} t_1 & t_2 & \dots & t_{n-1} & t_n \\ t_2 & t_3 & \dots & t_n & t_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_n & t_1 & \dots & t_{n-2} & t_{n-1} \end{pmatrix}.$$

Then, a function $f\colon D^n \to \mathbb{R}$ ($D \subset X$ is a non-empty, convex set) is T-Schur convex, if for all $x \in D^n$

$$f(Tx) \le f(x).$$

Borbála Fazekas: *Numerical solutions of the Lane–Emden-equation on the unit disc*

The Lane-Emden-equation is the following nonlinear second order partial differential equation

$$\text{(1)} \qquad -\Delta u = |u|^{p-1} \cdot u \quad \text{on } B,$$

$$u = 0 \quad \text{on } \partial B,$$

where $B \subset \mathbb{R}^2$ denotes the unit disc, $p > 1$. The solution u is in the Sobolev space $H_0^1(B)$. It can be proven analytically, that equation (1) has some radialsymmetric and also some radially nonsymmetric solutions. Our main task is to find these and also other solutions numerically.

György Gát: *Cesàro means with varying parameters of Walsh-Fourier series*

Let x be an element of the unit interval $I := [0, 1)$. The $\mathbb{N} \ni n$th Walsh function at $x \in I$ is

$$\omega_n(x) := (-1)^{\sum_{k=0}^{\infty} n_k x_k}, \quad n = \sum_{k=0}^{\infty} n_k 2^k, \quad x = \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}}.$$

Let $\hat{f}(k) := \int_0^1 f(x)\omega_k(x)dx$ be the kth Walsh-Fourier coefficient of the integrable function f and define the (C, α) (Cesàro) means of Walsh-Fourier series of f as

$$\sigma_n^\alpha f := \frac{1}{A_{n-1}^\alpha} \sum_{j=0}^{n-1} A_{n-j}^\alpha \hat{f}(j)\omega_j,$$

where $A_n^\alpha := \frac{(1+\alpha)\cdots(n+\alpha)}{n!}$ for parameter $\alpha \in \mathbb{R} \setminus \{-1, -2, \dots\}$. It is well-known, that for $\alpha = 1$ we have the Fejér means and the a.e. relation $\sigma_n^1 f \to f$. Meanwhile, for $\alpha = 0$, $\sigma_n^0 f$ is the nth partial sum of the Walsh-Fourier series of function f for what there exists a negative result, i.e. an integrable function f such as $\sigma_n^0 f \to f$ nowhere. In 2007 Akhobadze ([1]) introduced the notion of the Cesàro means of a Fourier series with varying parameters. That is, $\alpha = (\alpha_n)$ is a sequence. Taking into account the above it is an interesting situation to investigate the behavior of $\sigma_n^{\alpha_n} f$ when $\alpha_n \to 0$. In this talk, we show some recent almost everywhere convergence results of the kind above with respect to the Walsh-Fourier series.

Reference

[1] T. Akhobadze, *On the convergence of generalized Cesàro means of trigonometric Fourier series. I*, Acta Math. Hungar. **115** (2007), no. 1–2, 59–78.

Roman Ger: *A short proof of alienation of additivity from quadraticity*

Without a use of pexiderized versions of abstract polynomials theory we show that on 2-divisible groups the functional equation

$$f(x + y) + g(x + y) + g(x - y) = f(x) + f(y) + 2g(x) + 2g(y)$$

forces the unknown functions f and g to be additive and quadratic, respectively, modulo a constant.

Motivated by the observation that the equation

$$f(x+y)+f(x^2)=f(x)+f(y)+f(x)^2$$

implies both the additivity and multiplicativity of f we deal also with the alienation phenomenon of equations in a single and several variables.

Attila Gilányi: *Alienness of linear functional equations*

The concept of the alienness of functional equations was introduced by Jean Dhombres in his paper [2]. Investigations related to it were performed by several authors during the last years (cf., e.g., the recent survey [3]).

In this talk, we present a computer assisted approach to its consideration in connection with linear functional equations. Our studies are based on a computer program developed for the solution of linear functional equations of two variables described in [1] (cf. also [4], [5] and [6]).

References

[1] G.Gy. Borus and A. Gilányi, *Solving systems of linear functional equations with computer*, 4^{th} IEEE International Conference on Cognitive Infocommunications (CogInfoCom), IEEE, 2013, 559–562.

[2] J. Dhombres, *Relations de dépendance entre les équations fonctionnelles de Cauchy*, Aequationes Math. **35** (1988), 186–212.

[3] R. Ger and M. Sablik, *Alien functional equations: a selective survey of results*, in: J. Brzdęk et al. (Eds.), *Developments in Functional Equations and Related Topics*, Springer Optim. Appl. **124**, Springer, Cham, 2017, pp. 107–147.

[4] A. Gilányi, *Charakterisierung von monomialen Funktionen und Lösung von Funktionalgleichungen mit Computern*, Diss., Universität Karlsruhe, Karlsruhe, Germany, 1995.

[5] A. Gilányi, *Solving linear functional equations with computer*, Math. Pannon. **9** (1998), 57–70.

[6] L. Székelyhidi, *On a class of linear functional equations*, Publ. Math. Debrecen **29** (1982), 19–28.

Angshuman R. Goswami: *Decomposition of* Φ*-convex function*

A real valued function f defined on a real open interval I is called Φ-convex if, for all $x,y\in I$, $t\in[0,1]$ it satisfies

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y)+t\Phi\big((1-t)|x-y|\big)+(1-t)\Phi\big(t|x-y|\big),$$

where $\Phi:[0,\ell(I)[\to\mathbb{R}_+$ is a given non negative error function.

The main results of the paper offer various characterizations of Φ-convexity. One of the main result states that f is Φ-convex if and only if f can be decomposed into the sum of a convex function and a Φ-Hölder function.

Eszter Gselmann: *On a class of linear functional equations without the range condition* (Joint work with Gergely Kiss and Csaba Vincze)

The main purpose of this talk is to provide the general solution for a class of linear functional equations. Let $n \geq 2$ be an arbitrarily fixed integer, let further X and Y be linear spaces over the field $\mathbb{K}$ and let $\alpha_i, \beta_i \in \mathbb{K}$, $i = 1, \dots, n$ be arbitrarily fixed constants. We describe all those functions $f, f_{i,j} \colon X \times Y \to \mathbb{K}$, $i, j = 1, \dots, n$ that fulfill functional equation

$$f\left(\sum_{i=1}^{n} \alpha_i x_i, \sum_{i=1}^{n} \beta_i y_i\right) = \sum_{i,j=1}^{n} f_{i,j}(x_i, y_j), \quad x_i \in X, y_i \in Y, i = 1, \dots, n.$$

Additionally, necessary and sufficient conditions are also given which guarantee the solutions to be non-trivial.

This equation belongs to the class of *linear functional equations*, that was thoroughly investigated by L. Székelyhidi in [1, 2, 3]. At the same time, we *cannot* state that the functions involved are polynomials. This is because of the fact that the homomorphisms appearing in the argument of the unknown functions do not fulfill range condition appearing in the mentioned works of L. Székelyhidi.

References

[1] L. Székelyhidi, *On a class of linear functional equations*, Publ. Math. Debrecen **29** (1982), no. 1–2, 19–28.

[2] L. Székelyhidi, *On a linear functional equation*, Aequationes Math. **38** (1989), no. 2–3, 113–122.

[3] L. Székelyhidi, *Convolution Type Functional Equations on Topological Abelian Groups*, World Scientific Publishing Co., Teaneck, NJ, 1991.

Tibor Kiss: *Equality of Cauchy means and quasi-arithmetic means of two variables* (Joint work with Zsolt Páles)

The Cauchy Mean Value Theorem states that, having a nonempty open subinterval $I \subseteq \mathbb{R}$ and two differentiable functions $f, g : I \to \mathbb{R}$, for all $x, y \in I$ with $x \neq y$, there exists u in the interval determined by x and y such that

$$f'(u)(g(x) - g(y)) = g'(u)(f(x) - f(y))$$

holds.

It is easy to check that u has to be unique provided that $0 \notin g'(I)$ and f'/g' is invertible. In this latter case, the point u is called the Cauchy mean value of x and y, is denoted by $u = \mathscr{C}_{f,g}(x,y)$, and can be expressed as

$$\mathscr{C}_{f,g}(x,y) = \left(\frac{f'}{g'}\right)^{-1}\left(\frac{f(x)-f(y)}{g(x)-g(y)}\right).$$

The aim of the talk is to characterize those Cauchy means, for which there exists a continuous, strictly monotone function $\varphi : I \to \mathbb{R}$ such that

$$\mathscr{C}_{f,g}(x,y) = \varphi^{-1}\Big(\frac{\varphi(x)+\varphi(y)}{2}\Big), \quad x,y \in I,$$

or, shortly, which can be written as a quasi-arithmetic mean. Only conditions required in the definitions are used.

References

[1] J. Aczél, *A mean value property of the derivative of quadratic polynomials – without mean values and derivatives*, Math. Mag. **58** (1985), no. 1, 42–45.

[2] Z.M. Balogh, O.O. Ibrogimov and B.S. Mityagin, *Functional equations and the Cauchy mean value theorem*, Aequationes Math. **90** (2016), no. 4, 683–697.

[3] Sh. Haruki, *A property of quadratic polynomials*, Amer. Math. Monthly **86** (1979), no. 7, 577–579.

[4] T. Kiss and Zs. Páles, *On a functional equation related to two-variable weighted quasi-arithmetic means*, J. Difference Equ. Appl. **24** (2018), no. 1, 107–126.

[5] T. Kiss and Zs. Páles, *On a functional equation related to two-variable Cauchy means*, Math. Inequal. Appl. **22** (2019).

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[7] P.K. Sahoo and T. Riedel, *Mean Value Theorems and Functional Equations*, World Scientific Publishing Co., Inc., River Edge, NJ, 1998.

Radosław Łukasik: *A functional equation preserving the biadditivity* (Joint work with Paweł Wójcik)

Let S be a semigroup, H, X be groups, $A\colon S^2 \to X$, $B\colon H^2 \to X$ be biadditive functions. In this talk, we consider the functional equation

$$B(f(x), g(y)) = A(x,y), \quad x,y \in S,$$

for two unknown mappings $f, g\colon S \to H$.

This equation is a generalization of the equation

$$\langle f(x)|g(y)\rangle = \langle x|y\rangle, \quad x,y \in H,$$

where $f, g\colon H \to K$ are unknown functions, H, K are Hilbert spaces, which was solved in papers [1], [2].

References

[1] R. Łukasik, *A note on the orthogonality equation with two functions*, Aequationes Math. **90** (2016), no. 5, 961–965.
[2] R. Łukasik and P. Wójcik, *Decomposition of two functions in the orthogonality equation*, Aequationes Math. **90** (2016), no. 3, 495–499.

Gyula Maksa: *On the functional equation* $f(x+y)+g(xy) = h(x)+h(y)$

In this talk, after some historical remarks on the functional equation

$$f(x+y) + g(xy) = h(x) + h(y),$$

two applications of it are presented. One of them is in connection with functional equations involving means and their Gauss composition and the other one with the alienation of functional equations.

Janusz Morawiec: *Around a Kazimierz Nikodem result – part I* (Joint work with Thomas Zürcher)

Let $(X, \mathcal{A}, \mu)$ be a probability space and let $S\colon X \to X$ be a measurable transformation. Motivated by the paper of K. Nikodem ([1]), we concentrate on a functional equation generating measures that are absolutely continuous with respect to μ and ε-invariant under S. As a consequence of the investigation, we obtain a result on the existence and uniqueness of solutions $\varphi \in L^1([0,1])$ of the functional equation

$$\varphi(x) = \sum_{n=1}^{N} |f_n'(x)| \varphi(f_n(x)) + g(x),$$

where $g \in L^1([0,1])$ and $f_1, \ldots, f_N\colon [0,1] \to [0,1]$ are functions satisfying some extra conditions.

Reference

[1] K. Nikodem, *On ϵ-invariant measures and a functional equation*, Czechoslovak Math. J. **41** (1991), no. 4, 565–569.

Gergő Nagy: *Characterizations of centrality in C^*-algebras in terms of local monotonicity and local additivity of functions*

In this talk, we give characterizations of central elements in an arbitrary C^*-algebra $\mathcal{A}$ in terms of local properties of maps on $\mathcal{A}$ given by the function calculus. One of these properties is local monotonicity. We say that a real function f is locally monotone at the self-adjoint element $a \in \mathcal{A}$ if for any element $b \in \mathcal{A}$ satisfying $a \leq b$, one has $f(a) \leq f(b)$. We present a result stating that if f is defined on an open interval which is unbounded from above and it is strictly convex and increasing, then a is central if and only if f is locally monotone at a. That assertion significantly improves similar theorems by Ogasawara, Pedersen, Wu, Molnár and Virosztek. Another local property discussed in the talk is local additivity which is defined similarly as local monotonicity. A statement on local additivity analogous to the previous one is also presented as well as some applications of these results.

Andrzej Olbryś: *On a functional inequality connected with the concept of T-Schur convexity* (Joint work with Tomasz Szostok)

Following [2] where the concept of T-Schur convexity was introduced we examine functions $f : D \to \mathbb{R}$ which satisfy the functional inequality of the form

$$\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} t_{ij}x_j\right) \leq \sum_{i=1}^{n} f(x_i),$$

where

$$T = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ t_{21} & t_{22} & \dots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n1} & t_{n2} & \dots & t_{nn} \end{pmatrix}$$

is a doubly stochastic matrix and D stands for a convex subset of a real linear space.

References

[1] A.W. Marshall, I. Olkin and B.C. Arnold, *Inequalities: Theory of Majorization and Its Applications*, Second edition, Springer Series in Statistics, New York–Dordrecht–Heidelberg–London, 2011.
[2] A. Olbryś and T. Szostok, *On T-Schur convex maps*. Submitted.

Zsolt Páles: *On the homogenization of means* (Joint work with Paweł Pasteczka)

The aim of this talk is to introduce two notions of homogenization of means. In general, the homogenization is an operator which attaches a homogeneous mean to a given one.

To explain the main ideas, let I be an interval with $\inf I = 0$ and let $M\colon I^n \to I$ be an n-variable mean. We consider the following constructions:

$$M_{\$}(x) := \inf\left\{\tfrac{1}{t}M(tx) \mid tx \in I^n\right\} \quad \text{and} \quad M^{\$}(x) := \sup\left\{\tfrac{1}{t}M(tx) \mid tx \in I^n\right\},$$

$$M_{\#}(x) := \liminf_{t\to 0^+} \tfrac{1}{t}M(tx) \quad \text{and} \quad M^{\#}(x) := \limsup_{t\to 0^+} \tfrac{1}{t}M(tx).$$

One can see that $M_{\$}$ and $M^{\$}$ are the largest and smallest homogeneous means such that $M_{\$} \leq M \leq M^{\$}$ holds on I. Therefore, these means will be called the *lower and upper homogeneous envelopes of M*, respectively. We call $M_{\#}$ and $M^{\#}$ the *lower and upper (local) homogenization* of M, respectively. It is obvious that $M_{\$}$, $M^{\$}$, $M_{\#}$ and $M^{\#}$ are homogeneous means on $\mathbb{R}_+$, and we have the inequalities $M_{\$} \leq M_{\#} \leq M^{\#} \leq M^{\$}$. In the talk we investigate and compute these homogenizations in several important classes of means. Our results show that, under some regularity or convexity assumptions, the homogenization of quasiarithmetic means are power means, and homogenization of semideviation means are homogeneous semideviation means.

Maciej Sablik: *Further results on non-symmetric equations stemming from MVT*

We continue our research started in [2]. We consider the equation

$$\frac{f(x) - f(y)}{x - y} = g(\Phi(x,y))$$

where $\Phi\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a piecewise linear mapping, and both f and g are unknown. The problem has arised in connection with the paper [1].

References

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[2] M. Sablik, *An elementary method of solving functional equations*, Ann. Univ. Sci. Budapest. Sect. Comput. **48** (2018), 181–188.

Ekaterina Shulman: *Stability problems for set-valued mappings on groups*

Let G be a group and let $\mathcal{S}(X)$ be the structure of subspaces in a Banach space X. Let a map $F : G \to \mathcal{S}(X)$ be *subadditive*, i.e.,

$$F(gh) \subset F(g) + F(h) \quad \text{for every} \quad g, h \in G. \tag{1}$$

Assuming that dimensions of all subspaces $F(g)$, for $g \in G$, are finite and do not exceed n, we study conditions that imply that $\dim \sum\limits_{g\in G} F(g)$ is finite.

It was proved in [1] that if

(*) all $F(g)$ are invariant with respect to a bounded representation of finite multiplicity acting on X

then $\dim \sum_{g\in G} F(g) < Cn$ for some constant C.

In this talk we consider the situations when one of the conditions (1) or (*) is "satisfied up to a finite-dimensional subspace".

Reference

[1] E. Shulman, *Subadditive set-functions on semigroups, applications to group representations and functional equations*, J. Funct. Anal. **263** (2012), no. 5, 1468–1484.

László Székelyhidi: *Functions with finite dimensional difference spaces* (Joint work with Żywilla Fechner)

We consider complex valued functions on commutative groups with the property that all differences of a certain order belong to a given finite dimensional linear space. If all first order differences of a function belong to a given finite dimensional linear space, then, clearly, the function satisfies a Levi–Civita-type functional equation, hence it is an exponential polynomial. We generalize this observation for higher order differences.

Patrícia Szokol: *Preserving problems related to different means of positive operators* (Joint work with Gergő Nagy)

In this presentation, we mainly discuss the problem of describing the structure of transformations leaving norms of generalized weighted quasi-arithmetic means of invertible positive operators invariant. Under certain conditions, we present the solution of this problem which generalizes one of our former results containing its solution for weighted quasi-arithmetic means. Moreover, we investigate the relation between the generalized weighted quasi-arithmetic means and the Kubo-Ando means and show that the common members of their class and the set of the latter means are weighted arithmetic means.

Tomasz Szostok: *Some remarks on Steffensen's inequality*

We present some remarks concerning Steffensen's inequality using the method obtained in [1].

Reference

[1] T. Szostok, *Inequalities for convex functions via Stieltjes integral*, Lith. Math. J. **58** (2018), no. 1, 95–103.

Paweł Wójcik: *The Daugavet equation in some function spaces*

It is the pioneering result due to Daugavet that the norm identity $\|I + T\| = 1 + \|T\|$, which has become known as the Daugavet equation, holds for finite-rank operator $T\colon \mathcal{C}[0,1] \to \mathcal{C}[0,1]$. For finite-codimension subspaces of $\mathcal{C}([0,1];E)$ it has been proved by Kadets. The aim of this report is to discuss a new generalization of result of Kadets.

Amr Zakaria: *Equality of Bajraktarević means with Cauchy means* (Joint work with Zsolt Páles)

This talk offers a solution to the equality problem of two-variable Bajraktarević means (cf. [2], [3]) with two-variable Cauchy means (cf. [4]), i.e., we aim to solve the functional equation

$$\left(\frac{f}{g}\right)^{-1}\left(\frac{f(x)+f(y)}{g(x)+g(y)}\right) = \left(\frac{h'}{k'}\right)^{-1}\left(\frac{h(x)-h(y)}{k(x)-k(y)}\right), \quad x, y \in I,$$

where $f, g, h, k : I \to \mathbb{R}$ are unknown continuous functions such that h, k are differentiable, g, k' are nowhere zero on I, $\frac{f}{g}$ and $\frac{h'}{k'}$ are strictly monotone on I. For the necessity part of this result, we will assume that $f, g : I \to \mathbb{R}$ are eight and $h, k : I \to \mathbb{R}$ are nine times continuously differentiable. Our results extend and generalize those of Alzer and Ruscheweyh ([1]) who proved that the intersection of the classes of Gini means (that are homogeneous Bajraktarević means) with the class of Stolarsky means (that are homogeneous Cauchy means) is equal to the class of power means (that are homogeneous quasiarithmetic means).

References

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Thomas Zürcher: *Around a Kazimierz Nikodem result - part II* (Joint work with Janusz Morawiec)

In the first part, equations of the form

$$\varphi(x) = \sum_{n=1}^{N} |f_n'(x)|\varphi(f_n(x)) + g(x)$$

were considered. In this talk, we are changing the derivatives f_n' to some other functions g_n, looking for solutions $\varphi \in L^1([0,1])$ of

$$\varphi(x) = \sum_{n=1}^{N} |g_n(x)|\varphi(f_n(x)) + g(x).$$

This is not only a cosmetic change. We need new methods to tackle this kind of equations.

Marcin Zygmunt: *Remarks on the extensions of multiadditive functions on groups*

We present examples of multiadditive functions defined on a subgroup that cannot be extended to the whole group. We continue with a discussion on conditions for which such extensions are possible. This topic is related to [1, Theorem 2].

Reference

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2. Problems and remarks

1. Remark (*Open problem on the stability of monotonicity*) First I recall a notion and a result which was introduced and stated in my paper *On approximately convex functions* (Proc. Amer. Math. Soc. **131** (2003), no. 1, 243–252).

A *function* $p : I \to \mathbb{R}$ *is called* ε*-nondecreasing on the interval* I if

$$p(x) \le p(y) + \varepsilon$$

holds for all $x \le y$ in I.

The connection between ε-nondecreasing and nondecreasing functions is described in the next result. We also provide its simple proof.

THEOREM. *Let I be an open interval of $\mathbb{R}$, $p : I \to \mathbb{R}$, and ε be a non-negative number. Then p is ε-nondecreasing if and only if there exists a non-decreasing function $q : I \to \mathbb{R}$ such that $|p(x) - q(x)| \le \varepsilon/2$ holds for all $x \in I$.*

PROOF. Assume that q is nondecreasing such that $\|p - q\| \le \varepsilon/2$. Then for $x \le y$, we have

$$p(x) \le q(x) + |p(x) - q(x)| \le q(y) + \frac{\varepsilon}{2} \le p(y) + \frac{\varepsilon}{2} + |p(y) - q(y)| \le p(y) + \varepsilon.$$

Thus, p is ε-nondecreasing.

Conversely, assume that p is ε-nondecreasing and define

$$q(x) := \sup_{v \in I,\, v \le x} \Big(p(v) - \frac{\varepsilon}{2}\Big) \in I.$$

Then q is obviously nondecreasing. By its definition, we have that

$$p(x) - \frac{\varepsilon}{2} \le q(x).$$

On the other hand, using that p is ε-nondecreasing, $p(v) \le p(x) + \varepsilon$ for all $v \le x$, whence

$$q(x) = \sup_{v \in I,\, v \le x} \Big(p(v) - \frac{\varepsilon}{2}\Big) \le p(x) + \frac{\varepsilon}{2}.$$

The two inequalities obtained yield that $|p(x) - q(x)| \le \varepsilon/2$ for all $x \in I$. $\square$

The notion of ε-nondecreasingness can be extended to vector-variable and vector-valued functions in the following way: Let $D \subseteq \mathbb{R}^n$ be a convex set. A function $p \colon D \to \mathbb{R}^n$ is said to be ε-nondecreasing if, for all $x, y \in D$,

$$\langle p(x) - p(y), x - y \rangle \ge -\varepsilon \|x - y\|.$$

(Here the norm $\|\cdot\|$ is derived from the inner product $\langle\cdot,\cdot\rangle$.) In the case $\varepsilon = 0$, we simply say that p is nondecreasing. It is very well-known that the gradient of a differentiable convex function is always nondecreasing in the above sense. It is also easy to see that if $q\colon D \to \mathbb{R}^n$ is nondecreasing and $\|p - q\|_\infty \le \varepsilon/2$, then p is ε-nondecreasing.

We have the following

Open Problem. *Let $D \subseteq \mathbb{R}^n$ be an open convex set. Does there exist a constant $c_n \ge 0$ such that, for all ε-nondecreasing function $p\colon D \to \mathbb{R}^n$, there exists a nondecreasing function $q : D \to \mathbb{R}^n$ such that $\|p(x) - q(x)\| \le c_n\varepsilon$ for all $x \in D$.*

Zsolt Páles

2. Remark (*Remark to the talk of Professor Ger*) In the talk of Professor Roman Ger the alienation of the Cauchy functional equation and the quadratic functional equation was considered. Thus the following equation was studied.

$$f(x+y) + g(x+y) + g(x-y) = f(x) + f(y) + 2g(x) + 2g(y). \tag{1}$$

Putting

$$f_1 := f + g,\ f_2 := g,\ f_3 := f + 2g$$

we may write (1) in the form

$$f_1(x+y) + f_2(x-y) = f_3(x) + f_3(y).$$

Now, using the well known result of L. Székelyhidi, it is easy to show that $g = f_2$ is a polynomial function of order at most 2 and $f = f_1 - f_2$ is also a polynomial function of order at most 2. Therefore we have

$$f = F_2 + F_1 + c,$$

$$g = G_2 + G_1 + d,$$

where the functions F_1 and G_1 are additive and G_2, F_2 are diagonalizations of some 2−additive and symmetric functions. It is also well known that in such case (F_1, G_1) and (F_2, G_2) separately satisfy (1). Thus, using the fact that the functions F_1 and G_1 satisfy (1) and taking $x = y$ in (1), we get

$$F_1(2x) + G_1(2x) = 2F_1(x) + 4G_1(x),$$

i.e., $G_1 = 0$. Similarly,

$$F_2(2x) + G_2(2x) = 2F_2(x) + 4G_2(x)$$

yields $F_2 = 0$. Finally it is visible that $c = -2d$. This means that f must be additive and g must be quadratic (up to some constants) and the alienation of our equations is proved. This approach is weaker than Ger's result, since stronger assumptions on the domain and codomain of f and g are used here. Such assumptions are needed for the Székelyhidi's results to hold. However it should be noted that this approach is quite universal and may be used to study the alienation problem for a large class of linear functional equations.

Tomasz Szostok

3. Remark (*Remark on convolution-type functional equations*) Let G be a locally compact topological group, $\mathcal{C}(G)$ a locally convex topological vector space, and $\mathcal{C}(G)^* = \mathcal{M}_c(G)$ be its dual - a space of compactly supported Borel measures.

Every f in $\mathcal{C}(G)$ has a natural extension to $\mathcal{M}_c(G)$:

$$f(\mu) = \int_G f \, d\mu \qquad \text{e.g. } f(\delta_x) = f(x).$$

Then

$$f(\mu + \nu) = \int_G f \, d(\mu + \nu) = \int_G f \, d\mu + \int_G f \, d\nu = f(\mu) + f(\nu),$$
$$f(\lambda\mu) = \int_G f \, d(\lambda\mu) = \lambda \int_G f \, d\mu = \lambda f(\mu).$$

Hence $f : \mathcal{M}_c(G) \to \mathbb{C}$ is a linear homomorphism. It is continuous with respect to the weak*-topology: if (μ_i) is a net, weak*-convergent to μ, then

$$f(\mu) = \int_G f \, d\mu = \lim_i \int_G f \, d\mu_i = \lim_i f(\mu_i).$$

Hence every function f defines a linear functional on $\mathcal{M}_c(G)$.

Theorem. *Every linear functional $\Lambda : \mathcal{M}_c(G) \to \mathbb{C}$ is of the form*

$$\Lambda(\mu) = f(\mu)$$

with some f in $\mathcal{C}(G)$. In particular, f is uniquely determined by Λ.

We notice that $\mathcal{M}_c(G)$ is an algebra:

$$f(\mu * \nu) = \int_G f\, d(\mu * \nu) = \int_G \int_G f(xy)\, d\mu(x)\, d\mu(y).$$

E.g.

$$f(\delta_x * \delta_y) = f(xy).$$

Theorem. *The linear functional f is an algebra homomorphism if and only if f is an exponential.*

We observe that: f is additive if and only if $f(\mu * \nu) = f(\mu) + f(\nu)$.
f is a g-sine function if and only if $f(\mu * \nu) = f(\mu)g(\nu) + f(\nu)g(\mu)$.
f is quadratic if and only if $f(\mu * \nu) + f(\mu * \check{\nu}) = 2f(\mu) + 2f(\nu)$.
In general: let $\mathcal{F}$ be a nonempty subset in $\mathcal{M}_c(G)$ and we consider the system of convolution type functional equations

$$\mu * f(x) = 0 \text{ for each } \mu \in \mathcal{F}, x \in G.$$

Then f is a solution if and only if

$$\mu * f(\nu) = 0 \text{ for each } \mu \in \mathcal{F}, \nu \in \mathcal{M}_c(G).$$

László Székelyhidi

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