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INEQUALITIES OF HERMITE–HADAMARD TYPE FOR *GA*-CONVEX FUNCTIONS

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Abstract. Some inequalities of Hermite–Hadamard type for GA-convex functions defined on positive intervals are given.

1. Introduction

Let $I \subset (0, \infty)$ be an interval; a real-valued function $f \colon I \to \mathbb{R}$ is said to be GA-*convex* (*concave*) on I if

$$f\left(x^{1-\lambda} y^{\lambda}\right) \leq (\geq) (1-\lambda) f(x) + \lambda f(y) \tag{1.1}$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Since the condition (1.1) can be written as

$$f \circ \exp\left((1-\lambda)\ln x + \lambda \ln y\right) \leq (\geq) (1-\lambda) f \circ \exp(\ln x) + \lambda f \circ \exp(\ln y),$$

then we observe that $f \colon I \to \mathbb{R}$ is GA-convex (concave) on I if and only if $f \circ \exp$ is convex (concave) on $\ln I := \{\ln z, z \in I\}$. If $I = [a, b]$ then $\ln I = [\ln a, \ln b]$.

It is known that the function $f(x) = \ln(1+x)$ is GA-convex on $(0, \infty)$ (see [1]).

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For real and positive values of x, the *Euler gamma* function Γ and its *logarithmic derivative* ψ, the so-called *digamma function*, are defined by

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt \quad \text{and} \quad \psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}.$$

It has been shown in [17] that the function $f\colon (0,\infty) \to \mathbb{R}$ defined by

$$f(x) = \psi(x) + \frac{1}{2x}$$

is GA-concave on $(0,\infty)$ while the function $g\colon (0,\infty) \to \mathbb{R}$ defined by

$$g(x) = \psi(x) + \frac{1}{2x} + \frac{1}{12x^2}$$

is GA-convex on $(0,\infty)$.

If $[a,b] \subset (0,\infty)$ and the function $g\colon [\ln a, \ln b] \to \mathbb{R}$ is convex (concave) on $[\ln a, \ln b]$, then the function $f\colon [a,b] \to \mathbb{R}$, $f(t) = g(\ln t)$, is GA-convex (concave) on $[a,b]$.

Indeed, if $x, y \in [a,b]$ and $\lambda \in [0,1]$, then

$$\begin{aligned} f\left(x^{1-\lambda}y^\lambda\right) &= g\left(\ln\left(x^{1-\lambda}y^\lambda\right)\right) = g\left[(1-\lambda)\ln x + \lambda \ln y\right] \\ &\leq (\geq)\,(1-\lambda)\,g(\ln x) + \lambda g(\ln y) = (1-\lambda) f(x) + \lambda f(y), \end{aligned}$$

which shows that f is GA-convex (concave) on $[a,b]$.

We recall the classical Hermite–Hadamard inequality that states that

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}$$

for any convex function $f\colon [a,b] \to \mathbb{R}$.

For related results, see [2]–[5] and [7]–[15].

In [17] the authors obtained the following Hermite–Hadamard type inequality.

Theorem 1.1. *If $b > a > 0$ and $f\colon [a,b] \to \mathbb{R}$ is a differentiable GA-convex (concave) function on $[a,b]$, then*

$$\begin{aligned} f(I(a,b)) &\leq (\geq)\,\frac{1}{b-a}\int_a^b f(t)\,dt \\ &\leq (\geq)\,\frac{b-L(a,b)}{b-a} f(b) + \frac{L(a,b)-a}{b-a} f(a). \end{aligned} \tag{1.2}$$

The *identric mean* $I(a,b)$ is defined by

$$I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$

while the *logarithmic mean* is defined by

$$L(a,b) := \frac{b-a}{\ln b - \ln a}.$$

The differentiability of the function is not necessary in Theorem 1.1 for the first inequality from (1.2) to hold. A proof of this fact is proved below after some short preliminaries. The second inequality in (1.2) has been proved in [17] without differentiability assumption.

2. Preliminaries

We recall some facts on the lateral derivatives of a convex function.

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f\colon I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $f'_-(x) \le f'_+(x) \le f'_-(y) \le f'_+(y)$ which shows that both f'_- and f'_+ are nondecreasing functions on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f\colon I \to \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi\colon I \to [-\infty, \infty]$ such that $\varphi(\mathring{I}) \subset \mathbb{R}$ and

$$f(x) \ge f(a) + (x-a)\varphi(a) \text{ for any } x, a \in I.$$

It is also well known that if f is convex on I, then ∂f is nonempty, f'_-, $f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(x) \le \varphi(x) \le f'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on $\mathring{I}$, then $\partial f = \{f'\}$.

Now, since $f \circ \exp$ is convex on $[\ln a, \ln b]$, it follows that f has finite lateral derivatives on $(\ln a, \ln b)$ and by gradient inequality for convex functions we have

$$f \circ \exp(x) - f \circ \exp(y) \geq (x-y)\,\varphi(\exp y)\exp y, \tag{2.1}$$

where $\varphi(\exp y) \in \left[f'_-(\exp y), f'_+(\exp y)\right]$ for any $x, y \in (\ln a, \ln b)$.

If $s, t \in (a,b)$ and we take in (2.1) $x = \ln t, y = \ln s$, then we get

$$f(t) - f(s) \geq (\ln t - \ln s)\,\varphi(s)\,s, \tag{2.2}$$

where $\varphi(s) \in \left[f'_-(s), f'_+(s)\right]$.

Now, if we take the integral mean on $[a,b]$ in the inequality (2.2), we get

$$\frac{1}{b-a}\int_a^b f(t)\,dt - f(s) \geq \left(\frac{1}{b-a}\int_a^b \ln t\,dt - \ln s\right)\varphi(s)\,s$$

and since

$$\frac{1}{b-a}\int_a^b \ln t\,dt = \ln I(a,b),$$

then we get

$$\frac{1}{b-a}\int_a^b f(t)\,dt \geq f(s) + (\ln I(a,b) - \ln s)\,\varphi(s)\,s \tag{2.3}$$

for any $s \in (a,b)$ and $\varphi(s) \in \left[f'_-(s), f'_+(s)\right]$. This is an inequality of interest in itself.

Now, if we take in (2.3) $s = I(a,b) \in (a,b)$ then we get the first inequality in (1.2) for GA-convex functions.

If f is differentiable and GA-convex on (a,b), then we have from (2.3) the inequality

$$\frac{1}{b-a}\int_a^b f(t)\,dt \geq f(s) + (\ln I(a,b) - \ln s)\,f'(s)\,s \tag{2.4}$$

for any $s \in (a,b)$.

If we take in (2.4) $s = \frac{a+b}{2} = A(a,b)$, then we get

$$\frac{1}{b-a}\int_a^b f(t)\,dt \geq f(A(a,b)) - f'(A(a,b))\,A(a,b)\ln\left(\frac{A(a,b)}{I(a,b)}\right).$$

If we assume that $f'(A(a,b)) \leq 0$, then, since $I(a,b) \leq A(a,b)$, we get

$$\frac{1}{b-a}\int_a^b f(t)\,dt \geq f(A(a,b))$$

provided that f is differentiable and GA-convex on (a,b).

Also, if we take in (2.4) $s = L(a,b)$, then we get

$$\frac{1}{b-a}\int_a^b f(t)\,dt \geq f(L(a,b)) + f'(L(a,b))\,L(a,b)\ln\left(\frac{I(a,b)}{L(a,b)}\right). \tag{2.5}$$

If we assume that $f'(L(a,b)) \geq 0$, then we get from (2.5) that

$$\frac{1}{b-a}\int_a^b f(t)\,dt \geq f(L(a,b))$$

provided that f is differentiable and GA-convex on (a,b).

Now, if we take in (2.4) $s = \sqrt{ab} = G(a,b)$, then we get

$$\frac{1}{b-a}\int_a^b f(t)\,dt \geq f(G(a,b)) + f'(G(a,b))\,G(a,b)\ln\left(\frac{I(a,b)}{G(a,b)}\right). \tag{2.6}$$

Since

$$\begin{aligned}\ln\left(\frac{I(a,b)}{G(a,b)}\right) &= \ln I(a,b) - \ln G(a,b)\\ &= \frac{b\ln b - a\ln a}{b-a} - 1 - \frac{\ln a + \ln b}{2}\\ &= \frac{a+b}{2}\frac{\ln b - \ln a}{b-a} - 1 = \frac{A(a,b) - L(a,b)}{L(a,b)},\end{aligned}$$

then (2.6) is equivalent to

$$\frac{1}{b-a}\int_a^b f(t)\,dt \geq f(G(a,b)) + f'(G(a,b))\,G(a,b)\,\frac{A(a,b)-L(a,b)}{L(a,b)}.$$

If $f'(G(a,b)) \geq 0$, then we have

$$\frac{1}{b-a}\int_a^b f(t)\,dt \geq f(G(a,b))$$

provided that f is differentiable and GA-convex on (a,b).

Motivated by the above results we establish in this paper other inequalities of Hermite–Hadamard type for GA-convex functions. Applications for special means are also provided.

3. New results

We start with the following result that provides in the right side of (1.2) a bound in terms of the identric mean.

THEOREM 3.1. *Let* $f\colon (0,\infty) \supset [a,b] \to \mathbb{R}$ *be a GA-convex (concave) function on* $[a,b]$*. Then we have*

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le (\ge) \frac{(\ln b - \ln I(a,b))\, f(a) + (\ln I(a,b) - \ln a)\, f(b)}{\ln b - \ln a} = \frac{b-L(a,b)}{b-a} f(b) + \frac{L(a,b)-a}{b-a} f(a). \tag{3.1}$$

PROOF. Since f is a GA-convex (concave) function on $[a,b]$ then $f\circ\exp$ is convex (concave) and we have

$$\begin{aligned} f(t) &= f\circ\exp(\ln t) = f\circ\exp\left(\frac{(\ln b - \ln t)\ln a + (\ln t - \ln a)\ln b}{\ln b - \ln a}\right) \\ &\le (\ge) \frac{(\ln b - \ln t)\, f\circ\exp(\ln a) + (\ln t - \ln a)\, f\circ\exp(\ln b)}{\ln b - \ln a} \\ &= \frac{(\ln b - \ln t)\, f(a) + (\ln t - \ln a)\, f(b)}{\ln b - \ln a} \end{aligned} \tag{3.2}$$

for any $t \in [a,b]$.

This inequality is of interest in itself as well.

If we take the integral mean in (3.2), we get

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)\,dt \\ &\quad \le (\ge) \frac{\left(\ln b - \frac{1}{b-a}\int_a^b \ln t\,dt\right) f(a) + \left(\frac{1}{b-a}\int_a^b \ln t\,dt - \ln a\right) f(b)}{\ln b - \ln a} \end{aligned}$$

and since

$$\frac{1}{b-a}\int_a^b \ln t\,dt = \ln I\left(a,b\right),$$

then we obtain the desired result (3.1).

Now, we observe that

$$\begin{aligned}\frac{\ln b-\ln I\left(a,b\right)}{\ln b-\ln a} &= \frac{\ln b-\frac{b\ln b-a\ln a}{b-a}+1}{\ln b-\ln a}\\ &= \frac{\left(b-a\right)\ln b-b\ln b+a\ln a+b-a}{\left(b-a\right)\left(\ln b-\ln a\right)}\\ &= \frac{b-a-a\left(\ln b-\ln a\right)}{\left(b-a\right)\left(\ln b-\ln a\right)}\\ &= \frac{L\left(a,b\right)-a}{b-a}\end{aligned}$$

and, similarly

$$\frac{\ln I\left(a,b\right)-\ln a}{\ln b-\ln a} = \frac{b-L\left(a,b\right)}{b-a},$$

which proves the last part of (3.1). □

If $f\colon (0,\infty)\supset I\to\mathbb{R}$ is GA-convex (concave) on I, then we have the inequality

$$f\left(\sqrt{xy}\right)\leq(\geq)\,\frac{f\left(x\right)+f\left(y\right)}{2} \tag{3.3}$$

for any $x,y\in I$.

The following refinement of (3.3), which is an inequality of Hermite–Hadamard type, holds (see [16] for an extension to GA h-convex functions). For the sake of completeness we give here a short proof.

Lemma 3.2. *Let $f\colon (0,\infty)\supset[a,b]\to\mathbb{R}$ be a GA-convex (concave) function on $[a,b]$. Then we have*

$$f\left(\sqrt{ab}\right)\leq(\geq)\,\frac{1}{\ln b-\ln a}\int_a^b\frac{f\left(t\right)}{t}dt\leq(\geq)\,\frac{f\left(a\right)+f\left(b\right)}{2}. \tag{3.4}$$

Proof. By the definition of GA-convex (concave) functions on $[a,b]$ we have

$$f\left(a^{1-\lambda}b^{\lambda}\right)\leq(\geq)\,(1-\lambda)\,f\,(a)+\lambda f\,(b) \tag{3.5}$$

for any $\lambda\in[0,1]$.

Integrating the inequality (3.5) on $[0,1]$ we get

$$\int_0^1 f\left(a^{1-\lambda}b^{\lambda}\right)d\lambda\leq(\geq)\,f\,(a)\int_0^1(1-\lambda)\,d\lambda+f\,(b)\int_0^1\lambda\,d\lambda. \tag{3.6}$$

Since

$$\int_0^1(1-\lambda)\,d\lambda=\int_0^1\lambda\,d\lambda=\frac{1}{2}$$

and, by changing the variable $t=a^{1-\lambda}b^{\lambda}$, $\lambda\in[0,1]$, we have

$$\int_0^1 f\left(a^{1-\lambda}b^{\lambda}\right)d\lambda=\frac{1}{\ln b-\ln a}\int_a^b\frac{f\,(t)}{t}dt,$$

then by (3.6) we get the second inequality in (3.4).

By the inequality (3.3) we have

$$\begin{aligned} f\left(\sqrt{ab}\right)&=f\left(\sqrt{a^{1-\lambda}b^{\lambda}a^{\lambda}b^{1-\lambda}}\right)\\ &\leq(\geq)\,\frac{1}{2}\left[f\left(a^{1-\lambda}b^{\lambda}\right)+f\left(a^{\lambda}b^{1-\lambda}\right)\right] \end{aligned} \tag{3.7}$$

for any $\lambda\in[0,1]$.

Integrating the inequality (3.7) on $[0,1]$ we get

$$f\left(\sqrt{ab}\right)\leq(\geq)\,\frac{1}{2}\left[\int_0^1 f\left(a^{1-\lambda}b^{\lambda}\right)d\lambda+\int_0^1 f\left(a^{\lambda}b^{1-\lambda}\right)d\lambda\right]. \tag{3.8}$$

Since

$$\int_0^1 f\left(a^{\lambda}b^{1-\lambda}\right)d\lambda=\int_0^1 f\left(a^{1-\lambda}b^{\lambda}\right)d\lambda=\frac{1}{\ln b-\ln a}\int_a^b\frac{f\,(t)}{t}dt,$$

then by (3.8) we get the first inequality in (3.4). $\square$

Remark 3.3. The inequality (3.4) can be also written for any $d > c > 0$ with $c, d \in I$ as

$$f\left(\sqrt{cd}\right) \le (\ge) \int_0^1 f\left(c^{1-\lambda} d^{\lambda}\right) d\lambda \le (\ge) \frac{f(c) + f(d)}{2} \tag{3.9}$$

provided that f is a GA-convex (concave) function on I.

We have the following representation result:

Lemma 3.4. *Let $g\colon \mathbb{R} \supset [x, y] \to \mathbb{C}$ be a Lebesgue integrable function on $[x, y]$. Then for any $\lambda \in [0, 1]$ we have the representation*

$$\begin{aligned} \int_0^1 g[(1-t)x + ty]\, dt = (1-\lambda) \int_0^1 g\left[(1-t)\left((1-\lambda)x + \lambda y\right) + ty\right] dt \\ + \lambda \int_0^1 g\left[(1-t)x + t\left((1-\lambda)x + \lambda y\right)\right] dt. \end{aligned} \tag{3.10}$$

Proof. For $\lambda = 0$ and $\lambda = 1$ the equality (3.10) is obvious.
Let $\lambda \in (0, 1)$. Observe that

$$\begin{aligned} &\int_0^1 g\left[(1-t)\left(\lambda y + (1-\lambda)x\right) + ty\right] dt \\ &\qquad = \int_0^1 g\left[\left((1-t)\lambda + t\right) y + (1-t)(1-\lambda)x\right] dt \end{aligned}$$

and

$$\int_0^1 g\left[t\left(\lambda y + (1-\lambda)x\right) + (1-t)x\right] dt = \int_0^1 g\left[t\lambda y + (1-\lambda t)x\right] dt.$$

If we make the change of variable $u := (1-t)\lambda + t$, then we have $1 - u = (1-t)(1-\lambda)$ and $du = (1-\lambda)\, dt$. Then

$$\int_0^1 g\left[\left((1-t)\lambda + t\right) y + (1-t)(1-\lambda)x\right] dt = \frac{1}{1-\lambda} \int_\lambda^1 g\left[uy + (1-u)x\right] du.$$

If we make the change of variable $u := \lambda t$, then we have $du = \lambda dt$ and

$$\int_0^1 g\left[t\lambda y + (1-\lambda t)x\right] dt = \frac{1}{\lambda} \int_0^\lambda g\left[uy + (1-u)x\right] du.$$

Therefore

$$
\begin{aligned}
(1-\lambda)&\int_0^1 g\left[(1-t)\left(\lambda y+(1-\lambda)x\right)+ty\right]dt\\
&+\lambda\int_0^1 g\left[t\left(\lambda y+(1-\lambda)x\right)+(1-t)x\right]dt\\
&\quad=\int_\lambda^1 g\left[uy+(1-u)x\right]du+\int_0^\lambda g\left[uy+(1-u)x\right]du\\
&\quad=\int_0^1 g\left[uy+(1-u)x\right]du
\end{aligned}
$$

and the identity (3.10) is proved. □

Corollary 3.5. *Let $f\colon (0,\infty)\supset [a,b]\to\mathbb{C}$ be a Lebesgue integrable function on $[a,b]$. Then for any $\lambda\in[0,1]$ we have the representation*

$$
\begin{aligned}
\int_0^1 f\left(a^{1-s}b^s\right)ds&=(1-\lambda)\int_0^1 f\left(\left[a^{1-\lambda}b^\lambda\right]^{1-s}b^s\right)ds\\
&\qquad+\lambda\int_0^1 f\left(a^{1-s}\left[a^{1-\lambda}b^\lambda\right]^s\right)ds.
\end{aligned}
\tag{3.11}
$$

Proof. Using (3.10) we have

$$
\begin{aligned}
\int_0^1 f\left(a^{1-s}b^s\right)ds&=\int_0^1 f\circ\exp\left((1-s)\ln a+s\ln b\right)ds\\
&=(1-\lambda)\int_0^1 f\circ\exp\left[(1-s)\left((1-\lambda)\ln a+\lambda\ln b\right)+s\ln b\right]ds\\
&\quad+\lambda\int_0^1 f\circ\exp\left[(1-s)\ln a+s\left((1-\lambda)\ln a+\lambda\ln b\right)\right]ds\\
&=(1-\lambda)\int_0^1 f\circ\exp\left[(1-s)\ln\left[a^{1-\lambda}b^\lambda\right]+s\ln b\right]ds\\
&\quad+\lambda\int_0^1 f\circ\exp\left[(1-s)\ln a+s\ln\left[a^{1-\lambda}b^\lambda\right]\right]ds\\
&=(1-\lambda)\int_0^1 f\left(\left[a^{1-\lambda}b^\lambda\right]^{1-s}b^s\right)ds+\lambda\int_0^1 f\left(a^{1-s}\left[a^{1-\lambda}b^\lambda\right]^s\right)ds
\end{aligned}
$$

and the identity (3.11) is proved. □

We are able now to provide a refinement of (3.4) as follows:

THEOREM 3.6. *Let $f\colon (0,\infty) \supset [a,b] \to \mathbb{R}$ be a GA-convex (concave) function on $[a,b]$. Then for any $\lambda \in [0,1]$ we have*

$$
\begin{aligned}
f\left(\sqrt{ab}\right) &\le (\ge)\,(1-\lambda)\, f\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right) + \lambda f\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right) \\
&\le (\ge)\,\frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t}\,dt \\
&\le (\ge)\,\frac{1}{2}\left[f\left(a^{1-\lambda} b^{\lambda}\right) + (1-\lambda) f(b) + \lambda f(a)\right] \\
&\le (\ge)\,\frac{f(a)+f(b)}{2}.
\end{aligned}
\tag{3.12}
$$

PROOF. We prove the inequalities only for the GA-convex case.

Using the inequality (3.9) we have

$$
f\left(\sqrt{a^{1-\lambda} b^{\lambda} b}\right) \le \int_0^1 f\left(\left[a^{1-\lambda} b^{\lambda}\right]^{1-s} b^{s}\right) ds \le \frac{f\left(a^{1-\lambda} b^{\lambda}\right) + f(b)}{2},
$$

that is equivalent to

$$
f\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right) \le \int_0^1 f\left(\left[a^{1-\lambda} b^{\lambda}\right]^{1-s} b^{s}\right) ds \le \frac{f\left(a^{1-\lambda} b^{\lambda}\right) + f(b)}{2} \tag{3.13}
$$

for any $\lambda \in [0,1]$.

We also have

$$
f\left(\sqrt{a a^{1-\lambda} b^{\lambda}}\right) \le \int_0^1 f\left(a^{1-s}\left[a^{1-\lambda} b^{\lambda}\right]^{s}\right) ds \le \frac{f(a) + f\left(a^{1-\lambda} b^{\lambda}\right)}{2},
$$

that is equivalent to

$$
f\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right) \le \int_0^1 f\left(a^{1-s}\left[a^{1-\lambda} b^{\lambda}\right]^{s}\right) ds \le \frac{f(a) + f\left(a^{1-\lambda} b^{\lambda}\right)}{2} \tag{3.14}
$$

for any $\lambda \in [0,1]$.

If we multiply (3.13) by $1-\lambda$ and (3.14) by λ and add the obtained inequalities, we get, by the identity (3.11), that

$$
(1-\lambda)\, f\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right) + \lambda f\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right) \le \int_0^1 f\left(a^{1-s} b^{s}\right) ds
$$

$$\leq (1-\lambda)\frac{f\left(a^{1-\lambda}b^{\lambda}\right)+f(b)}{2}+\lambda\frac{f(a)+f\left(a^{1-\lambda}b^{\lambda}\right)}{2}$$
$$=\frac{1}{2}\left[f\left(a^{1-\lambda}b^{\lambda}\right)+(1-\lambda)f(b)+\lambda f(a)\right]$$

for any $\lambda\in[0,1]$, which proves the second and the third inequality in (3.12).

By the GA-convexity we have

$$(1-\lambda)f\left(a^{\frac{1-\lambda}{2}}b^{\frac{\lambda+1}{2}}\right)+\lambda f\left(a^{\frac{2-\lambda}{2}}b^{\frac{\lambda}{2}}\right)$$
$$\geq f\left[\left(a^{\frac{1-\lambda}{2}}b^{\frac{\lambda+1}{2}}\right)^{1-\lambda}\left(a^{\frac{2-\lambda}{2}}b^{\frac{\lambda}{2}}\right)^{\lambda}\right]=f\left(a^{\frac{1}{2}}b^{\frac{1}{2}}\right),$$

which proves the first inequality in (3.12).

By the GA-convexity we also have

$$\frac{1}{2}\left[f\left(a^{1-\lambda}b^{\lambda}\right)+(1-\lambda)f(b)+\lambda f(a)\right]$$
$$\leq\frac{1}{2}\left[(1-\lambda)f(a)+\lambda f(b)+(1-\lambda)f(b)+\lambda f(a)\right]$$
$$=\frac{f(a)+f(b)}{2},$$

which proves the last inequality in (3.12). □

COROLLARY 3.7. *With the assumptions of Theorem* 3.6 *we have*

$$f\left(\sqrt{ab}\right)\leq(\geq)\frac{1}{2}\left[f\left(a^{\frac{1}{4}}b^{\frac{3}{4}}\right)+f\left(a^{\frac{3}{4}}b^{\frac{1}{4}}\right)\right]$$
$$\leq(\geq)\frac{1}{\ln b-\ln a}\int_a^b\frac{f(t)}{t}dt$$
$$\leq(\geq)\frac{1}{2}\left[f\left(\sqrt{ab}\right)+\frac{f(b)+f(a)}{2}\right]$$
$$\leq(\geq)\frac{f(a)+f(b)}{2}.$$

4. Related results

The following result also holds:

THEOREM 4.1. *Let $f\colon (0,\infty) \supset [a,b] \to \mathbb{R}$ be a GA-convex (concave) function on $[a,b]$. Then for any $t \in [a,b]$ we have*

$$\begin{aligned}(4.1)\qquad \frac{1}{\ln b - \ln a}\int_a^b \frac{f(s)}{s}ds &\le (\ge)\, \frac{1}{2}\left[f(t) + \frac{f(b)(\ln b - \ln t) + f(a)(\ln t - \ln a)}{\ln b - \ln a}\right]\\ &\le (\ge)\, \frac{f(a)+f(b)}{2}.\end{aligned}$$

PROOF. We give a proof only for the GA-convex case.
From the inequality (2.2) we have that

$$f(t) - f(s) \ge (\ln t - \ln s)\, f'_+(s)\, s \tag{4.2}$$

for any $s \in (a,b)$ and $t \in [a,b]$.
We divide (4.2) by $s > 0$ and integrate on $[a,b]$ over s to get

$$f(t)\int_a^b \frac{1}{s}ds - \int_a^b \frac{f(s)}{s}ds \ge \left(\int_a^b f'_+(s)\,ds\right)\ln t - \int_a^b f'_+(s)\ln s\,ds \tag{4.3}$$

for any $t \in [a,b]$.
However,

$$\int_a^b \frac{1}{s}ds = \ln b - \ln a,\quad \int_a^b f'_+(s)\,ds = f(b) - f(a)$$

and

$$\begin{aligned}&\int_a^b f'_+(s)\ln s\,ds\\ &\qquad = f(s)\ln s\big|_a^b - \int_a^b \frac{f(s)}{s}ds = f(b)\ln b - f(a)\ln a - \int_a^b \frac{f(s)}{s}ds.\end{aligned}$$

Therefore, by (4.3) we get

$$f(t)(\ln b-\ln a)-\int_a^b \frac{f(s)}{s}ds \geq (f(b)-f(a))\ln t - f(b)\ln b + f(a)\ln a + \int_a^b \frac{f(s)}{s}ds,$$

which can be written as

$$f(t)(\ln b-\ln a)+f(b)(\ln b-\ln t)+f(a)(\ln t-\ln a) \geq 2\int_a^b \frac{f(s)}{s}ds$$

and the first inequality in (4.1) is proved.

Using (3.2) we have

$$\begin{aligned} f(t)+\frac{f(b)(\ln b-\ln t)+f(a)(\ln t-\ln a)}{\ln b-\ln a} &\leq \frac{(\ln b-\ln t)f(a)+(\ln t-\ln a)f(b)}{\ln b-\ln a} \\ &\quad + \frac{f(b)(\ln b-\ln t)+f(a)(\ln t-\ln a)}{\ln b-\ln a} = f(a)+f(b) \end{aligned}$$

for any $t \in [a,b]$. That proves the last part of (4.1). □

By taking the integral mean in the inequality (4.1) we have:

Corollary 4.2. *With the assumptions of Theorem* 4.1 *we have*

$$\tag{4.4} \begin{aligned} \frac{1}{\ln b-\ln a}\int_a^b \frac{f(s)}{s}ds &\leq (\geq)\, \frac{1}{2}\frac{1}{b-a}\int_a^b f(t)\,dt \\ &+\frac{1}{2}\frac{f(b)(\ln b-\ln I(a,b))+f(a)(\ln I(a,b)-\ln a)}{\ln b-\ln a} \leq (\geq)\, \frac{f(a)+f(b)}{2}. \end{aligned}$$

Since a simple calculation reveals (see the proof of Theorem 3.1) that

$$\frac{f(b)(\ln b-\ln I(a,b))+f(a)(\ln I(a,b)-\ln a)}{\ln b-\ln a} = \frac{L(a,b)-a}{b-a}f(b)+\frac{b-L(a,b)}{b-a}f(a),$$

then the inequality (4.4) is equivalent to

$$\frac{1}{\ln b-\ln a}\int_a^b \frac{f(s)}{s}ds \le (\ge) \frac{1}{2}\frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{2}\left[\frac{L(a,b)-a}{b-a}f(b)+\frac{b-L(a,b)}{b-a}f(a)\right] \le (\ge) \frac{f(a)+f(b)}{2}.$$

Remark 4.3. Taking specific values for $t \in [a,b]$ in (4.1) we get the following results:

$$\begin{aligned}\frac{1}{\ln b-\ln a}\int_a^b \frac{f(s)}{s}ds &\le (\ge) \frac{1}{2}f\left(\frac{a+b}{2}\right) \\ &\quad + \frac{1}{2}\left[\frac{f(b)\left(\ln b-\ln\frac{a+b}{2}\right)+f(a)\left(\ln\frac{a+b}{2}-\ln a\right)}{\ln b-\ln a}\right] \\ &\le (\ge) \frac{f(a)+f(b)}{2},\end{aligned}$$

$$\begin{aligned}\frac{1}{\ln b-\ln a}\int_a^b \frac{f(s)}{s}ds &\le (\ge) \frac{1}{2}\left[f\left(\sqrt{ab}\right)+\frac{f(a)+f(b)}{2}\right] \\ &\le (\ge) \frac{f(a)+f(b)}{2},\end{aligned}$$

$$\begin{aligned}\frac{1}{\ln b-\ln a}\int_a^b \frac{f(s)}{s}ds &\le (\ge) \frac{1}{2}f(I(a,b)) \\ &\quad + \frac{1}{2}\left[\frac{f(b)(\ln b-\ln I(a,b))+f(a)(\ln I(a,b)-\ln a)}{\ln b-\ln a}\right] \\ &= \frac{1}{2}\left[f(I(a,b))+\frac{L(a,b)-a}{b-a}f(b)+\frac{b-L(a,b)}{b-a}f(a)\right] \\ &\le (\ge) \frac{f(a)+f(b)}{2},\end{aligned}$$

and

$$\frac{1}{\ln b-\ln a}\int_a^b \frac{f(s)}{s}ds \le (\ge) \frac{1}{2}f(L(a,b)) + \frac{1}{2}\left[\frac{f(b)(\ln b-\ln L(a,b))+f(a)(\ln L(a,b)-\ln a)}{\ln b-\ln a}\right] \le (\ge) \frac{f(a)+f(b)}{2}.$$

Now, observe that

$$f(b)(\ln b - \ln t) + f(a)(\ln t - \ln a) = 0$$

iff

$$\ln t = \frac{f(b)\ln b - f(a)\ln a}{f(b) - f(a)} = \ln\left(\frac{b^{f(b)}}{a^{f(a)}}\right)^{\frac{1}{f(b)-f(a)}},$$

which is equivalent to

$$t = \left(\frac{b^{f(b)}}{a^{f(a)}}\right)^{\frac{1}{f(b)-f(a)}}.$$

Therefore, if

$$t = \left(\frac{b^{f(b)}}{a^{f(a)}}\right)^{\frac{1}{f(b)-f(a)}} \in [a,b],$$

then by (4.1) we get

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{f(s)}{s}ds \le (\ge) \frac{1}{2} f\left(\left(\frac{b^{f(b)}}{a^{f(a)}}\right)^{\frac{1}{f(b)-f(a)}}\right) \le (\ge) \frac{f(a)+f(b)}{2}.$$

The following result also holds.

THEOREM 4.4. *Let $f\colon (0,\infty) \supset [a,b] \to \mathbb{R}$ be a GA-convex (concave) function on $[a,b]$. Then for any $t \in [a,b]$ we have*

$$\frac{1}{2}\left[f(t) + \frac{f(b)b(\ln b - \ln t) + af(a)(\ln t - \ln a)}{b-a}\right] - \frac{1}{b-a}\int_a^b f(s)\,ds \tag{4.5}$$
$$\ge (\le) \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(s)\ln s\,ds - \left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)\ln t\right].$$

PROOF. We give a proof only for the GA-convex case.
Integrating (4.2) with respect to s we get

$$f(t)(b-a) - \int_a^b f(s)\,ds \ge \ln t\int_a^b f'_+(s)\,s\,ds - \int_a^b f'_+(s)\,s\ln s\,ds \tag{4.6}$$

for any $t \in [a,b]$.

Observe that, integrating by parts, we have

$$\int_a^b f'_+(s)\,s\,ds = bf(b) - af(a) - \int_a^b f(s)\,ds$$

and

$$\begin{aligned}\int_a^b f'_+(s)\,s\ln s\,ds &= f(b)\,b\ln b - f(a)\,a\ln a - \int_a^b (s\ln s)'\,f(s)\,ds\\ &= f(b)\,b\ln b - f(a)\,a\ln a - \int_a^b (\ln s + 1)\,f(s)\,ds\\ &= f(b)\,b\ln b - f(a)\,a\ln a - \int_a^b f(s)\ln s\,ds - \int_a^b f(s)\,ds.\end{aligned}$$

Using the inequality (4.6) we get

$$\begin{aligned}&f(t)(b-a) - \int_a^b f(s)\,ds\\ &\qquad\geq \ln t\left(bf(b) - af(a) - \int_a^b f(s)\,ds\right)\\ &\qquad\quad - f(b)\,b\ln b + f(a)\,a\ln a + \int_a^b f(s)\ln s\,ds + \int_a^b f(s)\,ds\\ &\qquad= bf(b)\ln t - af(a)\ln t - \ln t\int_a^b f(s)\,ds\\ &\qquad\quad - f(b)\,b\ln b + f(a)\,a\ln a + \int_a^b f(s)\ln s\,ds + \int_a^b f(s)\,ds,\end{aligned}$$

that is equivalent to

$$\begin{aligned}&f(t)(b-a) - bf(b)\ln t + af(a)\ln t + f(b)\,b\ln b - f(a)\,a\ln a\\ &\qquad\qquad - 2\int_a^b f(s)\,ds \geq \int_a^b f(s)\ln s\,ds - \ln t\int_a^b f(s)\,ds,\end{aligned}$$

i.e.,

$$\begin{aligned}&f(t)(b-a) + f(b)\,b(\ln b - \ln t) + af(a)(\ln t - \ln a)\\ &\qquad\qquad - 2\int_a^b f(s)\,ds \geq \int_a^b f(s)\ln s\,ds - \ln t\int_a^b f(s)\,ds\end{aligned}$$

for any $t \in [a,b]$ and the inequality (4.5) is proved. $\square$

Corollary 4.5. *Let $f\colon (0,\infty)\supset[a,b]\to\mathbb{R}$ be a GA-convex function on $[a,b]$. Then*

$$\frac{bf(b)(\ln b-\ln I(a,b))+af(a)(\ln I(a,b)-\ln a)}{b-a}-\frac{1}{b-a}\int_a^b f(s)\,ds \geq \frac{1}{b-a}\int_a^b f(s)\ln s\,ds-\left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)\ln I(a,b). \tag{4.7}$$

Moreover, if f is nondecreasing then

$$\frac{bf(b)(\ln b-\ln I(a,b))+af(a)(\ln I(a,b)-\ln a)}{b-a}-\frac{1}{b-a}\int_a^b f(s)\,ds \geq \frac{1}{b-a}\int_a^b f(s)\ln s\,ds-\left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)\ln I(a,b)\geq 0. \tag{4.8}$$

Proof. Integrating over t on $[a,b]$ and dividing by $b-a$ in (4.5) we get

$$\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(s)\,ds + \frac{f(b)\,b\left(\ln b-\frac{1}{b-a}\int_a^b\ln t\,dt\right)+af(a)\left(\frac{1}{b-a}\int_a^b\ln t\,dt-\ln a\right)}{b-a}\right] - \frac{1}{b-a}\int_a^b f(s)\,ds \geq (\leq)\,\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(s)\ln s\,ds - \left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)\frac{1}{b-a}\int_a^b \ln t\,dt\right],$$

that is equivalent to (4.7).

Now, if f is nondecreasing on $[a,b]$, then by Čebyšev inequality for synchronous functions, we have

$$\frac{1}{b-a}\int_a^b f(s)\ln s\,ds \geq \left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)\frac{1}{b-a}\int_a^b \ln t\,dt$$

that proves (4.8). □

COROLLARY 4.6. *Let $f\colon (0,\infty) \supset [a,b] \to \mathbb{R}$ be a GA-convex function on $[a,b]$. Then*

$$\frac{1}{2}\left[f\left(\exp\left(\mu_f\right)\right) + \frac{f\left(b\right)b\left(\ln b - \mu_f\right) + af\left(a\right)\left(\mu_f - \ln a\right)}{b-a}\right] \geq \frac{1}{b-a}\int_a^b f\left(s\right)ds$$

provided that

$$\mu_f := \frac{\int_a^b f\left(s\right)\ln s\,ds}{\int_a^b f\left(s\right)ds} \in \left[\ln a, \ln b\right].$$

PROOF. Follows from (4.5) by taking

$$\ln t = \frac{\int_a^b f\left(s\right)\ln s\,ds}{\int_a^b f\left(s\right)ds} \in \left[\ln a, \ln b\right]. \qquad \square$$

REMARK 4.7. If we take $t = \sqrt{ab}$ in (4.5), then we get

$$\frac{1}{2}\left[f\left(\sqrt{ab}\right) + \frac{f\left(b\right)b + af\left(a\right)}{2L\left(a,b\right)}\right] - \frac{1}{b-a}\int_a^b f\left(s\right)ds \geq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(s\right)\ln s\,ds - \left(\frac{1}{b-a}\int_a^b f\left(s\right)ds\right)\ln\sqrt{ab}\right].$$

If we take $t = I\left(a,b\right)$ in (4.5), then we get

$$\frac{1}{2}\left[f\left(I\left(a,b\right)\right) + \frac{f\left(b\right)b\left(\ln b - \ln I\left(a,b\right)\right) + af\left(a\right)\left(\ln I\left(a,b\right) - \ln a\right)}{b-a}\right] - \frac{1}{b-a}\int_a^b f\left(s\right)ds \geq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(s\right)\ln s\,ds - \left(\frac{1}{b-a}\int_a^b f\left(s\right)ds\right)\ln I\left(a,b\right)\right].$$

We use the following results obtained by the author in [5] and [6].

Lemma 4.8. *Let $h\colon [\alpha,\beta] \to \mathbb{R}$ be a convex function on $[\alpha,\beta]$. Then we have the inequalities*

$$\begin{aligned}(4.9)\qquad &\frac{1}{8}\left[h'_{+}\left(\frac{\alpha+\beta}{2}\right)-h'_{-}\left(\frac{\alpha+\beta}{2}\right)\right](\beta-\alpha)\\ &\le \frac{h(\alpha)+h(\beta)}{2}-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}h(t)\,dt\\ &\le \frac{1}{8}\left[h'_{-}(\beta)-h'_{+}(\alpha)\right](\beta-\alpha)\end{aligned}$$

and

$$\begin{aligned}(4.10)\qquad &\frac{1}{8}\left[h'_{+}\left(\frac{\alpha+\beta}{2}\right)-h'_{-}\left(\frac{\alpha+\beta}{2}\right)\right](\beta-\alpha)\\ &\le \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}h(t)\,dt-h\left(\frac{\alpha+\beta}{2}\right)\\ &\le \frac{1}{8}\left[h'_{-}(\beta)-h'_{+}(\alpha)\right](\beta-\alpha).\end{aligned}$$

The constant $\frac{1}{8}$ is the best possible in (4.9) *and* (4.10)*.*

Finally, we have

Theorem 4.9. *Let $f\colon (0,\infty)\supset [a,b]\to\mathbb{R}$ be a GA-convex (concave) function on $[a,b]$. Then we have*

$$\begin{aligned}(4.11)\qquad &\frac{1}{8}\left[f'_{+}\left(\sqrt{ab}\right)-f'_{-}\left(\sqrt{ab}\right)\right]\sqrt{ab}\,(\ln b-\ln a)\\ &\le(\ge)\,\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_{a}^{b}\frac{f(s)}{s}ds\\ &\le(\ge)\,\frac{1}{8}\left[f'_{-}(b)\,b-f'_{+}(a)\,a\right](\ln b-\ln a)\end{aligned}$$

and

$$\begin{aligned}(4.12)\qquad &\frac{1}{8}\left[f'_{+}\left(\sqrt{ab}\right)-f'_{-}\left(\sqrt{ab}\right)\right]\sqrt{ab}\,(\ln b-\ln a)\\ &\le(\ge)\,\frac{1}{\ln b-\ln a}\int_{a}^{b}\frac{f(s)}{s}ds-f\left(\sqrt{ab}\right)\\ &\le(\ge)\,\frac{1}{8}\left[f'_{-}(b)\,b-f'_{+}(a)\,a\right](\ln b-\ln a).\end{aligned}$$

PROOF. Consider the function $h\colon [\ln a, \ln b] \to \mathbb{R}$ defined by $h(t) = f \circ \exp(t)$. Since f is a GA-convex (concave) function on $[a, b]$, then we have the lateral derivatives

$$h'_{\pm}(t) = \left(f'_{\pm} \circ \exp(t)\right) \exp t, \quad t \in [\ln a, \ln b].$$

If we apply the inequality (4.9) for the convex function $f \circ \exp$ on the interval $[\ln a, \ln b]$, then we have

$$\begin{aligned}
&\frac{1}{8}\left[f'_{+}\circ \exp\left(\frac{\ln a + \ln b}{2}\right) - f'_{-}\circ \exp\left(\frac{\ln a + \ln b}{2}\right)\right] \exp\left(\frac{\ln a + \ln b}{2}\right)(\ln b - \ln a)\\
&\le \frac{f \circ \exp(\ln a) + f \circ \exp(\ln b)}{2} - \frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} f \circ \exp(t)\, dt\\
&\le \frac{1}{8}\left[\left(f'_{-} \circ \exp(\ln b)\right)\exp(\ln b) - \left(f'_{+} \circ \exp(\ln a)\right)\exp(\ln a)\right](\ln b - \ln a),
\end{aligned}$$

that is equivalent to

$$\begin{aligned}
\text{(4.13)} \qquad &\frac{1}{8}\left[f'_{+}\left(\sqrt{ab}\right) - f'_{-}\left(\sqrt{ab}\right)\right]\sqrt{ab}\,(\ln b - \ln a)\\
&\le \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} f \circ \exp(t)\, dt\\
&\le \frac{1}{8}\left[f'_{-}(b)\, b - f'_{+}(a)\, a\right](\ln b - \ln a).
\end{aligned}$$

If we change the variable $s = \exp t$, then $t = \ln s$ and $dt = \frac{ds}{s}$. Therefore

$$\int_{\ln a}^{\ln b} f \circ \exp(t)\, dt = \int_a^b \frac{f(s)}{s} ds$$

and by (4.13) we get the desired inequality (4.11).

The inequality (4.12) follows by (4.10). □

REMARK 4.10. If the function $f\colon (0, \infty) \supset I \to \mathbb{R}$ is differentiable and GA-convex on $[a, b] \subset \mathring{I}$, then we have the following inequalities:

$$\begin{aligned}
\text{(4.14)} \qquad 0 &\le \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(s)}{s} ds\\
&\le \frac{1}{8}\left[f'(b)\, b - f'(a)\, a\right](\ln b - \ln a)
\end{aligned}$$

and

$$0 \le \frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds - f\left(\sqrt{ab}\right) \le \frac{1}{8} \left[f'(b)\, b - f'(a)\, a\right] (\ln b - \ln a)\,. \tag{4.15}$$

5. Some applications

Let $p \neq 0$ and consider the convex function $g(t) = \exp(pt)$, $t \in \mathbb{R}$. Then the function $f\colon (0,\infty) \to \mathbb{R}$, $f(t) = g(\ln t) = \exp(p \ln t) = t^p$, is a GA-convex function on $(0,\infty)$. Observe that for $0 < a < b$ we have

$$\frac{1}{b-a} \int_a^b t^p dt = \begin{cases} \frac{1}{p+1} \frac{b^{p+1} - a^{p+1}}{b-a}, \ p \neq -1, \\ \frac{\ln b - \ln a}{b-a}, \ p = -1, \end{cases}$$
$$= \begin{cases} L_p^p(a,b)\,, \ p \neq -1, \\ L^{-1}(a,b)\,, \ p = -1, \end{cases}$$

where $L_p(a,b)$ $(p \neq -1)$ is the *p-logarithmic mean* and L is the logarithmic mean defined in the introduction.

Using the inequality

$$\frac{1}{b-a} \int_a^b f(t)\, dt \le \frac{b - L(a,b)}{b-a} f(b) + \frac{L(a,b) - a}{b-a} f(a)$$

for $f(t) = t^p$ $(p \neq 0)$, we get

$$L_p^p(a,b) \le \frac{b - L(a,b)}{b-a} b^p + \frac{L(a,b) - a}{b-a} a^p$$

for $p \neq 0$, where $L_{-1}^{-1}(a,b) := L^{-1}(a,b)$.

Observe that

$$\frac{1}{b-a} \int_a^b \frac{f(t)}{t} dt = \frac{1}{b-a} \int_a^b t^{p-1} dt$$
$$= \frac{1}{p} \frac{b^p - a^p}{b-a} = L_{p-1}^{p-1}(a,b)\,, \ p \neq 0.$$

If we use the inequality

$$\begin{aligned} f\left(\sqrt{ab}\right) &\le (1-\lambda) f\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right) + \lambda f\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right) \\ &\le \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \\ &\le \frac{1}{2}\left[f\left(a^{1-\lambda} b^{\lambda}\right) + (1-\lambda) f(b) + \lambda f(a)\right] \le \frac{f(a)+f(b)}{2} \end{aligned}$$

for $\lambda \in [0,1]$ and $f(t) = t^p$ $(p \neq 0)$, then we get

$$\begin{aligned} G^p(a,b) &\le (1-\lambda) G^p\left(a^{1-\lambda}, b^{\lambda+1}\right) + \lambda G^p\left(a^{2-\lambda}, b^{\lambda}\right) \\ &\le L(a,b) L_{p-1}^{p-1}(a,b) \\ &\le \frac{1}{2}\left[G^p\left(a^{2(1-\lambda)}, b^{2\lambda}\right) + (1-\lambda) b^p + \lambda a^p\right] \le \frac{a^p+b^p}{2} \end{aligned}$$

for $\lambda \in [0,1]$.

If we use the inequalities (4.14) and (4.15) for $f(t) = t^p$ $(p \neq 0)$, then we get

$$0 \le \frac{a^p+b^p}{2} - L(a,b) L_{p-1}^{p-1}(a,b) \le \frac{1}{8} p^2 \frac{L_{p-1}^{p-1}(a,b)}{L(a,b)} (b-a)^2$$

and

$$0 \le L(a,b) L_{p-1}^{p-1}(a,b) - G^p(a,b) \le \frac{1}{8} p^2 \frac{L_{p-1}^{p-1}(a,b)}{L(a,b)} (b-a)^2 .$$

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