Annales Mathematicae Silesianae **36** (2022), no. 2, 184–192
DOI: 10.2478/amsil-2022-0010

CLASSIFICATION OF ODOMETERS: A SHORT ELEMENTARY PROOF

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Abstract. The paper deals with odometers (i.e. adding machines) of general type. We give a characterization of self-conjugacies of odometers which enables us to present an elementary proof of a classification of odometers given by Buescu and Stewart in [2]. The paper might also serve as a very quick introduction to odometers.

1. Introduction

Minimal systems represent an important class of dynamical systems since they are topologically irreducible and in this sense they can be considered as fundamental building blocks in topological dynamics. Odometers constitute an important family of minimal systems with an especially regular structure. Topologically they are all the Cantor set. From the point of view of algebra and geometry, they are compact commutative topological groups with a nice geometric representation. Besides being interesting on their own, they appear frequently and naturally as minimal sets and attractors in various branches of dynamical systems theory including interval and one-dimensional dynamics, Hamiltonian dynamics, and others.

Received: 18.01.2022. Accepted: 12.05.2022. Published online: 28.05.2022.
(2020) Mathematics Subject Classification: 37B05, 37B10.
Key words and phrases: odometer, conjugacy, self-conjugacy, classification.
This work was supported by VEGA grant 1/0158/20.

The classification of odometers up to topological conjugacy was first given by Buescu and Stewart in 1995 (see [2]) and later reproved in a different way by Block and Keesling in 2004 (see [1]). Both papers employ advanced results and notions from topology, dynamical systems and ergodic theory. Among others, the former is based on results on topologically discrete spectrum of a dynamical system while the latter rests upon properties of regularly recurrent points. In Section 4, we provide a short elementary proof of the classification of odometers stated in Theorem 4.3. The proof purposefully employs only basic tools from real analysis and number theory. In the proof we also give details which might be useful in implementing computations with odometers.

Our approach is enabled thanks to a characterization of self-conjugacies of odometers as rotations which we give in the second section in Theorem 3.4. One can obtain this result as a consequence of Halmos-von Neumann theorem on minimal isometries. We provide a completely elementary proof of this fact in Section 3. This result is used in [6] in the proof of a classification of Floyd--Auslander systems.

For purposes of this paper, by a *dynamical system* (or just shortly a *system*) we mean a pair (X, f) where X is a compact metric space and $f\colon X \to X$ is a continuous map. In fact, we will be concerned almost exclusively with the case when X is a compact metric space with a special structure and f is a minimal homeomorphism. We say that a system (X, f), or a map f, is *(topologically) minimal* if there is no nonempty proper closed subset M of X with $f(M) \subseteq M$ or, equivalently, if every point in X has dense orbit (since X is compact, it does not matter whether we consider forward or full orbits — for a survey on minimality see [5]). Minimality can be seen as a topological irreducibility of the system.

We say that two systems (X, f) and (Y, g) are *(topologically) conjugate* if there is a homeomorphism $h\colon X \to Y$ such that $h \circ f = g \circ h$, the homeomorphism h is then called a *(topological) conjugacy*, moreover, if $(X, f) = (Y, g)$, it is called a *(topological) self-conjugacy*. Two conjugate systems are indistinguishable from the point of view of topological dynamics. On the other hand, to know all the self-conjugacies of a system yields an important information about the system.

2. Odometers

In this section we define the object of our interest, odometers, which are commonly also known as adding machines or groups of $\mathbf{s}$-adic numbers. For a standard reference we refer the reader to [4, Chap. 2, Sec. 10], for a recent survey from a dynamical point of view, see [3].

Let $\mathbf{s} = (s_0, s_1, \ldots)$ be a sequence of integers where each $s_i \geq 2$. We denote by $\Delta_\mathbf{s}$ all the sequences $(x_0, x_1, \ldots)$, where each $x_i \in \{0, 1, \ldots, s_i - 1\}$. We define a metric d on $\Delta_\mathbf{s}$ for $(x_0, x_1, \ldots) \neq (y_0, y_1, \ldots)$ as follows

$$d((x_0, x_1, \ldots), (y_0, y_1, \ldots)) := \frac{1}{j+1}, \qquad \text{where } j := \min\{i x_i \neq y_i\}$$

and $d((x_0, x_1, \ldots), (x_0, x_1, \ldots)) := 0$. (This is one of the three most commonly used mutually equivalent metrics on $\Delta_\mathbf{s}$.) The metric space $\Delta_\mathbf{s}$ is obviously the topological product of a sequence of the discrete spaces with cardinalities $s_0, s_1, \ldots$.

Next we define an addition on $\Delta_\mathbf{s}$ as an addition with carry over as follows

$$(x_0, x_1, \ldots) + (y_0, y_1, \ldots) := (z_0, z_1, \ldots),$$

where $z_0 := x_0 + y_0 \mod s_0$, $z_k := x_k + y_k + r_{k-1} \mod s_k$, for $k = 1, 2, \ldots$ with $r_0 := 0$ if $x_0 + y_0 < s_0$, $r_0 := 1$ if $x_0 + y_0 \geq s_0$, and $r_k := 0$ if $x_k + y_k + r_{k-1} < s_k$ and $r_k := 1$ if $x_k + y_k + r_{k-1} \geq s_k$, for $k = 1, 2, \ldots$. (We use mod to denote the remainder.)

We define $\alpha_\mathbf{s} \colon \Delta_\mathbf{s} \to \Delta_\mathbf{s}$ by

$$\alpha_\mathbf{s}(x_0, x_1, x_2, \ldots) := (x_0, x_1, x_2, \ldots) + (1, 0, 0, \ldots).$$

We call the system $(\Delta_\mathbf{s}, \alpha_\mathbf{s})$ an *odometer*. When we need to distinguish elements of different odometers, we add a subscript, i.e. we write $(1, 0, 0, \ldots)_\mathbf{s}$, and similarly.

The structure $(\Delta_\mathbf{s}, +)$ forms an abelian topological group with the neutral element $\overline{0} := (0, 0, \ldots)$. For convenience we extend this notation and write $1\overline{0}$ for $(1, 0, 0, \ldots)$, and so on. The additive group of integers is naturally embedded into $(\Delta_\mathbf{s}, +)$ as the orbit of $1\overline{0}$ under $\alpha_\mathbf{s}$, i.e. a number $n \in \mathbb{Z}$ corresponds to the sequence $n \times 1\overline{0} \in \Delta_\mathbf{s}$. For nonnegative integers we obtain $a_0 a_1 \ldots a_k \overline{0}_\mathbf{s} = n \times 1\overline{0}_\mathbf{s}$ for $n = a_0 + a_1 s_0 + \cdots + a_k s_0 \ldots s_{k-1}$; we denote the element of $\Delta_\mathbf{s}$ corresponding to the number n by $\mathbf{s}(n)$.

It is also well-known that $(\Delta_\mathbf{s}, \alpha_\mathbf{s})$ is a minimal dynamical system (see e.g. [3]) — this fact is crucial for our purposes.

Using the construction above, we obtain a great number of various minimal systems for different sequences $\mathbf{s}$. The question is whether they are truly distinct from the point of view of dynamics, i.e. which of these are topologically conjugate and which are not. As we already mentioned in the beginning of the paper, the complete answer to this question was given in [2], cf. also [1]. Theorem 4.3 restates the result.

3. Self-conjugacies of odometers

Let us consider maps given by an addition of any element of $\Delta_{\mathbf{s}}$, also called group rotations. Mostly to simplify formulations, we introduce the following notation: for arbitrary fixed $z = (z_0, z_1, \dots) \in \Delta_{\mathbf{s}}$, we define a map $+_z \colon \Delta_{\mathbf{s}} \to \Delta_{\mathbf{s}}$, the addition by z, as follows

$$+_z(x_0, x_1, \dots) = (x_0, x_1, \dots) + (z_0, z_1, \dots).$$

Obviously, $\alpha_{\mathbf{s}} = +_{1\bar{0}}$.

It is possible to obtain any of the maps $+_z \colon \Delta_{\mathbf{s}} \to \Delta_{\mathbf{s}}$ in the following way

$$+_z = \lim_{k\to\infty} \alpha_{\mathbf{s}}^{n_k} \tag{1}$$

for some monotone increasing sequence of nonnegative integers $(n_k)_{k=0}^{\infty}$. We can simply take the sequence $(n_k)_{n=0}^{\infty}$ such that $\alpha_{\mathbf{s}}^{n_0} = +_{(z_0,0,0,\dots)}$ thus $n_0 = z_0$, $\alpha_{\mathbf{s}}^{n_1} = +_{(z_0,z_1,0,0,\dots)}$ thus $n_1 = z_1 s_0 + z_0$, $\alpha_{\mathbf{s}}^{n_2} = +_{(z_0,z_1,z_2,0,\dots)}$ thus $n_2 = z_2 s_1 s_0 + z_1 s_0 + z_0$, etc.

Lemma 3.1. *Let $z \in \Delta_{\mathbf{s}}$. Then the map $+_z$ is an isometry.*

Proof. Let $x = (x_0, x_1, \dots), y = (y_0, y_1, \dots) \in \Delta_{\mathbf{s}}$ and let $d(x, y) = \frac{1}{j+1}$, i.e. $x_i = y_i$ for $i < j$ and $x_j \neq y_j$. We have

$$+_z(x)_0 = x_0 + z_0 \mod s_0 = y_0 + z_0 \mod s_0 = +_z(y)_0,$$

$$+_z(x)_i = x_i + z_i + r_{i-1} \mod s_i = y_i + z_i + r_{i-1} \mod s_i = +_z(y)_i,$$

$$+_z(x)_j = x_j + z_j + r_{j-1} \mod s_j \neq y_j + z_j + r_{j-1} \mod s_j = +_z(y)_j.$$

Hence $d(+_z(x), +_z(y)) = \frac{1}{j+1} = d(x, y)$. □

Proposition 3.2. *Let X be a dynamical system such that f is a minimal isometry. Then every self-conjugacy of (X, f) is an isometry.*

Proof. Let h be a self-conjugacy of (X, f) and fix arbitrary $x, y \in X$. By minimality of f, there is an increasing sequence of nonnegative integers $(n_k)_{k=0}^{\infty}$ such that $\lim_{k\to\infty} d(f^{n_k}(x), y) = 0$ hence $\lim_{k\to\infty} h(f^{n_k}(x)) = h(y)$.

Next, again by minimality of f, there is an increasing sequence of nonnegative integers $(m_l)_{l=0}^{\infty}$ such that $\lim_{l\to\infty} f^{m_l}(x) = h(x)$. We have

$$\begin{aligned} d(h(x), h(y)) &= \lim_{k\to\infty} d(h(x), h(f^{n_k}(x))) = \lim_{k\to\infty} d(h(x), f^{n_k}(h(x))) \\ &= \lim_{k\to\infty} \lim_{l\to\infty} d(f^{m_l}(x), f^{n_k}(f^{m_l}(x))) \\ &= \lim_{k\to\infty} \lim_{l\to\infty} d(f^{m_l}(x), f^{m_l}(f^{n_k}(x))) \\ &= \lim_{k\to\infty} d(x, f^{n_k}(x)) = d(x, y). \end{aligned}$$

□

LEMMA 3.3. *Let h be a self-conjugacy of $(\Delta_{\mathbf{s}}, \alpha_{\mathbf{s}})$. Then for all $x, y \in \Delta_{\mathbf{s}}$ we have $h(x + y) = h(x) + y = x + h(y)$.*

PROOF. Let us fix an arbitrary $y \in \Delta_{\mathbf{s}}$. Using (1), we can write $+_y = \lim_{k\to\infty} \alpha_{\mathbf{s}}^{n_k}$ for some sequence $(n_k)_{k=0}^{\infty}$. Then

$$\begin{aligned} h(x+y) = h(+_y(x)) &= h(\lim_{k\to\infty} \alpha_{\mathbf{s}}^{n_k}(x)) = \lim_{k\to\infty} h(\alpha_{\mathbf{s}}^{n_k}(x)) \\ &= \lim_{k\to\infty} \alpha_{\mathbf{s}}^{n_k}(h(x)) = +_y(h(x)) = h(x) + y. \end{aligned}$$

We obtain the second equality trivially from commutativity. □

THEOREM 3.4. *Let h be a self-conjugacy of $(\Delta_{\mathbf{s}}, \alpha_{\mathbf{s}})$. Then there is a $z \in \Delta_{\mathbf{s}}$ such that $h = +_z$, i.e. h is the addition by z.*

PROOF. By Proposition 3.2, h is an isometry. Denote $z := h(\overline{0})$. We will show that for every $x \in \Delta_{\mathbf{s}}$ we get $h(x) = +_z(x) = x + z = x + h(\overline{0})$. By Lemma 3.3, for all $x, y \in \Delta_{\mathbf{s}}$ we have $h(x + y) = x + h(y)$. Hence, taking $y := \overline{0}$, we have

$$h(x) = h(x + \overline{0}) = x + h(\overline{0}) = x + z = +_z(x)$$

for all $x \in \Delta_{\mathbf{s}}$. □

4. Conjugacies of odometers

The last theorem above allows us to easily describe all the possible conjugacies between two odometers and thus provide a new, simple, elementary and short proof of the classification of odometers.

The following notion will allow us to formulate the classification in a concise, easily applicable way. Take any sequence $\mathbf{a} = (a_i)_{i=0}^{\infty}$ of positive integers and assign to it its *multiplicity function* $M(\mathbf{a})$ from the set of all primes to the set of all nonnegative integers with infinity included as follows:

$$M(\mathbf{a})(p) := \sup\{n \in \mathbb{N} p^n | a_0 a_1 \cdots a_k \text{ for some } k \in \mathbb{N}\}$$

in other words, $M(\mathbf{a})(p)$ is the sum of the powers of p in prime decompositions of the a_i's.

We will obtain the classification as a corollary of the following structural proposition which turns out to be an easy consequence of Theorem 3.4.

Proposition 4.1. *Any conjugacy h between two odometers $(\Delta_{\mathbf{s}}, \alpha_{\mathbf{s}})$ and $(\Delta_{\mathbf{t}}, \alpha_{\mathbf{t}})$ can be written in the form $h = h_0 + c$ where h_0 is a conjugacy sending $\overline{0}_{\mathbf{s}}$ to $\overline{0}_{\mathbf{t}}$ and $c = h(\overline{0}_{\mathbf{s}})$.*

Proof. Consider an arbitrary conjugacy h between $(\Delta_{\mathbf{s}}, \alpha_{\mathbf{s}})$ and $(\Delta_{\mathbf{t}}, \alpha_{\mathbf{t}})$. Thanks to Theorem 3.4, we can complete its commutative diagram to the one in Figure 1 which directly yields the proposition. □

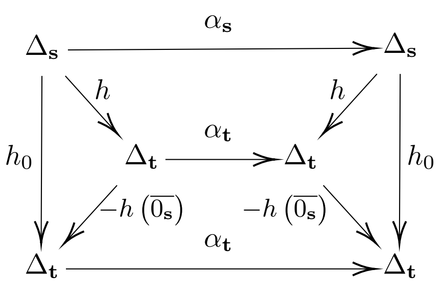

Figure 1.

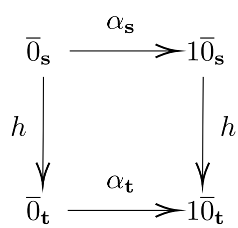

Figure 2.

To keep the proof of the theorem below more organized, we establish the following

Lemma 4.2. *Let X, Y be compact metric spaces, $A \subseteq X$ dense, $B \subseteq Y$ dense and $h\colon A \to B$ a bijection. Then the folowing two conditions are equivalent:*

(i) *h can be extended to a homeomorphism from X onto Y;*

(ii) *a sequence of points $a_i \in A$ converges in X iff the sequence of points $h(a_i)$ converges in Y.*

Proof. One direction is trivial. Let us prove now that (ii) $\Longrightarrow$ (i). Define the extension of h to X as follows: $H(x) := \lim h(a_i)$ for a chosen sequence of points $a_i \in A$ converging to x (to avoid a misunderstanding later, we denote the extension by H and not simply by h itself). This definition is independent on the choice of (a_i), since if we take another sequence (c_i), $c_i \in A$, converging to x then taking a mixed sequence $(a_0, c_0, a_1, c_1, \dots)$ shows that the limits of the images of the two sequences must be the same indeed. Thus H is well defined on the whole X due to density of A. Analogously, we define the extension H^{-1} of h^{-1} on the whole Y by $H^{-1}(y) := \lim h^{-1}(b_i)$ for a chosen sequence of points $b_i \in B$ converging to y. To see that H^{-1} is really the inverse of H it is sufficient to realize that $b_i = h(a_i)$ for some $a_i \in A$ and we get $H^{-1}(H(x)) = \lim h^{-1}(h(a_i)) = \lim a_i = x$ because $b_i \to H(x)$ implies $a_i \to x$. By the symmetric argument we obtain $H(H^{-1}(y)) = y$ and we have established that H is a bijection between X and Y.

Now we will show that H is also continuous. Take in X any converging sequence $x_i \to x$ and approach each x_i by a sequence of points from A, $a_{ij} \to x_i$. From each of these sequences we can choose a point a_{ij_i} such that $a_{ij_i} \to x$ and $h(a_{ij_i})$ is $\frac{1}{i}$-close to $H(x_i)$ thus showing that $H(x_i) \to H(x)$. Continuity of H^{-1} on Y is obtained by the symmetric argument (or we just use the fact that H is a continuous bijection from a compact to a Hausdorff space). □

Theorem 4.3 ([2, Theorem 7.6]). *Two odometers $(\Delta_{\mathbf{s}}, \alpha_{\mathbf{s}})$ and $(\Delta_{\mathbf{t}}, \alpha_{\mathbf{t}})$ are conjugate if and only if $M(\mathbf{s}) = M(\mathbf{t})$.*

Proof. According to Proposition 4.1, it is sufficient to characterize the conjugacies h sending $\overline{0}_{\mathbf{s}}$ to $\overline{0}_{\mathbf{t}}$. From the definition of conjugacy we directly obtain $h(1\overline{0}_{\mathbf{s}}) = 1\overline{0}_{\mathbf{t}}$, see Figure 2. Repeating this procedure, we get $h(n \times 1\overline{0}_{\mathbf{s}}) = n \times 1\overline{0}_{\mathbf{t}}$ for every $n \in \mathbb{N}$ (of course, for every $n \in \mathbb{Z}$ as well). Thus h is already defined on $\mathbb{N} \times 1\overline{0}_{\mathbf{s}}$, a dense subset of $\Delta_{\mathbf{s}}$. Moreover, the image $h(\mathbb{N} \times 1\overline{0}_{\mathbf{s}}) = \mathbb{N} \times 1\overline{0}_{\mathbf{t}}$ is a dense set in $\Delta_{\mathbf{t}}$. We will show that the map h extends continuously to the whole $\Delta_{\mathbf{s}}$ if and only if $M(\mathbf{s}) = M(\mathbf{t})$. To this end we use Lemma 4.2 which also automatically yields that in this case h is a homeomorphism.

First we will show that if $\alpha_{\mathbf{s}}$ is conjugate to $\alpha_{\mathbf{t}}$ then $M(\mathbf{s}) = M(\mathbf{t})$. Suppose that the last equality does not hold and fix a prime p such that $M(\mathbf{s})(p) \neq M(\mathbf{t})(p)$; we can assume without loss of generality that $M := M(\mathbf{s})(p) < M(\mathbf{t})(p)$. We are going to construct a sequence of positive integers converging in $\Delta_{\mathbf{s}}$ but diverging in $\Delta_{\mathbf{t}}$. In $\Delta_{\mathbf{s}}$, take the following sequence: $\mathbf{s}(u_0) := 1\overline{0}_{\mathbf{s}}$, $\mathbf{s}(u_1) := 11\overline{0}_{\mathbf{s}}$, $\mathbf{s}(u_2) := 111\overline{0}_{\mathbf{s}}, \dots$, which clearly converges to $\overline{1}_{\mathbf{s}}$; the sequence $(\mathbf{s}(u_i))_{i=0}^{\infty}$ represents the numbers $u_0 = 1$, $u_1 = 1 + s_0$, $u_2 = 1 + s_0 s_1, \dots$. Denote by k the last index such that $p | s_k$ (such an index always exists because our assumption says that $M = M(\mathbf{s})(p)$ is finite). Then for any $i > k$ we have $u_{i+1} - u_i = s_0 s_1 \dots s_i = p^M q$ with p, q coprime. Let l be the first index with $p^{M+1} | t_0 t_1 \dots t_l$. If the sequence $(\mathbf{t}(u_i))_{i=0}^{\infty}$ is convergent then it is eventually constant on the first l indices and necessarily $p^{M+1}|u_{i+1} - u_i$ for sufficiently large i which is not the case.

Now we will show the converse, i.e. $M(\mathbf{s}) = M(\mathbf{t})$ implies that $\alpha_{\mathbf{s}}$ is conjugate to $\alpha_{\mathbf{t}}$. Let there be a sequence of numbers $(u_i)_{i=0}^{\infty}$ such that $\mathbf{s}(u_i)$ converges in $\Delta_{\mathbf{s}}$ but $\mathbf{t}(u_i)$ diverges in $\Delta_{\mathbf{t}}$. Take the first index m on which $\mathbf{t}(u_i)$ is not eventually constant. Thus for sufficiently large i we have

$$u_i = v_0 + v_1 t_0 + \cdots + v_{m-1} t_0 \ldots t_{m-2} + v_m t_0 \ldots t_{m-1} + \cdots + v_n t_0 \ldots t_{n-1}$$

with $v_0, \ldots, v_{m-1}$ constant but v_m depending on i. It yields that there are arbitrarily large i such that

$$u_{i+1} = v_0 + v_1 t_0 + \cdots + v_{m-1} t_0 \ldots t_{m-2} + v'_m t_0 \ldots t_{m-1} + \cdots + v'_{n'} t_0 \ldots t_{n'-1}$$

with $v_m \neq v'_m$. Thus $u_{i+1} - u_i = t_0 \ldots t_{m-1}(v'_m - v_m + t_m r)$, with r a non-negative integer, and that is not divisible by $t_0 \ldots t_{m-1} t_m$. But since $\mathbf{s}(u_i)$ converges in $\Delta_{\mathbf{s}}$, $u_{i+1} - u_i$ will eventually become divisible by $s_0 \ldots s_j$ for arbitrarily large j and thus by $t_0 \ldots t_{m-1} t_m$ as well because $M(\mathbf{s}) = M(\mathbf{t})$. This contradiction finishes the proof. □

Remark 4.4. If we look at the proof above in more detail, it shows that it is exactly the index l on which eventual constancy of the sequence $(\mathbf{t}(u_i))$ is broken. This might be of practical importance in computations with sequences coded in different odometers.

Corollary 4.5. *For constant sequences* $\mathbf{s} = (s, s, \ldots)$ *and* $\mathbf{t} = (t, t, \ldots)$, *the odometers* $(\Delta_{\mathbf{s}}, \alpha_{\mathbf{s}})$ *and* $(\Delta_{\mathbf{t}}, \alpha_{\mathbf{t}})$ *are conjugate if and only if* s *and* t *have the same prime divisors.*

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