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ON A FUNCTIONAL EQUATION RELATED TO TWO-SIDED CENTRALIZERS

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Abstract. The main aim of this manuscript is to prove the following result. Let $n > 2$ be a fixed integer and R be a k-torsion free semiprime ring with identity, where $k \in \{2, n-1, n\}$. Let us assume that for the additive mapping $T\colon R \to R$

$$3T(x^n) = T(x)x^{n-1} + xT(x^{n-2})x + x^{n-1}T(x), \quad x \in R,$$

is also fulfilled. Then T is a two-sided centralizer.

In this paper R will denote an associative ring with center $Z(R)$. For an integer $n > 1$, a ring R is said to be n-torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. The expression $xy - yx$ will be marked by $[x, y]$. The ring R is prime if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$ implies $a = 0$. We indicate by $char(R)$ the characteristic of a prime ring R. Let X be a real or complex Banach space and let $\mathcal{L}(X)$ and $\mathcal{F}(X)$ denote the algebra of all bounded linear operators on X and the ideal of all finite rank operators in $\mathcal{L}(X)$, respectively. An algebra $\mathcal{A}(X) \subseteq \mathcal{L}(X)$ is said to be standard if $\mathcal{F}(X) \subseteq \mathcal{A}(X)$. Let us point out that any standard operator algebra is prime, which is a consequence of Hahn-Banach theorem. An additive mapping $D\colon R \to R$ is called a derivation, if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$. An additive mapping $D\colon R \to R$ is called a Jordan derivation if $D(x^2) = D(x)x + xD(x)$ is fulfilled

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for all $x \in R$. Every derivation is a Jordan derivation. The converse in general is not true. A classical result of Herstein [9] asserts that any Jordan derivation on a prime ring with $\mathrm{char}(R) \neq 2$ is a derivation.

A short proof of Herstein theorem can be found in [6]. Cusack [8] has generalized the theorem to 2-torsion free semiprime rings (an alternative proof can be found in [4]). Beidar, Brešar, Chebotar and Martindale [1] also generalized it considerably. Generalizations of Herstein theorem are also presented in [7]. An additive mapping $T\colon R \to R$ is called a left (right) centralizer if $T(xy) = T(x)y$ ($T(xy) = xT(y)$) holds for all $x, y \in R$. We call $T\colon R \to R$ a two-sided centralizer if T is both a left and a right centralizer. If $T\colon R \to R$ is a two-sided centralizer, where R is a semiprime ring with extended centroid C, then there exists an element $\lambda \in C$ such that $T(x) = \lambda x$ for all $x \in R$ (see [2, Theorem 2.3.2]). An additive mapping $T\colon R \to R$ is called a left (right) Jordan centralizer if $T(x^2) = T(x)x$ $\big(T(x^2) = xT(x)\big)$ holds for all $x \in R$. Zalar [17] has proved that any left (right) Jordan centralizer on a 2-torsion free semiprime ring is a left (right) centralizer. Lately several authors investigated centralizers on rings and algebras. Some of the results can be found in [3, 11, 12, 13, 14, 15, 16].

Let us start with the following result proved by M. Brešar in [5].

Theorem 1. *Let R be a 2-torsion free semiprime ring and let $D\colon R \to R$ be an additive mapping satisfying the equality*

$$D(xyx) = D(x)yx + xD(y)x + xyD(x) \tag{1}$$

for all $x, y \in R$. Then D is a derivation.

An additive mapping $D\colon R \to R$, where R is an arbitrary ring, satisfying equality (1) for all $x, y \in R$ is called a Jordan triple derivation. One can easily prove that any Jordan derivation on an arbitrary 2-torsion free ring is a Jordan triple derivation (see, for example, [6] for the details), which means that Theorem 1 generalizes Cusack's generalization of Herstein theorem.

Motivated by this result, Vukman and Kosi-Ulbl in [14] proved the following

Theorem 2. *Let R be a 2-torsion free semiprime ring with extended centroid C and let $T\colon R \to R$ be an additive mapping. Suppose that*

$$3T(xyx) = T(x)yx + xT(y)x + xyT(x) \tag{2}$$

holds for all $x, y \in R$. Then T is of the form $T(x) = \lambda x$ for all $x \in R$ and some fixed $\lambda \in C$.

Let $n > 2$ be a fixed integer and let $y = x^{n-2}$ in (2). Then we obtain

$$3T(x^n) = T(x)x^{n-1} + xT(x^{n-2})x + x^{n-1}T(x). \tag{3}$$

One of our main purposes is to investigate equation (3) for additive mappings $T\colon \mathcal{A}(X) \to \mathcal{L}(X)$, where X denotes a Banach space over $\mathcal{F} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathcal{L}(X)$ denotes the algebra of all bounded linear operators acting on X, and $\mathcal{A}(X)$ is a standard operator algebra.

Theorem 3. *Let $\mathcal{L}(X)$ be the algebra of all bounded linear operators on X and let $\mathcal{A}(X) \subseteq \mathcal{L}(X)$ be a standard operator algebra, where X is a Banach space over the real or complex field $\mathcal{F}$. Suppose that there exists an additive mapping $T\colon \mathcal{A}(X) \to \mathcal{L}(X)$ satisfying the equality*

$$3T(A^n) = T(A)A^{n-1} + AT(A^{n-2})A + A^{n-1}T(A) \tag{4}$$

for all $A \in \mathcal{A}(X)$ and a fixed integer $n > 2$. Then T is of the form $T(A) = \lambda A$ for all $A \in \mathcal{A}(X)$ and some fixed $\lambda \in \mathcal{F}$.

In the proof of Theorem 3 we shall use the result below, see Vukman [11].

Theorem 4. *Let R be a 2-torsion free semiprime ring and let $T\colon R \to R$ be an additive mapping satisfying*

$$2T(x^2) = T(x)x + xT(x)$$

for all $x \in R$. Then T is a two-sided centralizer.

It should be mentioned that in the proof of Theorem 3 we will use some methods similar to those used by Molnár in [10].

Proof of Theorem 3. We start with equality (4). Let us first consider $A \in \mathcal{F}(X)$ and a projection P such that $A = AP = PA$. Identity (4) with $A = P$ yields that

$$T(P)P = PT(P) = PT(P)P. \tag{5}$$

In equation (4) we set $A + \alpha P$ for A, $\alpha \in \mathcal{F}$, and obtain

$$3\sum_{i=0}^{n}\binom{n}{i}T\Big(A^{n-i}(\alpha P)^i\Big) = \Big(T(A) + \alpha T(P)\Big)\Big(\sum_{i=0}^{n-1}\binom{n-1}{i}A^{n-1-i}(\alpha P)^i\Big)$$

$$+\,(A+\alpha P)\left(\sum_{i=0}^{n-2}\binom{n-2}{i}T\left(A^{n-2-i}\,(\alpha P)^{i}\right)\right)(A+\alpha P)
\qquad (6)$$

$$+\left(\sum_{i=0}^{n-1}\binom{n-1}{i}A^{n-1-i}\,(\alpha P)^{i}\right)(T\,(A)+\alpha T(P))\,.$$

Collecting all expressions with coefficient α^{n-1} from equation (6) and using (5), we arrive at

$$3nT\,(A)=T\,(A)\,P+PT\,(A)+nAB+nBA+(n-2)\,PT\,(A)\,P, \qquad (7)$$

where B stands for $T(P)$. Right multiplication of (7) by P gives

$$3nT\,(A)\,P=T\,(A)\,P+PT\,(A)\,P+nAB+nBA+(n-2)\,PT\,(A)\,P. \qquad (8)$$

Similarly we obtain

$$3nPT\,(A)=PT\,(A)\,P+PT\,(A)+nAB+nBA+(n-2)\,PT\,(A)\,P. \qquad (9)$$

Combining (8) and (9) gives us

$$T(A)P=PT(A),$$

which reduces equality (7) to

$$3T\,(A)=T\,(A)\,P+AB+BA. \qquad (10)$$

Multiplying the above by P from the right gives

$$3T\,(A)\,P=T\,(A)\,P+AB+BA. \qquad (11)$$

Combining (10) with (11) we get

$$T(A)=T(A)P. \qquad (12)$$

From the above equality one can conclude that T maps $\mathcal{F}(X)$ into itself. Using (12), equality (10) reduces to

$$2T\,(A)=AB+BA. \qquad (13)$$

Multiplying (13) from the right and from the left by A, respectively, gives

$$2T\,(A)\,A=ABA+BA^{2}\quad\text{and}\quad 2AT\,(A)=A^{2}B+ABA, \qquad (14)$$

respectively. Going back to (6) and collecting all expressions with coefficient α^{n-2} gives

$$\begin{aligned}3n(n-1)T\left(A^2\right) &= 2(n-1)(T(A)A+AT(A))\\ &+2(n-2)(AT(A)P+PT(A)A)+(n-1)(n-2)\left(A^2B+BA^2\right)\\ &+(n-2)(n-3)PT\left(A^2\right)P+2ABA.\end{aligned}$$

Using (12) the above equality simplifies to

$$\begin{aligned}2\left(n^2+n-3\right)T\left(A^2\right) &= 2(2n-3)(T(A)A+AT(A))\\ &+(n-1)(n-2)\left(A^2B+BA^2\right)+2ABA.\end{aligned} \tag{15}$$

Combining (14) and (15) we get

$$2T(A^2)=T(A)A+AT(A). \tag{16}$$

Therefore we have an additive mapping $T\colon \mathcal{F}(X)\to\mathcal{F}(X)$ satisfying equation (16) for all $A\in\mathcal{F}(X)$. Since $\mathcal{F}(X)$ is prime, by Theorem 4 we may conclude that T is a two-sided centralizer on $\mathcal{F}(X)$. We continue our proof by showing that there exists an operator $C\in\mathcal{L}(X)$ such that

$$T(A)=CA \qquad (A\in\mathcal{F}(X)). \tag{17}$$

For any fixed $x\in X$ and $f\in X^*$ by $x\otimes f$ we denote an operator from $\mathcal{F}(X)$ defined by $(x\otimes f)y=f(y)x$ for all $y\in X$. For any $A\in\mathcal{F}(X)$ we have $A(x\otimes f)=((Ax)\otimes f)$. Let us choose f and y such that $f(y)=1$ and define $Cx=T(x\otimes f)y$. Obviously, C is linear. Using the fact that T is a left centralizer on $\mathcal{F}(X)$ we obtain

$$\begin{aligned}(CA)x &= C(Ax)=T((Ax)\otimes f)y\\ &= T(A(x\otimes f))y=T(A)(x\otimes f)y=T(A)x, \quad x\in X.\end{aligned}$$

Therefore we have $T(A)=CA$ for any $A\in\mathcal{F}(X)$. Since T is a right centralizer on $\mathcal{F}(X)$ we obtain $C(AP)=T(AP)=AT(P)=ACP$, where $A\in\mathcal{F}(X)$ and P is an arbitrary one-dimensional projection. Therefore $[A,C]P=0$. Using the fact that P is an arbitrary one-dimensional projection we get $[A,C]=0$ for all $A\in\mathcal{F}(X)$. This means C commutes with all operators from $\mathcal{F}(X)$. In other words,

$$Cx=\lambda x$$

is fulfilled for all $x \in X$ and some fixed $\lambda \in \mathcal{F}$. Combining the above equation with (17) it follows that T is of the form

$$T(A) = \lambda A$$

for any $A \in \mathcal{F}(X)$ and some fixed $\lambda \in \mathcal{F}$. We want to prove that the same equality holds on $\mathcal{A}(X)$ as well. For this purpose we introduce a mapping $T_1 \colon \mathcal{A}(X) \to \mathcal{L}(X)$ defined by $T_1(A) = \lambda A$. Let us investigate a mapping $T_0 = T - T_1$. One can easily find out that T_0 is additive, it satisfies (4) and $T_0(A) = 0$ for all $A \in \mathcal{F}(X)$. We will prove that $T_0(A) = 0$ for all $A \in \mathcal{A}(X)$ as well. Let us introduce an operator $S \in \mathcal{A}(X)$ defined by $S = A + PAP - (AP + PA)$, where $A \in \mathcal{A}(X)$ and $P \in \mathcal{F}(X)$ is an one-dimensional projection. From the definition of the operator S it follows immediately that $SP = PS = 0$ and $S - A \in \mathcal{F}(X)$. Thus we have $T_0(S) = T_0(A)$. So we can rewrite (4) as

$$3T_0(S^n) = T_0(S)S^{n-1} + ST_0(S^{n-2})S + S^{n-1}T_0(S).$$

Using the above and the facts that $T_0(P) = 0$ as well as $SP = PS = 0$, we obtain

$$\begin{aligned}
T_0(S)S^{n-1} + ST_0(S^{n-2})S + S^{n-1}T_0(S) &= 3T_0(S^n) = 3T_0(S^n + P) \\
&= 3T_0((S+P)^n) = T_0(S+P)(S+P)^{n-1} \\
&\quad + (S+P)T_0((S+P)^{n-2})(S+P) + (S+P)^{n-1}T_0(S+P) \\
&= T_0(S)S^{n-1} + T_0(S)P + ST_0(S^{n-2})S + ST_0(S^{n-2})P \\
&\quad + PT_0(S^{n-2})S + PT_0(S^{n-2})P + S^{n-1}T_0(S) + PT_0(S).
\end{aligned}$$

Since $T_0(S) = T_0(A)$, we actually have

$$T_0(A)P + ST_0(A^{n-2})P + PT_0(A^{n-2})S + PT_0(A^{n-2})P + PT_0(A) = 0. \tag{18}$$

Setting αA for A in (18), we obtain

$$\begin{aligned}
&\alpha\,(T_0(A)P + PT_0(A)) + \alpha^{n-2}PT_0(A^{n-2})P \\
&\qquad\qquad + \alpha^{n-1}\left(ST_0(A^{n-2})P + PT_0(A^{n-2})S\right) = 0.
\end{aligned}$$

This implies that $T_0(A)P + PT_0(A) = 0$. Multiplying by P on both sides gives $PT_0(A)P = 0$. Multiplying on the right gives $T_0(A)P = -PT_0(A)P = 0$. Since P is an arbitrary one-dimensional projection, one can conclude that $T_0(A) = 0$ for any $A \in \mathcal{A}(X)$. In other words, we have proved that T is of the form $T(A) = \lambda A$ for all $A \in \mathcal{A}(X)$ and some fixed $\lambda \in \mathcal{F}$. The proof of the theorem is complete. $\square$

CONJECTURE. *Let R be a semiprime ring with suitable torsion restrictions and let $T\colon R \to R$ be an additive mapping satisfying the equation*

$$3T(x^n) = T(x)x^{n-1} + xT(x^{n-2})x + x^{n-1}T(x)$$

for all $x \in R$ and a fixed integer $n > 2$. Then T is a two-sided centralizer.

The result below proves the above conjecture in the case when R has an identity element.

THEOREM 5. *Let $n > 2$ be a fixed integer and R be a k-torsion free semiprime ring with identity, where $k \in \{2, n-1, n\}$. Let us assume that, for the additive mapping $T\colon R \to R$,*

$$3T(x^n) = T(x)x^{n-1} + xT(x^{n-2})x + x^{n-1}T(x), \quad x \in R, \tag{19}$$

is also fulfilled. Then T is a two-sided centralizer.

PROOF. Let us start from equation (19). Using the same techniques as in Theorem 3, we obtain

$$\begin{aligned} 2\left(n^2+n-3\right)T\left(x^2\right) &= 2\left(2n-3\right)T\left(x\right)x + 2\left(2n-3\right)xT\left(x\right) \\ &\quad + \left(n^2-3n+2\right)ax^2 + \left(n^2-3n+2\right)x^2a + 2xax, \quad x \in R, \end{aligned} \tag{20}$$

and

$$2T\left(x\right) = xa + ax, \quad x \in R, \tag{21}$$

where a stands for $T\left(e\right)$. Comparing the steps of the proof of Theorem 3 with the beginning of the proof of Theorem 5 we see that equations (20) and (21) correspond to equations (15) and (13), respectively. In the procedure mentioned above we used the fact that R is n-torsion free. According to (21) we obtain

$$2T\left(x^2\right) = x^2a + ax^2, \quad x \in R. \tag{22}$$

Multiplying (21) by x first from the right and then from the left side we get

$$2T\left(x\right)x = xax + ax^2 \quad \text{and} \quad 2xT\left(x\right) = x^2a + xax, \quad x \in R. \tag{23}$$

Using (22) and (23) in (20) after some calculation we obtain

$$x^2a + ax^2 - 2xax = 0, \quad x \in R.$$

In the above calculation we used the assumption that the ring R is 2 and $(n-1)$-torsion free. Now let us rewrite the above equation in the form

$$[[a, x], x] = 0, \quad x \in R. \tag{24}$$

Putting $x + y$ in place of x in (24) we obtain

$$[[a, x], y] + [[a, y], x] = 0, \quad x, y \in R. \tag{25}$$

Putting xy in place of y in (25) we obtain

$$\begin{aligned} 0 &= [[a, x], xy] + [[a, xy], x] \\ &= [[a, x], x] y + x [[a, x], y] + [[a, x] y + x [a, y], x] \\ &= x [[a, x], y] + [[a, x], x] y + [a, x] [y, x] + x [[a, y], x] \\ &= [a, x] [y, x], \quad x, y \in R, \end{aligned}$$

where we also used (24) and (25). Thus we have

$$[a, x] [y, x] = 0, \quad x, y \in R.$$

Substituting y with ya in the above we obtain $[a, x] y [a, x] = 0$ for all $x, y \in R$. Since R is semiprime, it follows from the last equation that $[a, x] = 0$ for all $x \in R$. This means that $a \in Z(R)$ and (21) reduces to $T(x) = ax$, $x \in R$, since R is 2-torsion free. It follows immediately that T is a two-sided centralizer, which completes the proof. □

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