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ON ORTHOGONALLY ADDITIVE FUNCTIONS WITH BIG GRAPHS

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Abstract. Let E be a separable real inner product space of dimension at least 2 and V be a metrizable and separable linear topological space. We show that the set of all orthogonally additive functions mapping E into V and having big graphs is dense in the space of all orthogonally additive functions from E into V with the Tychonoff topology.

Let E be a real inner product space of dimension at least 2 and V a linear topological Hausdorff space. A function f mapping E into V is called *orthogonally additive*, if

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in E \text{ with } x \perp y.$$

It is well known, see [6, Corollary 10] (cf. also [4, Theorem 1]), that every orthogonally additive function f defined on E has the form

$$f(x) = a(\|x\|^2) + b(x) \quad \text{for } x \in E, \tag{1}$$

where a and b are additive functions uniquely determined by f.

We continue a study of topological properties of some sets of orthogonally additive functions.

Given a non-empty set S consider the set V^S of all functions from S into V with the usual Tychonoff topology; clearly V^S is a linear topological space.

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In what follows we consider $\mathrm{Hom}_\perp(E,V)$ and $\mathrm{Hom}(S,V)$ for $S \in \{\mathbb{R}, E\}$ with the topology induced by V^E and V^S, respectively, where

$$\mathrm{Hom}_\perp(E,V) = \{f : E \to V \mid f \text{ is orthogonally additive}\}$$

and

$$\mathrm{Hom}(S,V) = \{f : S \to V \mid f \text{ is additive}\}$$

for $S \in \{\mathbb{R}, E\}$. According to [2, Corollary 1]:

1. *If* $V \neq \{0\}$, *then* $\mathrm{Hom}(E,V)$ *is closed and nowhere dense in* $\mathrm{Hom}_\perp(E,V)$.

   This and [1, Theorem] gives:

2. *The set*

$$\{f \in \mathrm{Hom}_\perp(E,V) : f \text{ is injective and } f(E) = V\}$$

   is nowhere dense in $\mathrm{Hom}_\perp(E,V)$.

   Moreover, see [3, Corollaries 2.3 and 2.4]:

3. *If* $\mathrm{card}\, E \le \mathrm{card}\, V$, *then the set*

$$\{f \in \mathrm{Hom}_\perp(E,V) : f \text{ is injective and } f(E) \neq V\}$$

   is dense in $\mathrm{Hom}_\perp(E,V)$.

4. *If* $\mathrm{card}\, V \le \mathrm{card}\, E$, *then the set*

$$\{f \in \mathrm{Hom}_\perp(E,V) : f(E) = V \text{ and } f \text{ is not injective}\}$$

   is dense in $\mathrm{Hom}_\perp(E,V)$.

The main result of this note concerns density of the set of all orthogonally additive functions with big graphs. Following [5, p. 317] we say that a function f mapping a topological space X into a topological space Y has a *big graph* if $B \cap \mathrm{Graph}(f) \neq \emptyset$ for every Borel subset B of $X \times Y$ such that the projection $\pi_X(B)$ of B onto X has the cardinality $\mathrm{card}\, X$. We start with the following theorem.

THEOREM 1. *If* $\mathcal{R}$ *is a family of subsets of* $E \times V$ *such that*

$$\mathrm{card}\, \pi_E(R) > \aleph_0 \quad \text{and} \quad \mathrm{card}\, \pi_E(R) \ge \mathrm{card}\, \mathcal{R} \quad \text{for } R \in \mathcal{R},$$

then the set

$$\{f \in \mathrm{Hom}_\perp(E,V) : R \cap \mathrm{Graph}(f) \neq \emptyset \text{ for every } R \in \mathcal{R}\}$$

is dense in $\mathrm{Hom}_\perp(E,V)$.

PROOF. Since (see [2, Theorem 1]) the operator which assigns the function f defined by (1) to the variable (a,b) from $\mathrm{Hom}(\mathbb{R},V)\times\mathrm{Hom}(E,V)$ is a homeomorphism onto $\mathrm{Hom}_{\perp}(E,V)$, it is enough to prove that the set

$$\bigcap_{R\in\mathcal{R}}\bigcup_{x\in E}\big\{(a,b)\in\mathrm{Hom}(\mathbb{R},V)\times\mathrm{Hom}(E,V):(x,a(\|x\|^2)+b(x))\in R\big\}\tag{2}$$

is dense in $\mathrm{Hom}(\mathbb{R},V)\times\mathrm{Hom}(E,V)$.

We shall show more: if $a\in\mathrm{Hom}(\mathbb{R},V)$ and $\mathcal{U}\subset\mathrm{Hom}(E,V)$ is open and non-empty, then $\{a\}\times\mathcal{U}$ intersects set (2). To this end we may assume that

$$\mathcal{U}=\{b\in\mathrm{Hom}(E,V):b(z_n)\in U_n\text{ for }n\in\{1,\dots,N\}\}$$

with some open subsets $U_1,\dots,U_N$ of V, $z_1,\dots,z_N\in E$ and $N\in\mathbb{N}$.

Fix $b_0\in\mathcal{U}$. To prove that $\{a\}\times\mathcal{U}$ meets set (2) it is enough to show that there is a $b\in\mathrm{Hom}(E,V)$ such that (a,b) is in the set (2) and

$$b(z_n)=b_0(z_n)\quad\text{for }n\in\{1,\dots,N\}.\tag{3}$$

Let H be a base of the vector space E over the field $\mathbb{Q}$ of all rationals and let H_0 be a finite subset of H such that $z_1,\dots,z_N\in\mathrm{Lin}_{\mathbb{Q}}H_0$. Put

$$\gamma=\operatorname{card}\mathcal{R}$$

and let $(R_\alpha)_{\alpha<\gamma}$ be a transfinite sequence of all elements of $\mathcal{R}$. (We treat γ as an ordinal.) By transfinite induction we define an injective transfinite sequence $(x_\alpha)_{\alpha<\gamma}$ of vectors of E and a transfinite sequence $(v_\alpha)_{\alpha<\gamma}$ of vectors of V such that for every $\alpha<\gamma$ we have

$$H_0\cup\{x_\beta:\ \beta\le\alpha\}\text{ is linearly independent over }\mathbb{Q},\tag{4}$$

$$H_0\cap\{x_\beta:\ \beta\le\alpha\}=\emptyset,\tag{5}$$

and

$$(x_\alpha,v_\alpha)\in R_\alpha.\tag{6}$$

We proceed as follows. If $\delta<\gamma$, $(x_\alpha)_{\alpha<\delta}$ is an injective transfinite sequence of vectors of E and $(v_\alpha)_{\alpha<\delta}$ is a transfinite sequence of vectors of V such that (4)–(6) hold for every $\alpha<\delta$, then

$$\begin{aligned}\operatorname{card}\mathrm{Lin}_{\mathbb{Q}}(H_0\cup\{x_\alpha:\ \alpha<\delta\})&\le\aleph_0\cdot\max\{\aleph_0,\operatorname{card}\delta\}\\&=\max\{\aleph_0,\operatorname{card}\delta\}<\operatorname{card}\pi_E(R_\delta),\end{aligned}$$

and so the set

$$\pi_E(R_\delta)\backslash\mathrm{Lin}_{\mathbb{Q}}(H_0 \cup \{x_\alpha :\ \alpha < \delta\})$$

is non-empty; choosing a point x_δ from this set we see that $H_0 \cup \{x_\alpha :\ \alpha \le \delta\}$ consists of vectors linearly independent over $\mathbb{Q}$ and $(x_\delta, v_\delta) \in R_\delta$ for some $v_\delta \in V$.

Let $b\colon E \to V$ be an additive function such that

$$b|_{H_0} = b_0|_{H_0} \quad \text{and} \quad b(x_\alpha) = v_\alpha - a(\|x_\alpha\|^2) \quad \text{for } \alpha < \gamma.$$

Then (3) holds and

$$(x_\alpha, a(\|x_\alpha\|^2) + b(x_\alpha)) = (x_\alpha, v_\alpha) \in R_\alpha$$

for $\alpha < \gamma$ and so (a, b) is in the set (2). □

Corollary 1. *If E is separable and V is metrizable and separable, then the set*

$$\{f \in \mathrm{Hom}_\perp(E, V) : f \text{ has a big graph}\} \tag{7}$$

is dense in $\mathrm{Hom}_\perp(E, V)$.

Proof. If $\mathcal{R}$ denotes the family

$$\{B \subset E \times V : B \text{ is Borel and } \operatorname{card} \pi_E(B) = \operatorname{card} E\},$$

then (see, e.g., [5, Theorem 2.3.4])

$$\operatorname{card} \mathcal{R} = \mathfrak{c} = \operatorname{card} E = \operatorname{card} \pi_E(B)$$

for every $B \in \mathcal{R}$. □

Functions with big graphs have a lot of interesting properties. In particular the following (see [5, Theorems 2.5.6 and 2.8.3] and the proofs of Theorems 12.4.4 and 12.4.5 in [5]):

5. *Assume X and Y are Polish spaces.*
   (i) *If X has no isolated points, then the graph of any function with big graph mapping X into Y is dense in $X \times Y$.*
   (ii) *If X and Y have no isolated points and $f\colon X \to Y$ has a big graph, then neither* $\mathrm{Graph}(f)$, *nor* $(X \times Y) \setminus \mathrm{Graph}(f)$ *contains a second category set with the Baire property.*

(iii) *If X and Y are connected, then the graph of any function with big graph mapping X into Y is connected.*

Hence and from Corollary 1 we obtain the following corollary.

COROLLARY 2. *If E is separable and Hilbert and V is Polish, $V \neq \{0\}$, then the set*

$$\{f \in \mathrm{Hom}_\perp(E,V) : \mathrm{Graph}(f) \textit{ is dense and connected, and neither } \mathrm{Graph}(f), \textit{ nor } (E \times V) \setminus \mathrm{Graph}(f) \textit{ contains a second category set with the Baire property}\}$$

is dense in $\mathrm{Hom}_\perp(E,V)$.

At the end note also that:

6. *If $V \neq \{0\}$, then the complement of the set* (7) *is dense in* $\mathrm{Hom}_\perp(E,V)$.

In fact, an obvious modification of the proof of Theorem 1 shows that:

7. *The set*

$$\{f \in \mathrm{Hom}_\perp(E,V) : \ f(E) \textit{ is countable}\}$$

is dense in $\mathrm{Hom}_\perp(E,V)$.

The reader interested in further problems connected with orthogonal additivity is referred to the survey paper [7] by Justyna Sikorska.

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