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Annales Mathematicae Silesianae **31** (2017), 71–81
DOI: 10.1515/amsil-2017-0002

MULTIPLIERS OF UNIFORM TOPOLOGICAL ALGEBRAS

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Abstract. Let E be a complete uniform topological algebra with Arens-Michael normed factors $(E_\alpha)_{\alpha\in\Lambda}$. Then $M(E)\cong \varprojlim M(E_\alpha)$ within an algebra isomorphism φ. If each factor E_α is complete, then every multiplier of E is continuous and φ is a topological algebra isomorphism where $M(E)$ is endowed with its seminorm topology.

1. Preliminaries

A *topological algebra* is an algebra (over the complex field) which is also a Hausdorff topological vector space such that the multiplication is separately continuous. For a topological algebra E, by $\Delta(E)$ we denote the set of all nonzero continuous multiplicative linear functionals on E. An *approximate identity* in a topological algebra E is a net $(e_\omega)_{\omega\in\Omega}$ such that for each $x\in E$ we have $xe_\omega\to_\omega x$ and $e_\omega x\to_\omega x$.

Let E be an algebra, a function $p\colon E\to[0,\infty[$ is called a *pseudo-seminorm*, if there exists $0\lneqq k\leq 1$ such that $p(x+y)\leq p(x)+p(y)$, $p(\lambda x)=|\lambda|^k p(x)$ and $p(xy)\leq p(x)p(y)$ for all $x,y\in E$ and $\lambda\in\mathbb{C}$. The number k is called the *homogenity index of* p. If $k=1$ then p is called a *seminorm*. A pseudo-seminorm p is a pseudo-norm, if $p(x)=0$ implies $x=0$.

A *locally* m*-pseudoconvex algebra* is a topological algebra E whose topology is determined by a directed family $\{p_\alpha:\alpha\in\Lambda\}$ of pseudo-seminorms. For each $\alpha\in\Lambda$, $\ker(p_\alpha)=\{x\in E: p_\alpha(x)=0\}$, the quotient algebra $E_\alpha=$

Received: 26.09.2016. Accepted: 21.01.2017. Published online: 12.04.2017.
(2010) Mathematics Subject Classification: 46H05, 46K05.
Key words and phrases: multiplier algebra, locally m-pseudoconvex algebra, uniform topological algebra.

$E/\ker(p_\alpha)$ is a pseudo-normed algebra in the pseudo-norm $\overline{p}_\alpha(x_\alpha) = p_\alpha(x)$, $x_\alpha = x + \ker(p_\alpha)$. Let $f_\alpha \colon E \to E_\alpha$, $f_\alpha(x) = x + \ker(p_\alpha) = x_\alpha$, be the quotient map, f_α is a continuous homomorphism from E onto E_α. We endow the set Λ with the partial order: $\alpha \leq \beta$ if and only if $p_\alpha(x) \leq p_\beta(x)$ for all $x \in E$. Take $\alpha \leq \beta$ in Λ, since $\ker(p_\beta) \subset \ker(p_\alpha)$, we define the surjective continuous homomorphism $f_{\alpha\beta} \colon E_\beta \to E_\alpha$, $x_\beta = x + \ker(p_\beta) \to x_\alpha = x + \ker(p_\alpha)$. Thus $\{(E_\alpha, f_{\alpha\beta}), \alpha \leq \beta\}$ is a projective system of pseudo-normed algebras. We also define the algebra isomorphism (into) $\Phi \colon E \to \varprojlim E_\alpha$, $\Phi(x) = (f_\alpha(x))_{\alpha \in \Lambda}$, the canonical projections $\pi_\alpha \colon \prod_{\alpha \in \Lambda} E_\alpha \to E_\alpha$ and the restrictions to the projective limit $g_\alpha = \pi_\alpha /_{\varprojlim E_\alpha} \colon \varprojlim E_\alpha \to E_\alpha$. Since $g_\alpha \circ \Phi = f_\alpha$ and the quotient map f_α is surjective, it follows that the map g_α is surjective, this proves that the projective system $\{(E_\alpha, f_{\alpha\beta}), \alpha \leq \beta\}$ is perfect in the sense of [5, Definition 2.10] (see also [2, Definition 2.7]). Thus, if E is a locally m-pseudoconvex algebra (not necessarily complete), then its generalized Arens-Michael projective system $\{(E_\alpha, f_{\alpha\beta}), \alpha \leq \beta\}$ is perfect. If E is complete, then $E \cong \varprojlim E_\alpha$ within a topological algebra isomorphism.

A *locally m-convex algebra* is a topological algebra E whose topology is defined by a directed family $\{p_\alpha : \alpha \in \Lambda\}$ of seminorms. For each $\alpha \in \Lambda$, put $\Delta_\alpha(E) = \{f \in \Delta(E) : |f(x)| \leq p_\alpha(x), x \in E\}$. Let E be an algebra with involution $*$. A seminorm on E is called a *C^*-seminorm* if $p(x^*x) = p(x)^2$ for all $x \in E$. A complete locally m-convex *-algebra $\left(E, (p_\alpha)_{\alpha \in \Lambda}\right)$, for which each p_α is a C^*-seminorm, is called a *locally C^*-algebra.* A *uniform seminorm* on an algebra E is a seminorm p satisfying $p(x^2) = p(x)^2$ for all $x \in E$. A *uniform topological algebra* is a topological algebra whose topology is determined by a directed family of uniform seminorms. In that case, such a topological algebra is also named a *uniform locally convex algebra.* A *uniform normed algebra* is a normed algebra $(E, \|.\|)$ such that $\|x^2\| = \|x\|^2$ for all $x \in E$.

An algebra E is called *proper* if for any $x \in E$, $xE = Ex = \{0\}$ implies $x = 0$. If E has identity, then E is proper. Moreover, a topological algebra with approximate identity is proper. Also, a (Hausdorff) uniform topological algebra is proper. Let E be an algebra, a map $T \colon E \to E$ is called a *multiplier* if $T(x)y = xT(y)$ for all $x, y \in E$. We denote by $M(E)$ the set of all multipliers of E. It is known that if E is a proper algebra, then any multiplier T of E is linear with the property $T(xy) = T(x)y = xT(y)$ for all $x, y \in E$, and $M(E)$ is a commutative algebra with the identity map I of E as its identity. Let $(E, \|\cdot\|)$ be a uniform normed algebra, and let $M_c(E)$ be the algebra of all continuous multipliers of E with the operator norm $\|\cdot\|_{op}$. It is known that $\|\cdot\|_{op}$ has the square property and the map $l \colon (E, \|\cdot\|) \to (M_c(E), \|\cdot\|_{op})$, $l(x)(y) = xy$, is an isometric isomorphism (into). For information on the multiplier algebra in non-normed topological algebras, see also [3] and [4].

In the sequel, we will need the following elementary result called the *universal property of the quotient*: Let X, Y, Z be vector spaces, $f \colon X \to Y$ and

$g\colon X \to Z$ be linear maps. If the map g is surjective and $\ker(g) \subset \ker(f)$, then there exists a unique linear map $h\colon Z \to Y$ such that $f = h \circ g$.

2. Results

Proposition 2.1. *Let $\left(E, (p_\alpha)_{\alpha\in\Lambda}\right)$ be a locally m-pseudoconvex algebra with proper pseudo-normed factors $(E_\alpha)_{\alpha\in\Lambda}$. The following assertions are equivalent:*

(i) *$T\left(\ker\left(f_\alpha\right)\right) \subset \ker\left(f_\alpha\right)$ for all $T \in M(E)$ and $\alpha \in \Lambda$;*

(ii) *for each $T \in M(E)$, there exists a unique $(T_\alpha)_{\alpha\in\Lambda} \in \prod_{\alpha\in\Lambda} M(E_\alpha)$ such that $f_\alpha \circ T = T_\alpha \circ f_\alpha$ and $T_\alpha \circ f_{\alpha\beta} = f_{\alpha\beta} \circ T_\beta$ for all $\alpha \leq \beta$ in Λ; furthermore, T is continuous if and only if T_α is continuous for all $\alpha \in \Lambda$.*

Proof. Since the pseudo-normed factors $(E_\alpha)_{\alpha\in\Lambda}$ are proper, it follows that the algebra E is proper and so every multiplier of E (or E_α) is linear.

(ii) $\Rightarrow$ (i): If $T \in M(E)$ and $x \in \ker\left(f_\alpha\right)$, then $f_\alpha\left(T\left(x\right)\right) = T_\alpha\left(f_\alpha\left(x\right)\right) = 0$ and so $T\left(x\right) \in \ker\left(f_\alpha\right)$.

(i) $\Rightarrow$ (ii): Take $T \in M(E)$ and $\alpha \in \Lambda$. Since $T\left(\ker\left(f_\alpha\right)\right) \subset \ker\left(f_\alpha\right)$ and by using the universal property of the quotient (see Preliminaries), there exists a unique linear map $T_\alpha\colon E_\alpha \to E_\alpha$ such that $f_\alpha \circ T = T_\alpha \circ f_\alpha$. Let $\alpha \in \Lambda$ and $x, y \in E$,

$$\begin{aligned} T_\alpha\left(f_\alpha\left(x\right) f_\alpha\left(y\right)\right) &= T_\alpha\left(f_\alpha\left(xy\right)\right) = f_\alpha\left(T\left(xy\right)\right) \\ &= f_\alpha\left(xT\left(y\right)\right) = f_\alpha\left(x\right) f_\alpha\left(T\left(y\right)\right) = f_\alpha\left(x\right) T_\alpha\left(f_\alpha\left(y\right)\right) \end{aligned}$$

and similarly on the other side, so T_α is a multiplier of E_α. Let $\alpha \leq \beta$ in Λ, we have $T_\alpha \circ f_\alpha = f_\alpha \circ T$, then $T_\alpha \circ f_{\alpha\beta} \circ f_\beta = f_{\alpha\beta} \circ f_\beta \circ T = f_{\alpha\beta} \circ T_\beta \circ f_\beta$, hence $T_\alpha \circ f_{\alpha\beta} = f_{\alpha\beta} \circ T_\beta$ since the quotient map f_β is surjective. Suppose that T is continuous. Let O_α be an open set in E_α, we have

$$f_\alpha^{-1}\left(T_\alpha^{-1}\left(O_\alpha\right)\right) = \left(T_\alpha \circ f_\alpha\right)^{-1}\left(O_\alpha\right) = \left(f_\alpha \circ T\right)^{-1}\left(O_\alpha\right)$$

which is open in E since $f_\alpha \circ T$ is continuous, then $T_\alpha^{-1}\left(O_\alpha\right)$ is open in E_α. Conversely, suppose that T_α is continuous for all $\alpha \in \Lambda$. Since E is topologically isomorphic to a subalgebra of $\varprojlim E_\alpha$, T is continuous if and only if $f_\alpha \circ T$ is continuous for all $\alpha \in \Lambda$. Since $f_\alpha \circ T = T_\alpha \circ f_\alpha$ and T_α is continuous for all $\alpha \in \Lambda$, we deduce that T is continuous. □

Proposition 2.2. *Let* $\left(E,(p_\alpha)_{\alpha\in\Lambda}\right)$ *be a locally m-pseudoconvex algebra with proper pseudo-normed factors* $(E_\alpha)_{\alpha\in\Lambda}$. *The following assertions are equivalent:*

(j) $U\left(\ker\left(f_{\alpha\beta}\right)\right)\subset\ker\left(f_{\alpha\beta}\right)$ *for all* $U\in M(E_\beta)$ *and* $\alpha\le\beta$ *in* Λ;

(jj) *there exists a unique projective system* $\left\{(M(E_\alpha),h_{\alpha\beta}),\ \alpha\le\beta\right\}$ *such that* $h_{\alpha\beta}(U)\circ f_{\alpha\beta}=f_{\alpha\beta}\circ U$ *for all* $U\in M(E_\beta)$ *and* $\alpha\le\beta$ *in* Λ*; furthermore, if* E_α *is complete for all* $\alpha\in\Lambda$*, then* $h_{\alpha\beta}$ *is continuous for all* $\alpha\le\beta$ *in* Λ.

Proof. (jj) $\Rightarrow$ (j): Let $U\in M(E_\beta)$ and $x_\beta\in\ker\left(f_{\alpha\beta}\right)$, then we have $f_{\alpha\beta}\left(U\left(x_\beta\right)\right)=h_{\alpha\beta}\left(U\right)\left(f_{\alpha\beta}\left(x_\beta\right)\right)=0$ and so $U\left(x_\beta\right)\in\ker\left(f_{\alpha\beta}\right)$.

(j) $\Rightarrow$ (jj): Let $\alpha\le\beta$ in Λ and $U\in M(E_\beta)$. Since $U\left(\ker\left(f_{\alpha\beta}\right)\right)\subset\ker\left(f_{\alpha\beta}\right)$ and by using the universal property of the quotient (see Preliminaries), there exists a unique linear map $V\colon E_\alpha\to E_\alpha$ such that $V\circ f_{\alpha\beta}=f_{\alpha\beta}\circ U$. Let $x_\alpha=x+\ker(p_\alpha), y_\alpha=y+\ker(p_\alpha)\in E_\alpha$ where $x,y\in E$. Put $x_\beta=x+\ker(p_\beta)$ and $y_\beta=y+\ker(p_\beta)$, clearly $x_\beta,y_\beta\in E_\beta$. By definition of the map $f_{\alpha\beta}$, we get $f_{\alpha\beta}(x_\beta)=x_\alpha$ and $f_{\alpha\beta}(y_\beta)=y_\alpha$. We have

$$\begin{aligned}V(x_\alpha y_\alpha)&=V\left(f_{\alpha\beta}\left(x_\beta\right)f_{\alpha\beta}\left(y_\beta\right)\right)\\&=V\left(f_{\alpha\beta}\left(x_\beta y_\beta\right)\right)=f_{\alpha\beta}\left(U\left(x_\beta y_\beta\right)\right)\\&=f_{\alpha\beta}\left(x_\beta U\left(y_\beta\right)\right)=f_{\alpha\beta}\left(x_\beta\right)f_{\alpha\beta}\left(U\left(y_\beta\right)\right)\\&=f_{\alpha\beta}\left(x_\beta\right)V\left(f_{\alpha\beta}\left(y_\beta\right)\right)=x_\alpha V(y_\alpha)\end{aligned}$$

and similarly on the other side, so V is a multiplier of E_α. This shows the existence of the map $h_{\alpha\beta}\colon M(E_\beta)\to M(E_\alpha)$ such that $h_{\alpha\beta}\left(U\right)\circ f_{\alpha\beta}=f_{\alpha\beta}\circ U$ for all $U\in M(E_\beta)$ and $\alpha\le\beta$ in Λ. Let $\alpha\le\beta$ in Λ, $U_1,U_2\in M(E_\beta)$ and $\lambda\in\mathbb{C}$,

$$\begin{aligned}h_{\alpha\beta}\left(U_1+\lambda U_2\right)\circ f_{\alpha\beta}&=f_{\alpha\beta}\circ\left(U_1+\lambda U_2\right)\\&=\left(f_{\alpha\beta}\circ U_1\right)+\lambda\left(f_{\alpha\beta}\circ U_2\right)\\&=h_{\alpha\beta}\left(U_1\right)\circ f_{\alpha\beta}+\lambda h_{\alpha\beta}\left(U_2\right)\circ f_{\alpha\beta}\\&=\left(h_{\alpha\beta}\left(U_1\right)+\lambda h_{\alpha\beta}\left(U_2\right)\right)\circ f_{\alpha\beta},\end{aligned}$$

hence $h_{\alpha\beta}\left(U_1+\lambda U_2\right)=h_{\alpha\beta}\left(U_1\right)+\lambda h_{\alpha\beta}\left(U_2\right)$ since $f_{\alpha\beta}$ is surjective. Also,

$$\begin{aligned}h_{\alpha\beta}\left(U_1\circ U_2\right)\circ f_{\alpha\beta}&=f_{\alpha\beta}\circ U_1\circ U_2\\&=h_{\alpha\beta}\left(U_1\right)\circ f_{\alpha\beta}\circ U_2\\&=h_{\alpha\beta}\left(U_1\right)\circ h_{\alpha\beta}\left(U_2\right)\circ f_{\alpha\beta},\end{aligned}$$

then $h_{\alpha\beta}(U_1 \circ U_2) = h_{\alpha\beta}(U_1) \circ h_{\alpha\beta}(U_2)$ since $f_{\alpha\beta}$ is surjective. Let $\alpha \leq \beta \leq \gamma$ in Λ and $W \in M(E_\gamma)$,

$$\begin{aligned}(h_{\alpha\beta} \circ h_{\beta\gamma})(W) \circ f_{\alpha\gamma} &= h_{\alpha\beta}(h_{\beta\gamma}(W)) \circ f_{\alpha\beta} \circ f_{\beta\gamma} \\ &= f_{\alpha\beta} \circ h_{\beta\gamma}(W) \circ f_{\beta\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma} \circ W \\ &= f_{\alpha\gamma} \circ W = h_{\alpha\gamma}(W) \circ f_{\alpha\gamma},\end{aligned}$$

consequently $(h_{\alpha\beta} \circ h_{\beta\gamma})(W) = h_{\alpha\gamma}(W)$ since $f_{\alpha\gamma}$ is surjective. Thus $h_{\alpha\beta} \circ h_{\beta\gamma} = h_{\alpha\gamma}$. Let $\alpha \in \Lambda$, if E_α is complete, then every multiplier of E_α is continuous. Now by assuming that E_α is complete for all $\alpha \in \Lambda$, we will show that $h_{\alpha\beta}$ is continuous for all $\alpha \leq \beta$ in Λ (see also, the proof of Theorem 2.12 in [5]). For $\alpha \in \Lambda$ and $r \gneqq 0$, let $B_\alpha(0,r) = \{x_\alpha \in E_\alpha : \overline{p}_\alpha(x_\alpha) \leq r\}$. We denote by $\|\cdot\|_\alpha$ the operator pseudo-norm on $M(E_\alpha)$. Let $\alpha \leq \beta$ in Λ, $f_{\alpha\beta}$ is open by the open mapping theorem, so there is $\lambda \gneqq 0$ such that $\lambda B_\alpha(0,1) \subset f_{\alpha\beta}(B_\beta(0,1))$, i.e., $B_\alpha(0,1) \subset f_{\alpha\beta}(B_\beta(0,r))$ where $r = \lambda^{-k_\beta}$ and k_β is the homogenity index of $\overline{p}_\beta$. Let $U \in M(E_\beta)$,

$$\begin{aligned}\|h_{\alpha\beta}(U)\|_\alpha &= \sup\{\overline{p}_\alpha(h_{\alpha\beta}(U)(f_\alpha(x))) : f_\alpha(x) \in B_\alpha(0,1)\} \\ &\leq \sup\{\overline{p}_\alpha(h_{\alpha\beta}(U)(f_{\alpha\beta}(f_\beta(x)))) : f_\beta(x) \in B_\beta(0,r)\} \\ &= \sup\{\overline{p}_\alpha(f_{\alpha\beta}(U(f_\beta(x)))) : f_\beta(x) \in B_\beta(0,r)\} \\ &\leq \sup\left\{\overline{p}_\beta(U(f_\beta(x))) : f_\beta(x) \in B_\beta(0,r)\right\} \\ &\leq \sup\left\{\|U\|_\beta\, \overline{p}_\beta(f_\beta(x)) : f_\beta(x) \in B_\beta(0,r)\right\} \\ &= r\|U\|_\beta.\end{aligned}$$

Therefore $h_{\alpha\beta}$ is continuous. □

THEOREM 2.3. *Let $\left(E, (p_\alpha)_{\alpha\in\Lambda}\right)$ be a complete locally m-pseudoconvex algebra with proper pseudo-normed factors $(E_\alpha)_{\alpha\in\Lambda}$. Assume that E satisfies conditions (i) and (j). Then $M(E) \cong \varprojlim M(E_\alpha)$ within an algebra isomorphism φ. Furthermore, if each factor E_α is complete, then every multiplier of E is continuous and φ is a topological algebra isomorphism where $M(E)$ is endowed with its pseudo-seminorm topology.*

PROOF. Take $T \in M(E)$. By Propositions 2.1 and 2.2,

$$(T_\alpha)_{\alpha\in\Lambda} \in \prod_{\alpha\in\Lambda} M(E_\alpha),$$

$T_\alpha \circ f_{\alpha\beta} = f_{\alpha\beta} \circ T_\beta$ and $h_{\alpha\beta}(T_\beta) \circ f_{\alpha\beta} = f_{\alpha\beta} \circ T_\beta$ for all $\alpha \leq \beta$ in Λ. Hence $h_{\alpha\beta}(T_\beta) \circ f_{\alpha\beta} = T_\alpha \circ f_{\alpha\beta}$ and consequently $h_{\alpha\beta}(T_\beta) = T_\alpha$ since the map $f_{\alpha\beta}$ is surjective. This shows that $(T_\alpha)_{\alpha\in\Lambda} \in \varprojlim M(E_\alpha)$. Thus the map $\varphi\colon M(E) \to \varprojlim M(E_\alpha)$, $T \to (T_\alpha)_{\alpha\in\Lambda}$, is well defined. We will show that φ is an algebra isomorphism. Let $T, S \in M(E)$ and $\lambda \in \mathbb{C}$, $T_\alpha \circ f_\alpha = f_\alpha \circ T$ and $S_\alpha \circ f_\alpha = f_\alpha \circ S$, then $(T_\alpha + \lambda S_\alpha) \circ f_\alpha = f_\alpha \circ (T + \lambda S)$, so $(T + \lambda S)_\alpha = T_\alpha + \lambda S_\alpha$ by Proposition 2.1. Also, $T_\alpha \circ S_\alpha \circ f_\alpha = T_\alpha \circ f_\alpha \circ S = f_\alpha \circ T \circ S$, hence $(T \circ S)_\alpha = T_\alpha \circ S_\alpha$ by Proposition 2.1. Let $T \in M(E)$, if $T_\alpha = 0$ for all $\alpha \in \Lambda$, then $f_\alpha \circ T = T_\alpha \circ f_\alpha = 0$ for all $\alpha \in \Lambda$ and consequently $T = 0$. Let $(U_\alpha)_{\alpha\in\Lambda} \in \varprojlim M(E_\alpha)$ and define the map $T = \Phi^{-1} \circ \varprojlim U_\alpha \circ \Phi\colon E \to E$ where $\varprojlim U_\alpha$ is the multiplier of $\varprojlim E_\alpha$ defined by

$$\left(\varprojlim U_\alpha\right)(x_\alpha)_\alpha = (U_\alpha(x_\alpha))_\alpha$$

and $\Phi\colon E \to \varprojlim E_\alpha$ is the topological algebra isomorphism given by $\Phi(x) = (f_\alpha(x))_\alpha$. Clearly T is a multiplier of E, also $f_\alpha \circ T = f_\alpha \circ \Phi^{-1} \circ \varprojlim U_\alpha \circ \Phi = U_\alpha \circ f_\alpha$ for all $\alpha \in \Lambda$, so $\varphi(T) = (U_\alpha)_\alpha$. If E_α is complete for all $\alpha \in \Lambda$, then every multiplier of E_α is continuous, hence every multiplier of E is continuous by Proposition 2.1. The pseudo-seminorm topology on $M(E)$ is the topology defined by the family of pseudo-seminorms $q_\alpha(T) = \|T_\alpha\|_\alpha$, $\alpha \in \Lambda$, so φ is a topological algebra isomorphism. □

PROPOSITION 2.4. *Let $\left(E, (p_\alpha)_{\alpha\in\Lambda}\right)$ be a locally m-pseudoconvex algebra with approximate identity $(e_\omega)_{\omega\in\Omega}$. Then E satisfies conditions (i) and (j).*

PROOF. Let $T \in M(E)$, $x \in \ker(f_\alpha)$ and $\omega \in \Omega$,

$$\begin{aligned} f_\alpha(T(x)) &= f_\alpha(T(x - xe_\omega + xe_\omega)) \\ &= f_\alpha(T(x) - T(xe_\omega)) + f_\alpha(T(xe_\omega)) \\ &= f_\alpha(T(x) - T(x)e_\omega) + f_\alpha(xT(e_\omega)) \\ &= f_\alpha(T(x) - T(x)e_\omega) + f_\alpha(x) f_\alpha(T(e_\omega)) \\ &= f_\alpha(T(x) - T(x)e_\omega). \end{aligned}$$

Since $T(x)e_\omega \to_\omega T(x)$ and f_α is continuous, we deduce that $f_\alpha(T(x)) = 0$. Now we will show that $U(\ker(f_{\alpha\beta})) \subset \ker(f_{\alpha\beta})$ for all $U \in M(E_\beta)$ and $\alpha \leq \beta$ in Λ. Since $(e_\omega)_{\omega\in\Omega}$ is an approximate identity in E and $f_\beta\colon E \to E_\beta$ is a surjective continuous homomorphism, it follows that $(f_\beta(e_\omega))_{\omega\in\Omega}$ is an approximate identity in E_β (see [8, Theorem 4.1]). Let $U \in M(E_\beta)$, $x_\beta \in \ker(f_{\alpha\beta})$ and $\omega \in \Omega$,

$$\begin{aligned} f_{\alpha\beta}(U(x_\beta)) &= f_{\alpha\beta}(U(x_\beta - x_\beta f_\beta(e_\omega) + x_\beta f_\beta(e_\omega))) \\ &= f_{\alpha\beta}(U(x_\beta) - U(x_\beta f_\beta(e_\omega))) + f_{\alpha\beta}(U(x_\beta f_\beta(e_\omega))) \\ &= f_{\alpha\beta}(U(x_\beta) - U(x_\beta) f_\beta(e_\omega)) + f_{\alpha\beta}(x_\beta U(f_\beta(e_\omega))) \\ &= f_{\alpha\beta}(U(x_\beta) - U(x_\beta) f_\beta(e_\omega)) + f_{\alpha\beta}(x_\beta) f_{\alpha\beta}(U(f_\beta(e_\omega))) \\ &= f_{\alpha\beta}(U(x_\beta) - U(x_\beta) f_\beta(e_\omega)). \end{aligned}$$

Since $U(x_\beta) f_\beta(e_\omega) \to_\omega U(x_\beta)$ and $f_{\alpha\beta}$ is continuous, we deduce that

$$f_{\alpha\beta}(U(x_\beta)) = 0.$$ □

COROLLARY 2.5 ([5, Theorems 2.6 and 2.12]). *Let $\left(E, (p_\alpha)_{\alpha\in\Lambda}\right)$ be a complete locally m-pseudoconvex algebra with approximate identity. Suppose that each factor $E_\alpha = E/\ker(p_\alpha)$ in the generalized Arens-Michael decomposition of E is complete. Then every multiplier of E is continuous and $M(E) \cong \varprojlim M(E_\alpha)$ within a topological algebra isomorphism where $M(E)$ is endowed with its pseudo-seminorm topology.*

PROOF. It follows from Theorem 2.3 and Proposition 2.4. □

COROLLARY 2.6. *Let $\left(E, (p_\alpha)_{\alpha\in\Lambda}\right)$ be a locally C^*-algebra. Then every multiplier of E is continuous and $M(E) \cong \varprojlim M(E_\alpha)$ within a topological algebra isomorphism where $M(E)$ is endowed with its seminorm topology.*

PROOF. By [7, Theorem 2.6] and [10, Corollary 1.12], E has an approximate identity and each factor E_α is complete. □

Now we will describe multiplier algebras of complete uniform topological algebras.

PROPOSITION 2.7. *Let $\left(E, (p_\alpha)_{\alpha\in\Lambda}\right)$ be a uniform topological algebra. Then*

$$\ker(f_\alpha) = \cap\{\ker(\chi) : \chi \in \Delta_\alpha(E)\}$$

for all $\alpha \in \Lambda$ and

$$\ker(f_{\alpha\beta}) = \cap\{\ker(\mu \circ f_{\alpha\beta}) : \mu \in \Delta(E_\alpha)\}$$

for all $\alpha \leq \beta$ in Λ.

Proof. First, we will show that $\Delta_\alpha(E)$ and $\Delta(E_\alpha)$ are non empty sets. Let F_α be the completion of $(E_\alpha, \overline{p}_\alpha)$, F_α is a uniform Banach algebra. By [8, Lemma 5.1], F_α is commutative and semisimple. Then $\Delta(F_\alpha)$ is a non empty set since F_α is not a radical algebra, hence $\Delta_\alpha(E)$ and $\Delta(E_\alpha)$ are non empty sets (see [9, Proposition 7.5]).

By [1, Theorem 6], $p_\alpha(x) = \sup\{|\chi(x)| : \chi \in \Delta_\alpha(E)\}$ for all $x \in E$ and $\alpha \in \Lambda$, then $\ker(f_\alpha) = \ker(p_\alpha) = \cap\{\ker(\chi) : \chi \in \Delta_\alpha(E)\}$ for all $\alpha \in \Lambda$. Let $\alpha \leq \beta$ in Λ and $x_\beta \in E_\beta$,

$$\begin{aligned} x_\beta \in \ker(f_{\alpha\beta}) \;\Leftrightarrow\; f_{\alpha\beta}(x_\beta) = 0 \;\Leftrightarrow\; \mu(f_{\alpha\beta}(x_\beta)) = 0 \;\text{ for all } \mu \in \Delta(E_\alpha) \\ \Leftrightarrow\; x_\beta \in \cap\{\ker(\mu \circ f_{\alpha\beta}) : \mu \in \Delta(E_\alpha)\}. \end{aligned}$$ □

Proposition 2.8. *Let $\left(E, (p_\alpha)_{\alpha\in\Lambda}\right)$ be a uniform topological algebra. Then E satisfies conditions (i) and (j).*

Proof. By Proposition 2.7, $\ker(f_\alpha) = \cap\{\ker(\chi) : \chi \in \Delta_\alpha(E)\}$ for all $\alpha \in \Lambda$. If T is a multiplier of E, then $T(\ker(\chi)) \subset \ker(\chi)$ for all $\chi \in \Delta_\alpha(E)$ by [6, Theorem 2.9] and [8, Lemma 5.1], so

$$\begin{aligned} T(\ker(f_\alpha)) &= T(\cap\{\ker(\chi) : \chi \in \Delta_\alpha(E)\}) \\ &\subset \cap\{T(\ker(\chi)) : \chi \in \Delta_\alpha(E)\} \\ &\subset \cap\{\ker(\chi) : \chi \in \Delta_\alpha(E)\} = \ker(f_\alpha). \end{aligned}$$

By Proposition 2.7, $\ker(f_{\alpha\beta}) = \cap\{\ker(\mu \circ f_{\alpha\beta}) : \mu \in \Delta(E_\alpha)\}$ for all $\alpha \leq \beta$ in Λ. If U is a multiplier of E_β, then $U(\ker(\delta)) \subset \ker(\delta)$ for all $\delta \in \Delta(E_\beta)$ by [6, Theorem 2.9] and [8, Lemma 5.1], so $U(\ker(\mu \circ f_{\alpha\beta})) \subset \ker(\mu \circ f_{\alpha\beta})$ for all $\mu \in \Delta(E_\alpha)$, and consequently

$$\begin{aligned} U(\ker(f_{\alpha\beta})) &= U(\cap\{\ker(\mu \circ f_{\alpha\beta}) : \mu \in \Delta(E_\alpha)\}) \\ &\subset \cap\{U(\ker(\mu \circ f_{\alpha\beta})) : \mu \in \Delta(E_\alpha)\} \\ &\subset \cap\{\ker(\mu \circ f_{\alpha\beta}) : \mu \in \Delta(E_\alpha)\} = \ker(f_{\alpha\beta}). \end{aligned}$$ □

Theorem 2.9. *Let $\left(E, (p_\alpha)_{\alpha\in\Lambda}\right)$ be a complete uniform topological algebra. Then $M(E) \cong \varprojlim M(E_\alpha)$ within an algebra isomorphism φ. Furthermore, if each factor E_α is complete, then every multiplier of E is continuous and φ is a topological algebra isomorphism where $M(E)$ is endowed with its seminorm topology.*

Proof. It follows from Theorem 2.3 and Proposition 2.8. □

Remark. Let $\left(E,(p_\alpha)_{\alpha\in\Lambda}\right)$ be a complete uniform topological algebra which is also a symmetric *-algebra. Then $\chi(x^*)=\overline{\chi(x)}$ for all $x\in E$ and $\chi\in\Delta(E)$ (see [9, Lemma 6.4]). Take $x\in E$ and $\alpha\in\Lambda$. By [1, Theorem 6],

$$\begin{aligned} p_\alpha(x^*x) &= \sup\{|\chi(x^*x)| :\ \chi\in\Delta_\alpha(E)\} = \sup\left\{|\chi(x)|^2 :\ \chi\in\Delta_\alpha(E)\right\} \\ &= (\sup\{|\chi(x)| :\ \chi\in\Delta_\alpha(E)\})^2 = p_\alpha(x)^2. \end{aligned}$$

Therefore $\left(E,(p_\alpha)_{\alpha\in\Lambda}\right)$ is a locally C^*-algebra, and so each factor E_α is complete.

As an application of previous results, we deduce the Arhippainen unitization theorem [1, Theorem 4] on uniform topological algebras.

Proposition 2.10. *Let $\left(E,(p_\alpha)_{\alpha\in\Lambda}\right)$ be a uniform topological algebra, and let $M_c(E)$ be the algebra of all continuous multipliers of E. Then there is a family of seminorms $(q_\alpha)_{\alpha\in\Lambda}$ on $M_c(E)$ such that*

1. *$\left(M_c(E),(q_\alpha)_{\alpha\in\Lambda}\right)$ is a uniform topological algebra;*
2. *the map $L\colon E\to M_c(E)$, $L(x)(y)=xy$, is an algebra isomorphism (into) and $q_\alpha(L(x))=p_\alpha(x)$ for all $x\in E$ and $\alpha\in\Lambda$.*

Proof. 1. By Propositions 2.1, 2.2 and 2.8, we define the map $\psi\colon M_c(E)\to \varprojlim M_c(E_\alpha)$, $T\to(T_\alpha)_{\alpha\in\Lambda}$. As in the proof of Theorem 2.3, ψ is an injective homomorphism. We endow $M_c(E)$ with the topology defined by the family of seminorms $q_\alpha(T)=\|T_\alpha\|_\alpha$, $\alpha\in\Lambda$, where $\|.\|_\alpha$ is the operator norm on $M_c(E_\alpha)$. Let $T\in M_c(E)$,

$$q_\alpha\left(T^2\right)=\|\left(T^2\right)_\alpha\|_\alpha=\|(T_\alpha)^2\|_\alpha=\|T_\alpha\|_\alpha^2=q_\alpha(T)^2$$

since $\|.\|_\alpha$ has the square property. Let $T\in M_c(E)$ with $q_\alpha(T)=0$ for all $\alpha\in\Lambda$, then $T_\alpha=0$ for all $\alpha\in\Lambda$, so $T=0$ since ψ is injective.

2. Since E is proper, L is an algebra isomorphism (into). Let $x\in E$ and $\alpha\in\Lambda$, $(L(x))_\alpha\circ f_\alpha=f_\alpha\circ L(x)$, then

$$(L(x))_\alpha(f_\alpha(y))=(f_\alpha\circ L(x))(y)=f_\alpha(xy)=f_\alpha(x)f_\alpha(y)$$

for all $y\in E$. Since the map $l\colon(E_\alpha,\overline{p}_\alpha)\to(M_c(E_\alpha),\|.\|_\alpha)$, $l(x_\alpha)(y_\alpha)=x_\alpha y_\alpha$, is an isometric isomorphism (into), it follows that

$$\|(L(x))_\alpha\|_\alpha=\overline{p}_\alpha(f_\alpha(x))=p_\alpha(x),$$

so $q_\alpha(L(x))=p_\alpha(x)$. □

Proposition 2.11. *Let E be a uniform topological algebra without unit, and let E_e be the algebra obtained from E by adjoining the unit. Then the map $g\colon E_e \to M_c(E)$, $g((x,\lambda)) = L(x) + \lambda I$ is an algebra isomorphism (into).*

Proof. It is easy to show that g is an algebra homomorphism. Let $(x,\lambda) \in E_e$ with $g((x,\lambda)) = 0$, then $L(x) = -\lambda I$. Suppose $\lambda \neq 0$, $I = -\lambda^{-1} L(x) = L\left(-\lambda^{-1} x\right)$, so $-\lambda^{-1}x$ is a left unit in E. Since E is commutative, $-\lambda^{-1}x$ is a unit in E, a contradiction. Thus $L(x) = 0$ and consequently $x = 0$ since E is proper. □

Corollary 2.12 ([1, Theorem 4]). *Let $\left(E, (p_\alpha)_{\alpha\in\Lambda}\right)$ be a uniform topological algebra without unit. Then there is a family of seminorms $(s_\alpha)_{\alpha\in\Lambda}$ on E_e such that $\left(E_e, (s_\alpha)_{\alpha\in\Lambda}\right)$ is a uniform topological algebra and $s_\alpha((x,0)) = p_\alpha(x)$ for all $x \in E$ and $\alpha \in \Lambda$.*

Proof. For each $\alpha \in \Lambda$, we define a seminorm on E_e by

$$s_\alpha((x,\lambda)) = q_\alpha(L(x) + \lambda I) \qquad \text{for all } x \in E \text{ and } \lambda \in \mathbb{C}.$$

By Propositions 2.10 and 2.11, $\left(E_e, (s_\alpha)_{\alpha\in\Lambda}\right)$ is a uniform topological algebra and $s_\alpha((x,0)) = q_\alpha(L(x)) = p_\alpha(x)$ for all $x \in E$. □

Remark. We have

$$s_\alpha((x,\lambda)) = q_\alpha(L(x) + \lambda I) \leq q_\alpha(L(x)) + |\lambda| q_\alpha(I) = p_\alpha(x) + |\lambda|$$

for all $x \in E$ and $\lambda \in \mathbb{C}$. This shows that the topology on E_e defined by the family of seminorms $(s_\alpha)_{\alpha\in\Lambda}$ is weaker than the usual topology on E_e defined by the family of seminorms $(\tilde{p}_\alpha)_{\alpha\in\Lambda}$ where $\tilde{p}_\alpha((x,\lambda)) = p_\alpha(x) + |\lambda|$.

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