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LIE DERIVATIONS ON TRIVIAL EXTENSION ALGEBRAS

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Abstract. In this paper we provide some conditions under which a Lie derivation on a trivial extension algebra is proper, that is, it can be expressed as a sum of a derivation and a center valued map vanishing at commutators. We then apply our results for triangular algebras. Some illuminating examples are also included.

1. Introduction

Let $\mathfrak{A}$ be a unital algebra (over a commutative unital ring $\mathbf{R}$) and $\mathfrak{X}$ be an $\mathfrak{A}$-bimodule. A linear mapping $\mathcal{D}$ from $\mathfrak{A}$ into $\mathfrak{X}$ is said to be a *derivation* if

$$\mathcal{D}(ab) = \mathcal{D}(a)b + a\mathcal{D}(b), \quad a, b \in \mathfrak{A}.$$

A linear mapping $\mathcal{L}\colon \mathfrak{A} \to \mathfrak{X}$ is called a *Lie derivation* if

$$\mathcal{L}[a, b] = [\mathcal{L}(a), b] + [a, \mathcal{L}(b)], \quad a, b \in \mathfrak{A},$$

where $[\cdot,\cdot]$ stands for the Lie bracket. Trivially every derivation is a Lie derivation. If $\mathcal{D}\colon \mathfrak{A} \to \mathfrak{A}$ is a derivation and $\ell\colon \mathfrak{A} \to Z(\mathfrak{A})(:=$ the center of $\mathfrak{A})$ is a linear map, then $\mathcal{D} + \ell$ is a Lie derivation if and only if $\ell([a, b]) = 0$, for all

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$a, b \in \mathfrak{A}$. Lie derivations of this form are called *proper Lie derivations*. A problem that we are dealing with is studying those conditions on an algebra such that every Lie derivation on it is proper. We say that an algebra $\mathfrak{A}$ has Lie derivation property if every Lie derivation on $\mathfrak{A}$ is proper.

Martindale [10] was the first one who showed that every Lie derivation on certain primitive ring is proper. Cheung [3] initiated the study of various mappings on triangular algebras; in particular, he investigated the properness of Lie derivations on triangular algebras (see also [4, 9, 12]). Cheung's results [4] have recently extended by Du and Wang [5] for a generalized matrix algebras. Wang [14] studied Lie n-derivations on a unital algebra with a nontrivial idempotent. Lie triple derivations on a unital algebra with a nontrivial idempotent have recently investigated by Benkovič [2].

In this paper we study Lie derivations on a trivial extension algebra. Let $\mathfrak{X}$ be an $\mathfrak{A}$-bimodule, then the direct product $\mathfrak{A} \times \mathfrak{X}$ together with the pairwise addition, scalar product and the algebra multiplication defined by

$$(a, x)(b, y) = (ab, ay + xb), \quad a, b \in \mathfrak{A}, x, y \in \mathfrak{X},$$

is a unital algebra which is called a trivial extension of $\mathfrak{A}$ by $\mathfrak{X}$ and will be denoted by $\mathfrak{A} \ltimes \mathfrak{X}$. For example, every triangular algebra $\mathrm{Tri}(\mathcal{A}, \mathfrak{X}, \mathcal{B})$ is a trivial extension algebra. Indeed, it can be identified with the trivial extension algebra $(\mathcal{A} \oplus \mathcal{B}) \ltimes \mathfrak{X}$; (see Sec. 3).

Trivial extension algebras are known as a rich source of (counter-)examples in various situations in functional analysis. Some aspects of (Banach) algebras of this type have been investigated in [1] and [15]. Derivations into various duals of a trivial extension (Banach) algebra studied in [15]. Jordan (higher) derivations on a trivial extension algebra are discussed in [11] (see also [6], [8] and [7]).

The main aim of this paper is providing some conditions under which a trivial extension algebra has the Lie derivation property. We are mainly dealing with those $\mathfrak{A} \ltimes \mathfrak{X}$ for which $\mathfrak{A}$ enjoys a nontrivial idempotent p satisfying

$$(\star) \qquad pxq = x,$$

for all $x \in \mathfrak{X}$, where $q = 1 - p$. A triangular algebra is the main example of a trivial extension algebra satisfying $(\star)$.

In Section 2, we characterize the properness of a Lie derivation on $\mathfrak{A} \ltimes \mathfrak{X}$ (Theorem 2.2), from which we derive Theorem 2.3, providing some sufficient conditions ensuring the Lie derivation property for $\mathfrak{A} \ltimes \mathfrak{X}$. In Section 3, we apply our results for a triangular algebra, recovering some results of [4].

2. Proper Lie derivations on $\mathfrak{A} \ltimes \mathfrak{X}$

We commence with the following elementary lemma describing the structures of derivations and Lie derivations on a trivial extension algebra $\mathfrak{A} \ltimes \mathfrak{X}$.

Lemma 2.1. *Let $\mathfrak{A}$ be a unital algebra and $\mathfrak{X}$ be an $\mathfrak{A}$-bimodule. Then every linear map $\mathcal{L}\colon \mathfrak{A} \ltimes \mathfrak{X} \to \mathfrak{A} \ltimes \mathfrak{X}$ has the presentation*

$$\mathcal{L}(a,x) = (\mathcal{L}_{\mathfrak{A}}(a) + T(x), \mathcal{L}_{\mathfrak{X}}(a) + S(x)), \quad a \in \mathfrak{A}, x \in \mathfrak{X}, \tag{2.1}$$

for some linear mappings $\mathcal{L}_{\mathfrak{A}}\colon \mathfrak{A} \to \mathfrak{A}$, $\mathcal{L}_{\mathfrak{X}}\colon \mathfrak{A} \to \mathfrak{X}$, $T\colon \mathfrak{X} \to \mathfrak{A}$ and $S\colon \mathfrak{X} \to \mathfrak{X}$. Moreover,

- *$\mathcal{L}$ is a Lie derivation if and only if*
  (a) *$\mathcal{L}_{\mathfrak{A}}$ and $\mathcal{L}_{\mathfrak{X}}$ are Lie derivations;*
  (b) $T([a,x]) = [a, T(x)]$ and $[T(x), y] = [T(y), x]$;
  (c) $S([a,x]) = [\mathcal{L}_{\mathfrak{A}}(a), x] + [a, S(x)]$,
  for all $a \in \mathfrak{A}, x, y \in \mathfrak{X}$.
- *$\mathcal{L}$ is a derivation if and only if*
  (i) *$\mathcal{L}_{\mathfrak{A}}$ and $\mathcal{L}_{\mathfrak{X}}$ are derivations;*
  (ii) $T(ax) = aT(x)$, $T(xa) = T(x)a$ and $xT(y) + T(x)y = 0$;
  (iii) $S(ax) = aS(x) + \mathcal{L}_{\mathfrak{A}}(a)x$ and $S(xa) = S(x)a + x\mathcal{L}_{\mathfrak{A}}(a)$,
  for all $a \in \mathfrak{A}, x, y \in \mathfrak{X}$.

It can be simply verified that the center $Z(\mathfrak{A} \ltimes \mathfrak{X})$ of $\mathfrak{A} \ltimes \mathfrak{X}$ is

$$\begin{aligned} Z(\mathfrak{A} \ltimes \mathfrak{X}) &= \{(a,x); a \in Z(\mathfrak{A}), [b,x] = 0 = [a,y] \text{ for all } b \in \mathfrak{A}, y \in \mathfrak{X}\} \\ &= \pi_{\mathfrak{A}}(Z(\mathfrak{A} \ltimes \mathfrak{X})) \times \pi_{\mathfrak{X}}(Z(\mathfrak{A} \ltimes \mathfrak{X})), \end{aligned}$$

where $\pi_{\mathfrak{A}}\colon \mathfrak{A} \ltimes \mathfrak{X} \to \mathfrak{A}$ and $\pi_{\mathfrak{X}}\colon \mathfrak{A} \ltimes \mathfrak{X} \to \mathfrak{X}$ are the natural projections given by $\pi_{\mathfrak{A}}(a,x) = a$ and $\pi_{\mathfrak{X}}(a,x) = x$, respectively.

It should be noticed that, if $\mathfrak{A} \ltimes \mathfrak{X}$ satisfies $(\star)$, then the equality $[p,x] = 0$ implies $x = 0$, for any $x \in \mathfrak{X}$. This leads to $\pi_{\mathfrak{X}}(Z(\mathfrak{A} \ltimes \mathfrak{X})) = \{0\}$, and so

$$\begin{aligned} Z(\mathfrak{A} \ltimes \mathfrak{X}) &= \{(a,0); a \in Z(\mathfrak{A}), [a,x] = 0 \text{ for all } x \in \mathfrak{X}\} \\ &= \pi_{\mathfrak{A}}(Z(\mathfrak{A} \ltimes \mathfrak{X})) \times \{0\}. \end{aligned} \tag{2.2}$$

Further, the property $(\star)$ also implies the following simplifications on the module operations which will be frequently used in the sequel

$$qx = 0 = xp, \ px = x = xq, \ papx = ax, \ xqaq = xa, \quad a \in \mathfrak{A}, x \in \mathfrak{X}. \tag{2.3}$$

The following characterization theorem which is a generalization of [4, Theorem 6] studies the properness of a Lie derivation on $\mathfrak{A} \ltimes \mathfrak{X}$. Before proceeding, we recall that an $\mathfrak{A}$-bimodule $\mathfrak{X}$ is called 2-torsion free if $2x = 0$ implies $x = 0$, for any $x \in \mathfrak{X}$.

Theorem 2.2. *Suppose that the trivial extension algebra $\mathfrak{A} \ltimes \mathfrak{X}$ satisfies $(\star)$ and that both $\mathfrak{A}$ and $\mathfrak{X}$ are 2-torsion free. Then a Lie derivation $\mathcal{L}$ on $\mathfrak{A} \ltimes \mathfrak{X}$ of the form*

$$\mathcal{L}(a, x) = (\mathcal{L}_{\mathfrak{A}}(a) + T(x), \mathcal{L}_{\mathfrak{X}}(a) + S(x)), \quad a \in \mathfrak{A}, x \in \mathfrak{X},$$

is proper if and only if there exists a linear map $\ell_{\mathfrak{A}} \colon \mathfrak{A} \to Z(\mathfrak{A})$ satisfying the following conditions:

(i) $\mathcal{L}_{\mathfrak{A}} - \ell_{\mathfrak{A}}$ *is a derivation on* $\mathfrak{A}$.

(ii) $[\ell_{\mathfrak{A}}(pap), x] = 0 = [\ell_{\mathfrak{A}}(qaq), x]$ *for all* $a \in \mathfrak{A}, x \in \mathfrak{X}$.

Proof. By Lemma 2.1 every Lie derivation on $\mathfrak{A} \ltimes \mathfrak{X}$ can be expressed in the from

$$\mathcal{L}(a, x) = (\mathcal{L}_{\mathfrak{A}}(a) + T(x), \mathcal{L}_{\mathfrak{X}}(a) + S(x)),$$

where $\mathcal{L}_{\mathfrak{A}} \colon \mathfrak{A} \to \mathfrak{A}$, $\mathcal{L}_{\mathfrak{X}} \colon \mathfrak{A} \to \mathfrak{X}$ are Lie derivations and $T \colon \mathfrak{X} \to \mathfrak{A}$, $S \colon \mathfrak{X} \to \mathfrak{X}$ are linear mappings satisfying

$$T([a, x]) = [a, T(x)], \quad [T(x), y] = [T(y), x], \quad \text{and} \quad S([a, x]) = [\mathcal{L}_{\mathfrak{A}}(a), x] + [a, S(x)],$$

for all $a \in \mathfrak{A}, x, y \in \mathfrak{X}$.

To prove "if" part, we set

$$\mathcal{D}(a, x) = \big((\mathcal{L}_{\mathfrak{A}} - \ell_{\mathfrak{A}})(a) + T(x), \mathcal{L}_{\mathfrak{X}}(a) + S(x)\big) \quad \text{and} \quad \ell(a, x) = (\ell_{\mathfrak{A}}(a), 0), \quad a \in \mathfrak{A}, x \in \mathfrak{X}.$$

Then clearly $\mathcal{L} = \mathcal{D} + \ell$. That ℓ is linear and $\ell(\mathfrak{A} \ltimes \mathfrak{X}) \subseteq Z(\mathfrak{A} \ltimes \mathfrak{X})$ follows trivially from $\ell_{\mathfrak{A}}(\mathfrak{A}) \subseteq Z(\mathfrak{A})$ and (2.2). It remains to show that $\mathcal{D}$ is a derivation on $\mathfrak{A} \ltimes \mathfrak{X}$. To do this we use Lemma 2.1. It should be mentioned that in the rest of proof we frequently making use the equalities in (2.3). First we have,

$$\begin{aligned} S(ax) &= S([pap, x]) \\ &= [\mathcal{L}_{\mathfrak{A}}(pap), x] + [pap, S(x)] \end{aligned}$$

$$= [(\mathcal{L}_{\mathfrak{A}} - \ell_{\mathfrak{A}})(pap), x] + [\ell_{\mathfrak{A}}(pap), x] + aS(x)$$

$$= [\big((\mathcal{L}_{\mathfrak{A}} - \ell_{\mathfrak{A}})(p)ap + p(\mathcal{L}_{\mathfrak{A}} - \ell_{\mathfrak{A}})(a)p + pa(\mathcal{L}_{\mathfrak{A}} - \ell_{\mathfrak{A}})(p)\big), x]$$

$$+ [\ell_{\mathfrak{A}}(pap), x] + aS(x)$$

$$= (\mathcal{L}_{\mathfrak{A}} - \ell_{\mathfrak{A}})(a)x + [\ell_{\mathfrak{A}}(pap), x] + aS(x), \tag{2.4}$$

for all $a \in \mathfrak{A}, x \in \mathfrak{X}$. Now the condition $[\ell_{\mathfrak{A}}(pap), x] = 0$ implies that $S(xa) = (\mathcal{L}_{\mathfrak{A}} - \ell_{\mathfrak{A}})(a)x + aS(x)$. With a similar procedure as above, from $[\ell_{\mathfrak{A}}(qaq), x] = 0$ we get $S(xa) = x(\mathcal{L}_{\mathfrak{A}} - \ell_{\mathfrak{A}})(a) + S(x)a$ for all $a \in \mathfrak{A}, x \in \mathfrak{X}$.

From the equality

$$T(x) = T([p, x]) = [p, T(x)] = pT(x) - T(x)p, \quad x \in \mathfrak{X},$$

we arrive at $yT(x) = 0 = T(x)y$ and so $yT(x) + T(y)x = 0$ for all $y, x \in \mathfrak{X}$. It also follows that $pT(x)p = 0$, $qT(x)q = 0$ and $qT(x)p = 0$ for all $x \in \mathfrak{X}$; note that $\mathfrak{A}$ is 2-torsion free.

The equality

$$0 = T([qap, x]) = [qap, T(x)] = qapT(x) - T(x)qap, \quad a \in \mathfrak{A}, x \in \mathfrak{X},$$

gives $qapT(x) = T(x)qap$ for all $a \in \mathfrak{A}, x \in \mathfrak{X}$. The latter relation together with the equality

$$T(ax) = T[pa, x] = paT(x) - T(x)pa, \quad a \in \mathfrak{A}, x \in \mathfrak{X},$$

lead us to $T(ax) = pT(ax)q = paT(x)q = aT(x)$ for all $a \in \mathfrak{A}, x \in \mathfrak{X}$. By a similar argument we get $T(xa) = T(x)a$ for all $a \in \mathfrak{A}, x \in \mathfrak{X}$.

Next, we set $\phi(a) = \mathcal{L}_{\mathfrak{X}}(paq)$, then ϕ is a derivation. Indeed, for each $a, b \in \mathfrak{A}$,

$$\phi(ab) = \mathcal{L}_{\mathfrak{X}}(pabq) = \mathcal{L}_{\mathfrak{X}}([pa, pbq]) + \mathcal{L}_{\mathfrak{X}}([paq, bq])$$

$$= \mathcal{L}_{\mathfrak{X}}(pa)pbq - pbq\mathcal{L}_{\mathfrak{X}}(pa) + pa\mathcal{L}_{\mathfrak{X}}(pbq) - \mathcal{L}_{\mathfrak{X}}(pbq)pa$$

$$+ \mathcal{L}_{\mathfrak{X}}(paq)bq - bq\mathcal{L}_{\mathfrak{X}}(paq) + paq\mathcal{L}_{\mathfrak{X}}(bq) - \mathcal{L}_{\mathfrak{X}}(bq)paq$$

$$= a\phi(b) + \phi(a)b.$$

As $\mathfrak{X}$ is 2-torsion free, the identity

$$\mathcal{L}_{\mathfrak{X}}(qap) = \mathcal{L}_{\mathfrak{X}}([qap, p]) = [\mathcal{L}_{\mathfrak{X}}(qap), p] + [qap, \mathcal{L}_{\mathfrak{X}}(p)] = -\mathcal{L}_{\mathfrak{X}}(qap),$$

implies that $\mathcal{L}_{\mathfrak{X}}(qap) = 0$ for all $a \in \mathfrak{A}$.

As $\mathcal{L}_{\mathfrak{X}}([pap, qaq]) = 0$, for all $a \in \mathfrak{A}$, we get,

$$\mathcal{L}_{\mathfrak{X}}(pap)qaq = -pap\mathcal{L}_{\mathfrak{X}}(qaq). \tag{2.5}$$

Substituting a with $qaq + p$ (resp. $pap + q$) in (2.5), leads to $p\mathcal{L}_{\mathfrak{X}}(qaq)q = -\mathcal{L}_{\mathfrak{X}}(p)a$ (resp. $p\mathcal{L}_{\mathfrak{X}}(pap)q = a\mathcal{L}_{\mathfrak{X}}(p)$), for all $a \in \mathfrak{A}$. We use the latter relations to prove that $\mathcal{L}_{\mathfrak{X}}$ is the sum of an inner derivation (implemented by $\mathcal{L}_{\mathfrak{X}}(p)$) and ϕ, and so it is a derivation. Indeed, for each $a \in \mathfrak{A}$,

$$\begin{aligned}\mathcal{L}_{\mathfrak{X}}(a) &= \mathcal{L}_{\mathfrak{X}}(pap) + \mathcal{L}_{\mathfrak{X}}(qaq) + \mathcal{L}_{\mathfrak{X}}(paq)\\ &= p\mathcal{L}_{\mathfrak{X}}(pap)q + p\mathcal{L}_{\mathfrak{X}}(qaq)q + \phi(a)\\ &= a\mathcal{L}_{\mathfrak{X}}(p) - \mathcal{L}_{\mathfrak{X}}(p)a + \phi(a).\end{aligned}$$

Now Lemma 2.1 confirms that $\mathcal{D}$ is a derivation on $\mathfrak{A} \ltimes \mathfrak{X}$, and so $\mathcal{L}$ is proper, as claimed.

For the converse, suppose that $\mathcal{L}$ is proper, that is, $\mathcal{L} = \mathcal{D} + \ell$, where $\mathcal{D}$ is a derivation and ℓ is a center valued linear map on $\mathfrak{A} \ltimes \mathfrak{X}$. Then, from (2.2), we get $\ell(\mathfrak{A} \ltimes \mathfrak{X}) \subseteq \pi_{\mathfrak{A}}(Z(\mathfrak{A} \ltimes \mathfrak{X})) \times \{0\}$, and this implies that ℓ has the presentation $\ell(a, x) = (\ell_{\mathfrak{A}}(a), 0)$ with $[\ell_A(a), x] = 0$, for all $a \in \mathfrak{A}, x \in \mathfrak{X}$, for some linear map $\ell_{\mathfrak{A}} : \mathfrak{A} \to Z(\mathfrak{A})$. On the other hand, $\mathcal{L} - \ell = \mathcal{D}$ is a derivation on $\mathfrak{A} \ltimes \mathfrak{X}$ and so, by Lemma 2.1, $\mathcal{L}_{\mathfrak{A}} - \ell_{\mathfrak{A}}$ is a derivation on $\mathfrak{A}$, as required. □

Applying Theorem 2.2, we come to the next main result providing some sufficient conditions ensuring the Lie derivation property for $\mathfrak{A} \ltimes \mathfrak{X}$. Note that we use the projection maps $\pi_{p\mathfrak{A}p} : \mathfrak{A} \ltimes \mathfrak{X} \to \mathfrak{A}$ and $\pi_{q\mathfrak{A}q} : \mathfrak{A} \ltimes \mathfrak{X} \to \mathfrak{A}$ defined by $\pi_{p\mathfrak{A}p}(a, x) = pap$ and $\pi_{q\mathfrak{A}q}(a, x) = qaq$, respectively. Before proceeding, we also introduce an auxiliary subalgebra $\mathcal{W}_{\mathfrak{A}}$ associated to $\mathfrak{A}$. For an algebra $\mathfrak{A}$, we denote by $\mathcal{W}_{\mathfrak{A}}$ the smallest subalgebra of $\mathfrak{A}$ contains all commutators and idempotents. We are especially dealing with those algebras satisfying $\mathcal{W}_{\mathfrak{A}} = \mathfrak{A}$. Some known examples of algebras satisfying $\mathcal{W}_{\mathfrak{A}} = \mathfrak{A}$ are: the full matrix algebra $\mathfrak{A} = M_n(A), n \geqslant 2$, where A is a unital algebra, and also every simple unital algebra $\mathfrak{A}$ with a nontrivial idempotent.

THEOREM 2.3. *Suppose that the trivial extension algebra $\mathfrak{A} \ltimes \mathfrak{X}$ satisfies $(\star)$ and that both $\mathfrak{A}$ and $\mathfrak{X}$ are 2-torsion free. Then $\mathfrak{A} \ltimes \mathfrak{X}$ has Lie derivation property if the following two conditions are satisfied:*

(I) *$\mathfrak{A}$ has Lie derivation property.*

(II) *One of the following three conditions hold:*

  (i) *$\mathcal{W}_{p\mathfrak{A}p} = p\mathfrak{A}p$ and $\mathcal{W}_{q\mathfrak{A}q} = q\mathfrak{A}q$.*

(ii) $Z(p\mathfrak{A}p) = \pi_{p\mathfrak{A}p}(Z(\mathfrak{A} \ltimes \mathfrak{X}))$ *and* $p\mathfrak{A}q$ *is faithful as a right* $q\mathfrak{A}q$*-module.*
(iii) $Z(q\mathfrak{A}q) = \pi_{q\mathfrak{A}q}(Z(\mathfrak{A} \ltimes \mathfrak{X}))$ *and* $p\mathfrak{A}q$ *is faithful as a left* $p\mathfrak{A}p$*-module.*

Proof. Let $\mathcal{L}$ be a Lie derivation on $\mathfrak{A} \ltimes \mathfrak{X}$ with the presentation as given in Lemma 2.1. Since $\mathcal{L}_{\mathfrak{A}}$ is a Lie derivation and (by (I)) $\mathfrak{A}$ has Lie derivation property, there exists a linear map $\ell_{\mathfrak{A}} : \mathfrak{A} \to Z(\mathfrak{A})$ such that $\mathcal{L}_{\mathfrak{A}} - \ell_{\mathfrak{A}}$ is a derivation on $\mathfrak{A}$ (and so $\ell_{\mathfrak{A}}$ vanishes on commutators of $\mathfrak{A}$). It is enough to show that, under either conditions of (II), $\ell_{\mathfrak{A}}$ satisfies Theorem 2.2(ii); that is, $[\ell_{\mathfrak{A}}(pap), x] = 0 = [\ell_{\mathfrak{A}}(qaq), x]$ for all $a \in \mathfrak{A}, x \in \mathfrak{X}$.

To prove the conclusion, we consider the subset $\mathfrak{A}' = \{pap : [\ell_{\mathfrak{A}}(pap), x] = 0,\ \text{for all } x \in \mathfrak{X}\}$ of $p\mathfrak{A}p$. We are going to show that $\mathfrak{A}'$ is a subalgebra of $p\mathfrak{A}p$ including all idempotents and commutators of $p\mathfrak{A}p$. First, we shall prove that $\mathfrak{A}'$ is a subalgebra. That $\mathfrak{A}'$ is an $\mathbf{R}$-submodule of $\mathfrak{A}$ follows from the linearity of $\ell_{\mathfrak{A}}$. The following identity confirms that $\mathfrak{A}'$ is closed under multiplication

$$(2.6)\qquad [\ell_{\mathfrak{A}}(papbp), x] = [\ell_{\mathfrak{A}}(pap), bx] + [\ell_{\mathfrak{A}}(pbp), ax], \quad a, b \in \mathfrak{A}, x \in \mathfrak{X}.$$

To prove (2.6), note that from the identity (2.4) we have

$$(2.7)\qquad S(ax) = (\mathcal{L}_{\mathfrak{A}} - \ell_{\mathfrak{A}})(a)x + [\ell_{\mathfrak{A}}(pap), x] + aS(x), \quad a \in \mathfrak{A}, x \in \mathfrak{X}.$$

Applying (2.7) for ab we have,

$$(2.8)\qquad S(abx) = (\mathcal{L}_{\mathfrak{A}} - \ell_{\mathfrak{A}})(ab)x + [\ell_{\mathfrak{A}}(pabp), x] + abS(x).$$

On the other hand, since $a[\ell_{\mathfrak{A}}(pbp), x] = [\ell_{\mathfrak{A}}(pbp), ax]$, we have,

$$\begin{aligned} S(abx) &= (\mathcal{L}_{\mathfrak{A}} - \ell_{\mathfrak{A}})(a)bx + [\ell_{\mathfrak{A}}(pap), bx] + aS(bx) \\ &= (\mathcal{L}_{\mathfrak{A}} - \ell_{\mathfrak{A}})(a)bx + [\ell_{\mathfrak{A}}(pap), bx] + a(\mathcal{L}_{\mathfrak{A}} - \ell_{\mathfrak{A}})(b)x \\ &\quad + [\ell_{\mathfrak{A}}(pbp), ax] + abS(x). \end{aligned}$$

Using the fact that $\mathcal{L}_{\mathfrak{A}} - \ell_{\mathfrak{A}}$ is a derivation, then a comparison of the latter equation and (2.8) leads to

$$[\ell_{\mathfrak{A}}(pabp), x] = [\ell_{\mathfrak{A}}(pap), bx] + [\ell_{\mathfrak{A}}(pbp), ax]$$

for all $a, b \in \mathfrak{A}, x \in \mathfrak{X}$, which trivially implies (2.6).

Next, we claim that $\mathfrak{A}'$ contains all idempotents of $p\mathfrak{A}p$. First note that, if one puts $a = b$ in (2.6), then

$$(2.9)\qquad [\ell_{\mathfrak{A}}((pap)^2), x] = [\ell_{\mathfrak{A}}(pap), 2ax], \quad a \in \mathfrak{A}, x \in \mathfrak{X}.$$

This follows that

$$\begin{aligned}[\ell_{\mathfrak{A}}((pap)^3), x] &= [\ell_{\mathfrak{A}}((pap)^2(pap)), x] \\ &= [\ell_{\mathfrak{A}}(pap), 3a^2x], \quad a \in \mathfrak{A}, x \in \mathfrak{X}. \end{aligned} \tag{2.10}$$

Suppose that $pap \in p\mathfrak{A}p$ is an idempotent, that is, $(pap)^2 = pap$. By (2.9) and (2.10), we arrive at

$$\begin{aligned}[\ell_{\mathfrak{A}}(pap), x] &= [\ell_{\mathfrak{A}}(3(pap)^2 - 2(pap)^3), x] \\ &= 3[\ell_{\mathfrak{A}}(pap), 2ax] - 2[\ell_{\mathfrak{A}}(pap), 3a^2x] \\ &= [\ell_{\mathfrak{A}}(pap), \big(6(pap) - 6(pap)^2\big)x] = 0; \end{aligned}$$

and this says that the idempotent pap lies in $\mathfrak{A}'$.

Further, that $\mathfrak{A}'$ contains all commutatorts follows trivially from the fact that $\ell_{\mathfrak{A}}$ vanishes on commutators. We thus have proved that $\mathfrak{A}'$ is a subalgebra of $p\mathfrak{A}p$ contains all idempotents and commutators. Now the assumption $\mathcal{W}_{p\mathfrak{A}p} = p\mathfrak{A}p$ in (i) gives $\mathfrak{A}' = p\mathfrak{A}p$, that is, $[\ell_{\mathfrak{A}}(pap), x] = 0$ for every $a \in \mathfrak{A}, x \in \mathfrak{X}$. A similar argument shows that, if $\mathcal{W}_{q\mathfrak{A}q} = q\mathfrak{A}q$, then $[\ell_{\mathfrak{A}}(qaq), x] = 0$ for every $a \in \mathfrak{A}, x \in \mathfrak{X}$.

Now suppose that $Z(p\mathfrak{A}p) = \pi_{p\mathfrak{A}p}(Z(\mathfrak{A} \ltimes \mathfrak{X}))$ and $p\mathfrak{A}q$ is faithful as a right $q\mathfrak{A}q$-module. In this case, to prove $[\ell_{\mathfrak{A}}(pap), x] = 0 = [\ell_{\mathfrak{A}}(qaq), x]$ for all $a \in \mathfrak{A}, x \in \mathfrak{X}$, we actually shall show that $[Z(\mathfrak{A}), \mathfrak{X}] = 0$. To this end, as the algebra $\mathfrak{A}$ enjoys the Peirce decomposition $\mathfrak{A} = p\mathfrak{A}p + p\mathfrak{A}q + q\mathfrak{A}p + q\mathfrak{A}q$, a direct verification reveals that

$$\begin{aligned} Z(\mathfrak{A}) = \{a \in \mathfrak{A};\ & pap \in Z(p\mathfrak{A}p), qaq \in Z(q\mathfrak{A}q), \\ & papm = mqaq, npap = qaqn \text{ for all } m \in p\mathfrak{A}q, n \in q\mathfrak{A}p\}. \end{aligned}$$

Combining the latter equality to that in (2.2) we arrive at

$$\begin{aligned} Z(\mathfrak{A} \ltimes \mathfrak{X}) = \{(a, 0);\ & a \in \mathfrak{A}, pap \in Z(p\mathfrak{A}p), qaq \in Z(q\mathfrak{A}q), papm = mqaq, \\ & npap = qaqn, [a, x] = 0 \text{ for all } m \in p\mathfrak{A}q, n \in q\mathfrak{A}p, x \in \mathfrak{X}\}. \end{aligned}$$

Let $a \in Z(\mathfrak{A})$. Since $pap \in Z(p\mathfrak{A}p)$ and $Z(p\mathfrak{A}p) = \pi_{p\mathfrak{A}p}(Z(\mathfrak{A} \ltimes \mathfrak{X}))$, there exists an element $(a', 0) \in Z(\mathfrak{A} \ltimes \mathfrak{X})$ such that $pap = \pi_{p\mathfrak{A}p}(a', 0) = pa'p$. It follows that $mqaq = papm = pa'pm = mqa'q$ for each $m \in p\mathfrak{A}q$. Since $p\mathfrak{A}q$ is a faithful right $q\mathfrak{A}q$-module, we get $qaq = qa'q$, and so $a = pap + qaq = pa'p + qa'q = a'$. In particular, $(a, 0) \in Z(\mathfrak{A} \ltimes \mathfrak{X})$ and so $[a, x] = 0$ for all $x \in \mathfrak{X}$, as claimed.

Similarly, if $Z(q\mathfrak{A}q) = \pi_{q\mathfrak{A}q}(Z(\mathfrak{A} \ltimes \mathfrak{X}))$ and $p\mathfrak{A}q$ is faithful as a left $p\mathfrak{A}p$-module, then the equality $[Z(\mathfrak{A}), \mathfrak{X}] = 0$ holds, which completes the proof. □

As the following example demonstrates, the Lie derivation property of $\mathfrak{A}$ in Theorem 2.3 is essential.

Example 2.4. Let $\mathfrak{A}$ be a unital algebra with a nontrivial idempotent p, which does not have Lie derivation property. Let $\mathcal{L}_{\mathfrak{A}}$ be a non-proper Lie derivation on $\mathfrak{A}$ and $\mathfrak{X}$ be an $\mathfrak{A}$-bimodule such that $pxq = x$ and $[\mathcal{L}_{\mathfrak{A}}(a), x] = 0$, for all $a \in \mathfrak{A}, x \in \mathfrak{X}$. Then a direct verification shows that $\mathcal{L}(a, x) = (\mathcal{L}_{\mathfrak{A}}(a), 0)$, is actually a non-proper Lie derivation on $\mathfrak{A} \ltimes \mathfrak{X}$.

To see a concrete example of a pair $\mathfrak{A}$, $\mathfrak{X}$ satisfying the aforementioned conditions, let $\mathfrak{A}$ be the triangular matrix algebra as given in [4, Example 8] and let $\mathfrak{X} = \mathbb{R}$ equipped with the module operations: $x \cdot (a_{ij}) = xa_{11}$, $(a_{ij}) \cdot x = a_{44}x$, $((a_{ij}) \in \mathfrak{A}$ and $x \in \mathbb{R})$.

The above example and Theorem 2.3 confirm that, the Lie derivation property of $\mathfrak{A}$ plays a key role for the Lie derivation property of $\mathfrak{A} \ltimes \mathfrak{X}$. In this respect, Lie derivation property of a unital algebra containing a nontrivial idempotent has already studied by Benkovič [2, Theorem 5.3] (see also the case $n = 2$ of a result given by Wang [14, Theorem 2.1]). About the Lie derivation property of a unital algebra with a nontrivial idempotent, we quote the following result from the first and third authors [13], which extended the aforementioned results.

Proposition 2.5 ([13, Corollary 4.3]). *Let $\mathfrak{A}$ be a 2-torsion free unital algebra with a nontrivial idempotent p and $q = 1-p$. Then $\mathfrak{A}$ has Lie derivation property if the following three conditions hold:*

(I) *$Z(q\mathfrak{A}q) = Z(\mathfrak{A})q$ and $p\mathfrak{A}q$ is a faithful left $p\mathfrak{A}p$-module; or $\mathcal{W}_{p\mathfrak{A}p} = p\mathfrak{A}p$ and $p\mathfrak{A}q$ is a faithful left $p\mathfrak{A}p$-module; or $p\mathfrak{A}p$ has Lie derivation property and $\mathcal{W}_{p\mathfrak{A}p} = p\mathfrak{A}p$.*

(II) *$Z(p\mathfrak{A}p) = Z(\mathfrak{A})p$ and $q\mathfrak{A}p$ is a faithful right $q\mathfrak{A}q$-module; or $\mathcal{W}_{q\mathfrak{A}q} = q\mathfrak{A}q$ and $q\mathfrak{A}p$ is a faithful right $q\mathfrak{A}q$-module; or $q\mathfrak{A}q$ has Lie derivation property and $\mathcal{W}_{q\mathfrak{A}q} = q\mathfrak{A}q$.*

(III) *One of the following assertions holds:*
   (i) *Either $p\mathfrak{A}p$ or $q\mathfrak{A}q$ does not contain nonzero central ideals.*
   (ii) *$p\mathfrak{A}p$ and $q\mathfrak{A}q$ are domain.*
   (iii) *Either $p\mathfrak{A}q$ or $q\mathfrak{A}p$ is strongly faithful.*

It should also be remarked that if $p\mathfrak{A}q\mathfrak{A}p = 0$ and $q\mathfrak{A}p\mathfrak{A}q = 0$, then the condition (III) in the above proposition is superfluous and can be dropped from the hypotheses, (see also [2, Remark 5.4]). One may apply Proposition 2.5 to show that, the algebra $\mathfrak{A} = B(X)$, of all bounded operators on a

Banach space X with dimension greater than 2, as well as, the full matrix algebra $\mathfrak{A} = M_n(A)$, $n \geqslant 2$, where A is a 2-torsion free unital algebra, have the Lie derivation property.

Illustrating Theorem 2.3 and Proposition 2.5, in the following we give an example of a trivial extension algebra, which is not a triangular algebra, having Lie derivation property .

Example 2.6. We consider the next subalgebra $\mathfrak{A}$ of $M_4(\mathbb{R})$ with a nontrivial idempotent p as follows;

$$\mathfrak{A} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & u & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} | a,\, b,\, c,\, d,\, u \in \mathbb{R} \right\}, \quad p = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

One can directly check that $p\mathfrak{A}p \cong \mathbb{R}$ and $q\mathfrak{A}q \cong \mathbb{R}^3$ (where the algebras $\mathbb{R}$ and $\mathbb{R}^3$ are equipped with their natural pointwise multiplications). In particular, $p\mathfrak{A}p$, $q\mathfrak{A}q$ have Lie derivation property, $\mathcal{W}_{p\mathfrak{A}p} = p\mathfrak{A}p$, $\mathcal{W}_{q\mathfrak{A}q} = q\mathfrak{A}q$ and $p\mathfrak{A}p$ does not contain nonzero central ideals. Thus, by virtue of Proposition 2.5, $\mathfrak{A}$ has Lie derivation property.

Further, $\mathfrak{X} = \mathbb{R}$ is an $\mathfrak{A}$-bimodule furnished with the module operations as

$$(a_{ij}) \cdot x = a_{33}x, \quad x \cdot (a_{ij}) = xa_{22}, \quad (a_{ij}) \in \mathfrak{A}, x \in \mathbb{R}.$$

Then clearly the trivial extension algebra $\mathfrak{A} \ltimes \mathbb{R}$ satisfies the condition $(\star)$; that is, $pxq = x$ for all $x \in \mathbb{R}$. So Theorem 2.3 guarantees that $\mathfrak{A} \ltimes \mathbb{R}$ has Lie derivation property. It is worthwhile mentioning that $\mathfrak{A} \ltimes \mathbb{R}$ is not a triangular algebra. This can be directly verified that, there is no nontrivial idempotent $P \in \mathfrak{A} \ltimes \mathbb{R}$ such that $P(\mathfrak{A} \ltimes \mathbb{R})Q \neq 0$ and $Q(\mathfrak{A} \ltimes \mathbb{R})P = 0$, where $Q = 1 - P$ (see [3]).

3. Application to triangular algebras

We recall that a triangular algebra $\mathrm{Tri}(\mathcal{A}, \mathfrak{X}, \mathcal{B})$ is an algebra of the form

$$\mathrm{Tri}(\mathcal{A}, \mathfrak{X}, \mathcal{B}) = \left\{ \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \mid a \in \mathcal{A},\ x \in \mathfrak{X},\ b \in \mathcal{B} \right\},$$

whose algebra operations are just like 2×2-matrix operations; where $\mathcal{A}$ and $\mathcal{B}$ are unital algebras and $\mathfrak{X}$ is an $(\mathcal{A}, \mathcal{B})$-bimodule; that is, a left $\mathcal{A}$-module and a right $\mathcal{B}$-module. One can easily check that $\mathrm{Tri}(\mathcal{A}, \mathfrak{X}, \mathcal{B})$ is isomorphic to the trivial extension algebra $(\mathcal{A} \oplus \mathcal{B}) \ltimes \mathfrak{X}$, where the algebra $\mathcal{A} \oplus \mathcal{B}$ has its usual pairwise operations and $\mathfrak{X}$ as an $(\mathcal{A} \oplus \mathcal{B})$-bimodule is equipped with the module operations

$$(a \oplus b)x = ax \quad \text{and} \quad x(a \oplus b) = xb, \quad a \in \mathcal{A}, b \in \mathcal{B}, x \in \mathfrak{X}.$$

Furthermore, the triangular algebra $\mathrm{Tri}(\mathcal{A}, \mathfrak{X}, \mathcal{B}) \cong (\mathcal{A} \oplus \mathcal{B}) \ltimes \mathfrak{X}$ satisfies the condition $(\star)$. Indeed, $p = (1_{\mathcal{A}}, 0)$ is a nontrivial idempotent, $q = (0, 1_{\mathcal{B}})$ and a direct verification shows that $pxq = x$, for all $x \in \mathfrak{X}$. Further, in this case for $\mathfrak{A} = \mathcal{A} \oplus \mathcal{B}$ we have,

$$p\mathfrak{A}p \cong \mathcal{A}, \quad p\mathfrak{A}q = 0, \quad q\mathfrak{A}p = 0 \quad \text{and} \quad q\mathfrak{A}q \cong \mathcal{B}.$$

It should be mentioned that in this case, for a Lie derivation $\mathcal{L}$ on $(\mathcal{A} \oplus \mathcal{B}) \ltimes \mathfrak{X}$ with the presentation

$$\mathcal{L}(a\oplus b, x) = (\mathcal{L}_{\mathcal{A}\oplus\mathcal{B}}(a\oplus b) + T(x),\ \mathcal{L}_{\mathfrak{X}}(a\oplus b) + S(x),) \quad (a\oplus b) \in \mathcal{A}\oplus\mathcal{B}, x \in \mathfrak{X},$$

as given in Lemma 2.1, we conclude that $T = 0$. Indeed, by Lemma 2.1(b), $T([a\oplus b, x]) = [a\oplus b, T(x)]$ for all $a \in \mathcal{A}, b \in \mathcal{B}, x \in \mathfrak{X}$. Using the latter relation for $a = 1, b = 0$ implies that $T(x) = 0$ for all $x \in \mathfrak{X}$.

A quick look at the proof of Theorem 2.2 reveals that, in this special case, as $T = 0$ and $q\mathfrak{A}p = 0$, we do not need the 2-torsion freeness of $\mathfrak{A}$ and $\mathfrak{X}$ in Theorems 2.2 and 2.3.

A direct verification also reveals that, the direct sum $\mathfrak{A} = \mathcal{A} \oplus \mathcal{B}$ has Lie derivation property if and only if both $\mathcal{A}$ and $\mathcal{B}$ have Lie derivation property.

Now, by the above observations, as an immediate consequence of Theorem 2.3, we directly arrive at the following result of Cheung (see [4, Theorem 11]).

Corollary 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras and let $\mathfrak{X}$ be an $(\mathcal{A}, \mathcal{B})$-bimodule. Then the triangular algebra $\mathcal{T} = \mathrm{Tri}(\mathcal{A}, \mathfrak{X}, \mathcal{B})$ has Lie derivation property if the following two conditions are satisfied:*

(I) *$\mathcal{A}$ and $\mathcal{B}$ have Lie derivation property.*

(II) *$\mathcal{W}_{\mathcal{A}} = \mathcal{A}$ and $\mathcal{W}_{\mathcal{B}} = \mathcal{B}$.*

It should be remarked here that, in [4, Theorem 11], Cheung combined his hypotheses with some "faithfulness" conditions and the equalities $Z(\mathcal{A}) = \pi_{\mathcal{A}}(Z(\mathcal{T}))$ and/or $Z(\mathcal{B}) = \pi_B(Z(\mathcal{T}))$. Combining the conditions "$\mathfrak{X}$ is faithful as a left $\mathcal{A}$-module" and "$\mathfrak{X}$ is faithful as a right $\mathcal{B}$-module" with those in the above corollary provides some more sufficient conditions ensuring the Lie derivation property for the triangular algebra $\mathrm{Tri}(\mathcal{A}, \mathfrak{X}, \mathcal{B})$. His results can be satisfactorily extended to a trivial extension algebra $\mathfrak{A} \ltimes \mathfrak{X}$ by employing the hypothesis "$\mathfrak{X}$ is loyal" instead of "$\mathfrak{X}$ is faithful".

We recall that, in the case where a unital algebra $\mathfrak{A}$ has a nontrivial idempotent p, an $\mathfrak{A}$-bimodule $\mathfrak{X}$ is said to be left loyal if $a\mathfrak{X} = 0$ implies that $pap = 0$, right loyal if $\mathfrak{X}a = 0$ implies that $qaq = 0$, and it is called loyal if it is both left and right loyal.

Note that for a triangular algebra $(\mathcal{A} \oplus \mathcal{B}) \ltimes \mathfrak{X}$, the loyalty of $\mathfrak{X}$ is nothing but the faithfulness of $\mathfrak{X}$ as an $(\mathcal{A}, \mathcal{B})$-module in the sense of Cheung [4]. Combining "the loyalty of $\mathfrak{X}$" with the current hypotheses of Theorem 2.3 and the equalities $Z(p\mathfrak{A}p) = \pi_{p\mathfrak{A}p}(Z(\mathfrak{A} \ltimes \mathfrak{X}))$, $Z(q\mathfrak{A}q) = \pi_{q\mathfrak{A}q}(Z(\mathfrak{A} \ltimes \mathfrak{X}))$, provide some more sufficient conditions seeking the Lie derivation property for a trivial extension algebra $\mathfrak{A} \ltimes \mathfrak{X}$. In the case where $\mathfrak{X}$ is a loyal $\mathfrak{A}$-module, the existence of an isomorphism between $\pi_{p\mathfrak{A}p}(Z(\mathfrak{A} \ltimes \mathfrak{X}))$ and $\pi_{q\mathfrak{A}q}(Z(\mathfrak{A} \ltimes \mathfrak{X}))$ is the key tool. Indeed, using the same argument as in [4, Proposition 3] (see also [2, Proposition 2.1]), it can be shown that, there exists a unique algebra isomorphism $\tau \colon \pi_{p\mathfrak{A}p}(Z(\mathfrak{A} \ltimes \mathfrak{X})) \to \pi_{q\mathfrak{A}q}(Z(\mathfrak{A} \ltimes \mathfrak{X}))$ satisfying $papx = x\tau(pap)$ for all $a \in \mathfrak{A}, x \in \mathfrak{X}$.

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