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WILD PRIMES OF A HIGHER DEGREE SELF-EQUIVALENCE OF A NUMBER FIELD

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Abstract. Let $\ell > 2$ be a prime number. Let K be a number field containing a unique ℓ-adic prime and assume that its class is an ℓth power in the class group C_K. The main theorem of the paper gives a sufficient condition for a finite set of primes of K to be the wild set of some Hilbert self-equivalence of K of degree ℓ.

1. Introduction

Let $n > 1$ be a natural number and let K as well as L be number fields containing a primitive nth root of unity. *A Hilbert symbol equivalence of degree* n between K and L is defined as a triple

$$f\colon \mu_n(K) \to \mu_n(L), \quad T\colon \Omega(K) \to \Omega(L), \quad t\colon \dot{K}/\dot{K}^n \to \dot{L}/\dot{L}^n,$$

where f is an isomorphism between the groups of nth roots of unity, t is an isomorphism between the nth power class groups and T is a bijection between the sets of all primes of K and L with (f, T, t) preserving Hilbert symbols of degree n in the sense that

$$(a, b)_{\mathfrak{p}}^{f} = (ta, tb)_{T\mathfrak{p}} \quad \text{for all} \quad a, b \in \dot{K}/\dot{K}^n, \ \ \mathfrak{p} \in \Omega(K).$$

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In the case $K = L$ and $f = \mathrm{id}$, the Hilbert equivalence (f, T, t) is called *Hilbert self-equivalence of degree n of K* or just *self-equivalence of degree n of K* and denoted by (T, t).

Hilbert equivalence of degree 2 was introduced in [11] as a necessary and sufficient condition for Witt equivalence of two global fields. The main result of [11] states that *two global fields of characteristic not 2 have isomorphic Witt rings if and only if there exists a Hilbert equivalence of degree 2 between these fields.*

Hilbert equivalence of degree $n > 2$, was introduced in [2] and was used for a classification of Milnor rings modulo n. In [16] it was proven that *for two number fields containing nth roots of unity, n square free, there exists a Hilbert equivalence of degree n if and only if there exists an isomorphism of graded rings $\mathsf{K}(K)/n\mathsf{K}(K) \cong \mathsf{K}(L)/n\mathsf{K}(L)$ such that $\{-1\}_n \longmapsto \{-1\}_n$,* where $\mathsf{K}(F)$ denotes the Milnor ring of F, (cf. [9]).

A finite prime $\mathfrak{p} \in \Omega(K)$ is called *a tame prime* of the Hilbert equivalence (f, T, t) of degree n if

$$\mathrm{ord}_{\mathfrak{p}}\, a \equiv \mathrm{ord}_{T\mathfrak{p}}\, ta \pmod{n} \quad \text{for all} \quad a \in \dot{K}/\dot{K}^n.$$

A finite prime $\mathfrak{p} \in \Omega(K)$ is said to be *wild* if it is not a tame prime of (f, T, t). The set $\mathcal{W}(f, T, t)$ of all wild primes of (f, T, t) is called the *wild set* of (f, T, t). The Hilbert equivalence (f, T, t) is said to be *tame* if all finite primes are tame. In [5] and [4] the reader can find characterizations of tame Hilbert equivalence of degree 2 and degree $\ell > 2$, ℓ being a prime number, respectively.

In the case of a Hilbert equivalence which is not tame it is natural to ask what the wild set of this equivalence could be. In [13] Somodi estimated the size of the wild set of Hilbert equivalence of degree 2, whereas in [14] and [15] he gave a description of the wild set of Hilbert self-equivalence of degree 2 of the rational number field $\mathbb{Q}$ as well as the Gaussian field $\mathbb{Q}(i)$. Somodi's results were generalized to a wider family of number fields in [7].

In this paper we consider self-equivalences of degree ℓ, where ℓ is a prime number $\neq 2$, of number fields K satisfying two conditions:

(C1) *The field K contains a primitive ℓth root of unity.*

(C2) *The field K has exactly one ℓ-adic prime $\mathfrak{r}$ and its class* $\mathrm{cl}\,\mathfrak{r}$ *is an ℓth power in the ideal class group C_K of K.*

For a number field K satisfying (C1) and (C2) we find a sufficient condition for a finite set of finite primes of K to be the wild set of some Hilbert self-equivalence of degree ℓ of K.

The main result of the paper is as follows.

THEOREM 1.1. *Let $\ell > 2$ be a prime number and let K be a number field satisfying* (C1) *and* (C2). *If $W = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$ is a set of finite primes of K*

and the classes $\operatorname{cl}\mathfrak{p}_1, \dots \operatorname{cl}\mathfrak{p}_k$ *are* ℓ*th powers in* C_K*, then there exists a Hilbert self-equivalence* (T,t) *of degree* ℓ *of* K *such that* $\mathcal{W}(T,t) = W$*.*

In the proof we use the technique developed in [3] and [4]. Basic facts and notions connected with this technique are presented in Sections 2 and 3.

The main theorem is proven in two steps. In the first one, presented in Section 3, we construct a Hilbert self-equivalence of degree ℓ with one wild prime adjusting the construction applied in [7] to the case of $\ell > 2$. The second step, carried out in Section 4, uses the analysis of the behaviour of wild sets under a composition of self-equivalences of degree ℓ.

2. Singular elements

Throughout the paper, let $\ell > 2$ be a prime number, K be a number field and $\zeta \in K$ be a fixed primitive ℓth root of unity. For any $\mathfrak{p} \in \Omega(K)$ we write $K_\mathfrak{p}$ for the completion of K at $\mathfrak{p}$, $(\,\cdot\,,\,\cdot\,)_\mathfrak{p} \colon \dot{K}_\mathfrak{p} \times \dot{K}_\mathfrak{p} \to \mu_\ell(K)$ for the Hilbert symbol of degree ℓ with values in the cyclic group $\mu_\ell(K)$ generated by ζ and $\left(\frac{\cdot}{\mathfrak{p}}\right)_\ell$ for the ℓth power residue symbol. Since $(x_1,x_2)_\mathfrak{p} = (y_1,y_2)_\mathfrak{p}$, if $x_i\dot{K}_\mathfrak{p}^\ell = y_i\dot{K}_\mathfrak{p}^\ell$, $i = 1,2$, we may consider $(\,\cdot\,,\,\cdot\,)_\mathfrak{p}$ as a mapping defined on $\dot{K}_\mathfrak{p}/\dot{K}_\mathfrak{p}^\ell \times \dot{K}_\mathfrak{p}/\dot{K}_\mathfrak{p}^\ell$.

If A is a finite abelian group, then the quotient A/A^ℓ is an elementary abelian ℓ-group. The group A/A^ℓ can be equipped with the structure of a linear space over the ℓ-element field $\mathbb{F}_\ell$. In this sense we will use the notion of linear independence of elements of A/A^ℓ. The ℓ-rank $\operatorname{rk}_\ell A$ of a group A is $\dim_{\mathbb{F}_\ell} A/A^\ell$. Where it is not misleading, we will use frequently the same symbol for $x \in A$ and for its canonical image in A/A^ℓ. We write $\langle\, x_1, \dots . x_n \,\rangle$ for the subgroup of A generated by the elements $x_1, \dots, x_n \in A$.

A finite nonempty set $S \subset \Omega(K)$ is called a *Hasse set* if it contains all infinite primes of K. We write $\mathcal{O}_S$, U_S, C_S, h^S for the ring of S-integers, the group of S-units, the S-class group of K and the order of C_S, respectively. The class of the ideal $\mathfrak{a}$ of $\mathcal{O}_S$ in the group C_S we denote by $\operatorname{cl}\mathfrak{a}$.

A Hasse set $S \subset \Omega(K)$ is said to be *sufficiently large*, if:

– S contains all ℓ-adic primes of K,
– the S-class number h^S is not divisible by ℓ, i.e. $\operatorname{rk}_\ell C_S = 0$.

For a Hasse set S of K we denote

$$E_S = \{a \in \dot{K} : \operatorname{ord}_\mathfrak{p} a \equiv 0 \pmod{\ell}, \quad \forall\, \mathfrak{p} \in \Omega(K) \setminus S\,\}.$$

It is easy to check that E_S is a subgroup of the multiplicative group $\dot{K}$. Elements of E_S are called *S-singular*.

We identify elements of the group U_S/U_S^ℓ with their images under the natural embedding $U_S/U_S^\ell \to E_S/\dot{K}^\ell$, so we may consider U_S/U_S^ℓ as a subgroup of the group $E_S/\dot{K}^\ell$. By Dirichlet Unit Theorem its ℓ-rank equals

$$\operatorname{rk}_\ell U_S/U_S^\ell = \#S. \tag{2.1}$$

PROPOSITION 2.1. *If S is a Hasse set of primes of K, then*

$$\operatorname{rk}_\ell E_S/\dot{K}^\ell = \#S + \operatorname{rk}_\ell C_S(K).$$

PROOF. The mapping

$$E_S \to {}_\ell C_S, \quad x \mapsto \operatorname{cl} \prod_{\mathfrak{p}\notin S} \mathfrak{p}^{\frac{1}{\ell}\operatorname{ord}_\mathfrak{p} x}$$

is a group epimorphism with the kernel $U_S\dot{K}^\ell$. It suffices to notice that $U_S\dot{K}^\ell/\dot{K}^\ell \cong U_S/U_S^\ell$ and then apply (2.1). □

Contraction of the diagonal homomorphism $\operatorname{diag}_S\colon \dot{K}/\dot{K}^\ell \to \prod_{\mathfrak{p}\in S} \dot{K}_\mathfrak{p}/\dot{K}_\mathfrak{p}^\ell$ to the subgroup $E_S/\dot{K}^\ell$, gives the homomorphism $E_S/\dot{K}^\ell \to \prod_{\mathfrak{p}\in S} \dot{K}_\mathfrak{p}/\dot{K}_\mathfrak{p}^\ell$, which we denote by i_S.

Let

$$\Delta_S = \{a \in E_S : a \in \dot{K}_\mathfrak{p}^\ell, \quad \forall\, \mathfrak{p} \in S\}.$$

The group $\Delta_S/\dot{K}^\ell$ is the kernel of i_S.

REMARK 2.2. Suppose $S \subset S'$ are Hasse sets of K. Then $E_S \subseteq E_{S'}$, $\Delta_{S'} \subseteq \Delta_S$ and there exists a natural group epimorphism $C_S \to C_{S'}$, which induces the group epimorphism $C_S/C_S^\ell \to C_{S'}/C_{S'}^\ell$ with the kernel equal to the subgroup of C_S/C_S^ℓ generated by the set $\{\operatorname{cl}\mathfrak{p}\, C_S^\ell\colon \mathfrak{p} \in S' \setminus S\}$. Thus

$$\operatorname{rk}_\ell C_{S'} = \operatorname{rk}_\ell C_S - \operatorname{rk}_\ell\langle\{\operatorname{cl}\mathfrak{p}\, C_S^\ell\colon \mathfrak{p} \in S' \setminus S\}\rangle$$

and

$$\operatorname{rk}_\ell E_{S'}/\dot{K}^\ell = \#\, S' + (\operatorname{rk}_\ell C_S - \operatorname{rk}_\ell\langle\{\operatorname{cl}\mathfrak{p}\, C_S^\ell\colon \mathfrak{p} \in S' \setminus S\}\rangle).$$

LEMMA 2.3. *Let S be a Hasse set of K containing all ℓ-adic primes and let $\mathfrak{p} \in \Omega(K) \setminus S$. Then*

$$\operatorname{cl}\mathfrak{p} \in C_S^\ell \iff \left(\frac{b}{\mathfrak{p}}\right)_\ell = 1 \quad \textit{for every } b \in \Delta_S.$$

PROOF. ($\Rightarrow$) By assumption there exists $x_\mathfrak{p} \in \dot{K}$ such that $x_\mathfrak{p}\mathcal{O}_S = \mathfrak{p} \cdot J^\ell$ for some S-ideal J of the field K. Fix $b \in \Delta_S$. Since for every prime $\mathfrak{q} \notin S \cup \{\mathfrak{p}\}$ the elements b and $x_\mathfrak{p}$ are $\mathfrak{q}$-adic units modulo $\dot{K}_\mathfrak{q}^\ell$, we have

$$(b, x_\mathfrak{p})_\mathfrak{q} = 1 \quad \text{for all } \mathfrak{q} \notin S \cup \{\mathfrak{p}\}.$$

Also

$$(b, x_\mathfrak{p})_\mathfrak{q} = 1 \quad \text{for all } \mathfrak{q} \in S,$$

because $b \in \dot{K}_\mathfrak{q}^\ell$ for all $\mathfrak{q} \in S$.

By Hilbert reciprocity law, $(b, x_\mathfrak{p})_\mathfrak{p} = 1$, i.e. $\left(\frac{b}{\mathfrak{p}}\right)_\ell = 1$.

($\Leftarrow$) Let $S_1 = S \cup \{\mathfrak{p}\}$. Since $b \in \dot{K}_\mathfrak{p}^\ell$ for every $b \in \Delta_S$ (by assumption), $\Delta_{S_1} = \Delta_S$. Thus $\operatorname{rk}_\ell C_S = \operatorname{rk}_\ell C_{S_1}$, and $\operatorname{cl}\mathfrak{p} \in C_S^\ell$. □

LEMMA 2.4. *Let S be a Hasse set of K containing all ℓ-adic primes and let $\mathfrak{q}_1, \dots, \mathfrak{q}_m \in \Omega(K) \setminus S$ be finite primes. Then the classes $\operatorname{cl}\mathfrak{q}_1, \dots, \operatorname{cl}\mathfrak{q}_m$ are linearly independent in the factor group C_S/C_S^ℓ if and only if there exist elements $b_1, \dots, b_m \in \Delta_S$ linearly independent in the group $\Delta_S/\dot{K}^\ell$ such that*

$$\left(\frac{b_i}{\mathfrak{q}_i}\right)_\ell = \zeta, \quad \left(\frac{b_j}{\mathfrak{q}_i}\right)_\ell = 1 \quad \textit{for all } i, j \in \{1, \dots, m\},\ i \neq j.$$

PROOF. ($\Leftarrow$) Suppose the classes $\mathfrak{q}_1, \dots, \mathfrak{q}_m$ are linearly dependent in the group C_S/C_S^ℓ. Then

$$x\mathcal{O}_S(K) = \mathfrak{q}_1^{\epsilon_1} \dots \mathfrak{q}_m^{\epsilon_m} \mathfrak{I}^\ell$$

for some element $x \in \dot{K}$, fractional S-ideal $\mathfrak{I}$ and $\epsilon_1, \dots, \epsilon_m \in \{0, \dots, \ell - 1\}$ not all equal to zero.

We may assume $\epsilon_1 > 0$. The element b_1 is a $\mathfrak{q}_j$-adic unit (modulo $\dot{K}_{\mathfrak{q}_j}^\ell$) for every $j \in \{1, \dots, m\}$, so by the well known relationship between Hilbert

symbol of degree ℓ and ℓth power residue symbol (cf. [10]) we get the following equalities

$$(b_1,\, x)_{\mathfrak{q}_1} = \left(\frac{b_1}{\mathfrak{q}_1}\right)_\ell^{\operatorname{ord}_{\mathfrak{q}_1}(x)} = \zeta^{\epsilon_1} \neq 1, \quad (b_1,\, x)_{\mathfrak{q}_j} = \left(\frac{b_1}{\mathfrak{q}_j}\right)_\ell^{\operatorname{ord}_{\mathfrak{q}_j}(x)} = 1$$

for every $j \in \{2, \dots, m\}$.

Since $b_1 \in \dot{K}_{\mathfrak{p}}^{\ell}$ for every $\mathfrak{p} \in S$, we have $(b_1,\, x)_{\mathfrak{p}} = 1$. Every prime $\mathfrak{q}$ of K not in $S \cup \{\mathfrak{q}_1, \dots, \mathfrak{q}_m\}$ is a finite non-ℓ-adic prime and the elements x, b_1 are $\mathfrak{q}$-adic units (modulo $\dot{K}_{\mathfrak{q}}^{\ell}$), so $(b_1,\, x)_{\mathfrak{q}} = 1$. This contradicts Hilbert reciprocity law.

($\Rightarrow$) We use induction on m. For $m = 1$ the statement follows from Lemma 2.3.

Suppose $m > 1$. By Lemma 2.3 there exists $b_1 \in \Delta_S$ such that

$$\left(\frac{b_1}{\mathfrak{q}_1}\right)_\ell = \zeta.$$

Let $S_1 = S \cup \{\mathfrak{q}_1\}$. Then $\operatorname{rk}_\ell C_{S_1} = \operatorname{rk}_\ell C_S - 1$, $\Delta_{S_1} \subseteq \Delta_S$ and $b_1 \notin \Delta_{S_1}$. Moreover, the classes $\operatorname{cl}\mathfrak{q}_2, \dots, \operatorname{cl}\mathfrak{q}_m$ are linearly independent in the group $C_{S_1}/C_{S_1}^{\ell}$.

By induction hypothesis we may find $b_2, \dots, b_m \in \Delta_{S_1}$, linearly independent in the group $\Delta_{S_1}/\dot{K}^{\ell}$, such that

$$\left(\frac{b_i}{\mathfrak{q}_i}\right)_\ell = \zeta, \quad \left(\frac{b_i}{\mathfrak{q}_j}\right)_\ell = 1 \quad \text{for all } i, j \in \{2, \dots, m\},\ i \neq j.$$

Obviously, $\left(\frac{b_i}{\mathfrak{q}_1}\right)_\ell = 1$ for $i = 2, \dots, m$. If necessary, we multiply b_1 by a product of appropriate powers of b_i, $i \in \{2, \dots, m\}$, to get $\left(\frac{b_1}{\mathfrak{q}_i}\right)_\ell = 1$ for $i = 2, \dots, m$. □

Corollary 2.5. *Let S be a Hasse set of primes of K containing all ℓ-adic primes and let $\mathfrak{q}_1, \dots, \mathfrak{q}_m \in \Omega(K) \setminus S$ be finite primes such that $\operatorname{cl}\mathfrak{q}_1, \dots, \operatorname{cl}\mathfrak{q}_m$ is a basis of the factor group C_S/C_S^{ℓ}. Then*

$$E_S/\dot{K}^{\ell} = U_S\dot{K}^{\ell}/\dot{K}^{\ell} \oplus \Delta_S/\dot{K}^{\ell} = U_S\dot{K}^{\ell}/\dot{K}^{\ell} \oplus \langle b_1\dot{K}^{\ell}, \dots, b_m\dot{K}^{\ell}\rangle,$$

where $b_1, \dots, b_n$ are chosen as in the lemma above.

3. Hilbert symbol bilinear spaces

In this section we will recall basic properties of Hilbert symbol of degree ℓ and prove new facts which will appear to be crucial in the next sections. Let us keep the assumptions on ℓ, K and ζ made at the beginning of the previous section.

Since $\ell > 2$, every infinite prime $\mathfrak{p}$ of K is complex and the $\mathfrak{p}$-adic Hilbert symbol is trivial, i.e. $(a,\, b)_{\mathfrak{p}} = 1$ for all $a, b \in \dot{K}_{\mathfrak{p}}$.

If $\mathfrak{p}$ is a finite and non-ℓ-adic prime, then according to the formula for the value of Hilbert symbol (cf. [10, Theorem 5.4]) we have:

3.1. *If u, v are $\mathfrak{p}$-adic units, then* $(u,\, v)_{\mathfrak{p}} = 1$.

3.2. *If u is a $\mathfrak{p}$-adic unit and $\pi_{\mathfrak{p}}$ is a uniformizer, i.e.* $\operatorname{ord}_{\mathfrak{p}} \pi_{\mathfrak{p}} = 1$, *then*

$$(\pi_{\mathfrak{p}},\, u)_{\mathfrak{p}} = 1 \iff u \in U_{\mathfrak{p}}^{\ell}.$$

3.3. *There exists a $\mathfrak{p}$-adic unit $u_{\mathfrak{p}}$ such that*

$$(u_{\mathfrak{p}},\, x)_{\mathfrak{p}} = \zeta^{\operatorname{ord}_{\mathfrak{p}} x} \quad \textit{for every } x \in \dot{K}_{\mathfrak{p}}.$$

For an ℓ-adic prime $\mathfrak{p}$ a simple formula for the value of the Hilbert symbol of degree ℓ is not known. Nevertheless, in this case there also exists a $\mathfrak{p}$-adic unit $u_{\mathfrak{p}}$ satisfying 3.3 (see [1, Example 2.12]).

In both cases the element $u_{\mathfrak{p}}$ is $\mathfrak{p}$-primary, i.e. the extension $K_{\mathfrak{p}}(\sqrt[\ell]{u_{\mathfrak{p}}})/K_{\mathfrak{p}}$ is unramified. We call $u_{\mathfrak{p}}$ the $\mathfrak{p}$-*adic primary unit normalized with respect to* ζ and denote by $u_{\mathfrak{p}}(\zeta)$ or just $u_{\mathfrak{p}}$ if ζ is fixed. Hilbert symbol is nondegenerate, so for a fixed ζ the element $u_{\mathfrak{p}}(\zeta)$ is uniquely determined up to ℓth power in the field $K_{\mathfrak{p}}$.

For primes $\mathfrak{p}$ and $\mathfrak{p}'$ of K a local isomorphism $\varphi\colon \dot{K}_{\mathfrak{p}}/\dot{K}_{\mathfrak{p}}^{\ell} \to \dot{K}_{\mathfrak{p}'}/\dot{K}_{\mathfrak{p}'}^{\ell}$ is said to preserve Hilbert symbols, if

$$(x,\, y)_{\mathfrak{p}} = (\varphi(x),\, \varphi(y))_{\mathfrak{p}'} \quad \text{for all } x, y \in \dot{K}_{\mathfrak{p}}.$$

If $\mathfrak{p}$ and $\mathfrak{p}'$ are finite primes and

$$\operatorname{ord}_{\mathfrak{p}} x \equiv \operatorname{ord}_{\mathfrak{p}'} \varphi(x) \pmod{\ell} \quad \text{for every } x \in \dot{K}_{\mathfrak{p}}/\dot{K}_{\mathfrak{p}}^{\ell},$$

then the isomorphism φ is called *tame*. Otherwise φ is called *wild*.

There exists a relatively easy necessary and sufficient condition for the existence of a local isomorphism preserving Hilbert symbols (cf. [3, Lemma 2.2]).

PROPOSITION 3.4. *Suppose* $\mathfrak{p}$ *and* $\mathfrak{p}'$ *are finite primes of the field* K*. There exists a local isomorphism* $\dot{K}_{\mathfrak{p}}/\dot{K}^{\ell}_{\mathfrak{p}} \to \dot{K}_{\mathfrak{p}'}/\dot{K}^{\ell}_{\mathfrak{p}'}$ *preserving Hilbert symbols if and only if* $\operatorname{rk}_\ell \dot{K}_{\mathfrak{p}}/\dot{K}^{\ell}_{\mathfrak{p}} = \operatorname{rk}_\ell \dot{K}_{\mathfrak{p}'}/\dot{K}^{\ell}_{\mathfrak{p}'}$.

By [10, Ch.3, Prop.1.5]:

$$\operatorname{rk}_\ell \dot{K}_{\mathfrak{p}}/\dot{K}^{\ell}_{\mathfrak{p}} = \begin{cases} 0, & \text{if } \mathfrak{p} \text{ is complex,} \\ 1, & \text{if } \mathfrak{p} \text{ is real } (\ell = 2), \\ [K_{\mathfrak{p}} : \mathbb{Q}_\ell\,] + 2, & \text{if } \mathfrak{p} \text{ is } \ell\text{-adic,} \\ 2, & \text{in other cases.} \end{cases}$$

As a consequence we get the following.

PROPOSITION 3.5. *Let* $\mathfrak{p}$ *and* $\mathfrak{p}'$ *be primes of* K*. If there exists an isomorphism* $\dot{K}_{\mathfrak{p}}/\dot{K}^{\ell}_{\mathfrak{p}} \to \dot{K}_{\mathfrak{p}'}/\dot{K}^{\ell}_{\mathfrak{p}'}$ *preserving Hilbert symbols, then:*

(i) $\mathfrak{p}$ *is finite* $\Leftrightarrow$ $\mathfrak{p}'$ *is finite.*
(ii) $\mathfrak{p}$ *is* ℓ*-adic* $\Leftrightarrow$ $\mathfrak{p}'$ *is* ℓ*-adic and* $[K_{\mathfrak{p}} : \mathbb{Q}_\ell\,] = [K_{\mathfrak{p}'} : \mathbb{Q}_\ell\,]$.
(iii) $\mathfrak{p}$ *is real* $\Leftrightarrow$ $\mathfrak{p}'$ *is real.*

LEMMA 3.6. *Let* $\mathfrak{p}$ *and* $\mathfrak{p}'$ *be primes of* K*. The isomorphism* $\varphi\colon \dot{K}_{\mathfrak{p}}/\dot{K}^{\ell}_{\mathfrak{p}} \to \dot{K}_{\mathfrak{p}'}/\dot{K}^{\ell}_{\mathfrak{p}'}$ *preserving Hilbert symbols is tame if and only if* $\varphi(u_{\mathfrak{p}}\dot{K}^{\ell}_{\mathfrak{p}}) = u_{\mathfrak{p}'}\dot{K}^{\ell}_{\mathfrak{p}'}$.

PROOF. Obviously, if $\varphi(u_{\mathfrak{p}}\dot{K}^{\ell}_{\mathfrak{p}}) = u_{\mathfrak{p}'}\dot{K}^{\ell}_{\mathfrak{p}'}$, then φ is tame. Suppose that φ preserves Hilbert symbols and is tame. Let $\varphi(u_{\mathfrak{p}}\dot{K}^{\ell}_{\mathfrak{p}}) = w\dot{K}^{\ell}_{\mathfrak{p}'}$. Then

$$\begin{aligned}(u_{\mathfrak{p}'}, \varphi(x))_{\mathfrak{p}'} &= \zeta^{\operatorname{ord}_{\mathfrak{p}'}\varphi(x)} = \zeta^{\operatorname{ord}_{\mathfrak{p}} x} \\ &= (u_{\mathfrak{p}}, x)_{\mathfrak{p}} = (w, \varphi(x))_{\mathfrak{p}'} \quad \text{for every } x \in \dot{K}_{\mathfrak{p}}.\end{aligned}$$

Since Hilbert symbol is nondegenerate, we have

$$\varphi(u_{\mathfrak{p}}\dot{K}^{\ell}_{\mathfrak{p}}) = w\dot{K}^{\ell}_{\mathfrak{p}'} = u_{\mathfrak{p}'}\dot{K}^{\ell}_{\mathfrak{p}'}. \qquad \square$$

LEMMA 3.7. *Let* $\mathfrak{p}$ *be a prime of* K *and let* H, H' *be subgroups of* $\dot{K}_{\mathfrak{p}}/\dot{K}^{\ell}_{\mathfrak{p}}$*. Any group isomorphism* $\tau\colon H \to H'$ *preserving Hilbert symbols can be extended to a group automorphism of* $\dot{K}_{\mathfrak{p}}/\dot{K}^{\ell}_{\mathfrak{p}}$ *preserving Hilbert symbols. Moreover, if* τ *is tame on* H*, i.e.* $\operatorname{ord}_{\mathfrak{p}} a \equiv \operatorname{ord}_{\mathfrak{p}} \tau(a) \pmod{\ell}$ *for every* $a \in H$ *and* $\tau(u_{\mathfrak{p}}\dot{K}^{\ell}_{\mathfrak{p}}) = u_{\mathfrak{p}}\dot{K}^{\ell}_{\mathfrak{p}}$*, when* $u_{\mathfrak{p}}\dot{K}^{\ell}_{\mathfrak{p}} \in H$*, then the isomorphism* τ *can be extended to a tame automorphism preserving Hilbert symbols.*

PROOF. From the standard properties of Hilbert symbols, it follows that the mapping

$$\beta_{\mathfrak{p}}\colon \dot{K}_{\mathfrak{p}}/\dot{K}_{\mathfrak{p}}^{\ell} \times \dot{K}_{\mathfrak{p}}/\dot{K}_{\mathfrak{p}}^{\ell} \to \mathbb{F}_{\ell}, \quad (x,y)_{\mathfrak{p}} = \zeta^{\beta_{\mathfrak{p}}(x,y)} \quad \text{for } x,y \in \dot{K}_{\mathfrak{p}}/\dot{K}_{\mathfrak{p}}^{\ell}$$

is nondegenerate bilinear and antisymmetric, so it is also alternating. To get the statement one can use a standard prolongation theorem from bilinear algebra. □

LEMMA 3.8. *Let $\mathfrak{r}$ be an ℓ-adic prime of the field K. The linear mapping $\varphi\colon \dot{K}_{\mathfrak{r}}/\dot{K}_{\mathfrak{r}}^{\ell} \to \dot{K}_{\mathfrak{r}}/\dot{K}_{\mathfrak{r}}^{\ell}$ uniquely defined by $\varphi(u_{\mathfrak{r}}\dot{K}_{\mathfrak{r}}^{\ell}) = u_{\mathfrak{r}}\pi_{\mathfrak{r}}\dot{K}_{\mathfrak{r}}^{\ell}$, $\varphi(\pi_{\mathfrak{r}}\dot{K}_{\mathfrak{r}}^{\ell}) = \pi_{\mathfrak{r}}\dot{K}_{\mathfrak{r}}^{\ell}$, and $\varphi(v\dot{K}_{\mathfrak{r}}^{\ell}) = v\dot{K}_{\mathfrak{r}}^{\ell}$ for every $v\dot{K}_{\mathfrak{r}}^{\ell} \in \langle u_{\mathfrak{r}}\dot{K}_{\mathfrak{r}}^{\ell}, \pi_{\mathfrak{r}}\dot{K}_{\mathfrak{r}}^{\ell}\rangle^{\perp}$ is an automorphism of the bilinear space $(\dot{K}_{\mathfrak{r}}/\dot{K}_{\mathfrak{r}}^{\ell}, \beta_{\mathfrak{r}})$.*

PROOF. It is a routine matter to check that φ is a group automorphism. Thus it suffices to show that it preserves Hilbert symbols. From properties of Hilbert symbol of degree ℓ it follows that $(x,x)_{\mathfrak{r}} = 1$ for every $x \in \dot{K}_{\mathfrak{r}}/\dot{K}_{\mathfrak{r}}^{\ell}$. Moreover,

$$\begin{aligned}
&(\pi_{\mathfrak{r}}, u_{\mathfrak{r}})_{\mathfrak{r}} = (\pi_{\mathfrak{r}}, \pi_{\mathfrak{r}})_{\mathfrak{r}}(\pi_{\mathfrak{r}}, u_{\mathfrak{r}})_{\mathfrak{r}} = (\pi_{\mathfrak{r}}, \pi_{\mathfrak{r}}u_{\mathfrak{r}})_{\mathfrak{r}},\\
&(\pi_{\mathfrak{r}}, v)_{\mathfrak{r}} = 1 = (u_{\mathfrak{r}}\pi_{\mathfrak{r}}, v)_{\mathfrak{r}} \quad \text{for every } v\dot{K}_{\mathfrak{r}}^{\ell} \in \langle u_{\mathfrak{r}}\dot{K}_{\mathfrak{r}}^{\ell}, \pi_{\mathfrak{r}}\dot{K}_{\mathfrak{r}}^{\ell}\rangle^{\perp},\\
&(u_{\mathfrak{r}}, v)_{\mathfrak{r}} = 1 = (u_{\mathfrak{r}}\pi_{\mathfrak{r}}, v)_{\mathfrak{r}} \quad \text{for every } v\dot{K}_{\mathfrak{r}}^{\ell} \in \langle u_{\mathfrak{r}}\dot{K}_{\mathfrak{r}}^{\ell}, \pi_{\mathfrak{r}}\dot{K}_{\mathfrak{r}}^{\ell}\rangle^{\perp}.
\end{aligned}$$

□

LEMMA 3.9. *Let $\mathfrak{p},\mathfrak{q}$ be non-ℓ-adic finite primes of K. The linear mapping $\varphi\colon \dot{K}_{\mathfrak{p}}/\dot{K}_{\mathfrak{p}}^{\ell} \to \dot{K}_{\mathfrak{q}}/\dot{K}_{\mathfrak{q}}^{\ell}$ uniquely defined by $\varphi(u_{\mathfrak{p}}\dot{K}_{\mathfrak{p}}^{\ell}) = u_{\mathfrak{q}}\pi_{\mathfrak{q}}\dot{K}_{\mathfrak{q}}^{\ell}$, $\varphi(\pi_{\mathfrak{p}}\dot{K}_{\mathfrak{p}}^{\ell}) = \pi_{\mathfrak{q}}\dot{K}_{\mathfrak{q}}^{\ell}$ is an isomorphism of the bilinear spaces $(\dot{K}_{\mathfrak{p}}/\dot{K}_{\mathfrak{p}}^{\ell}, \beta_{\mathfrak{p}})$ and $(\dot{K}_{\mathfrak{q}}/\dot{K}_{\mathfrak{q}}^{\ell}, \beta_{\mathfrak{q}})$.*

PROOF. Analogously as in the previous lemma. □

REMARK 3.10. *If (T,t) is a Hilbert self-equivalence of degree ℓ of K, then for any $\mathfrak{p} \in \Omega(K)$ the isomorphism t induces a local group isomorphism $t_{\mathfrak{p}}\colon \dot{K}_{\mathfrak{p}}/\dot{K}_{\mathfrak{p}}^{\ell} \to \dot{K}_{T\mathfrak{p}}/\dot{K}_{T\mathfrak{p}}^{\ell}$ preserving Hilbert symbols of degree ℓ (see* [2, Lemma 2.3]*). Notice that $t_{\mathfrak{p}}$ is tame if and only if $\mathfrak{p}$ is a tame prime of the self-equivalence (T,t).*

Assume that (T,t) is a self-equivalence and S is a Hasse set of primes of the field K. The group I_S of S-ideals of K is a free group generated by the set of all prime ideals and T maps finite primes to finite primes, so the bijection $T\colon \Omega(K) \to \Omega(K)$ induces a group automorphism $\widetilde{T}\colon I_S \to I_S$ such that $\widetilde{T}(\prod_{\mathfrak{p}\notin S}\mathfrak{p}^{e_{\mathfrak{p}}}) = \prod_{\mathfrak{p}\notin S}T(\mathfrak{p})^{e_{\mathfrak{p}}}$.

PROPOSITION 3.11. *Assume that the Hilbert self-equivalence (T,t) is tame outside a Hasse set S. The above defined isomorphism $\widetilde{T}\colon I_S \to I_S$ induces a group automorphism $\widehat{T}$ of the group C_S/C_S^ℓ such that $\widehat{T}(\operatorname{cl}\mathfrak{p}C_S^\ell) = \operatorname{cl}T(\mathfrak{p})C_S^\ell$ for every prime $\mathfrak{p} \notin S$.*

PROOF. For any S-ideals $\mathfrak{a}$, $\mathfrak{b}$ of K, we have

$$\operatorname{cl}\mathfrak{a}\,C_S^\ell = \operatorname{cl}\mathfrak{b}\,C_S^\ell \iff x\mathcal{O}_S\cdot\mathfrak{a} = \mathfrak{b}\cdot\mathfrak{J}^\ell \quad \text{for some } x\in\dot{K},\mathfrak{J}\in I_S,$$

that is,

$$\operatorname{cl}\mathfrak{a}\,C_S^\ell = \operatorname{cl}\mathfrak{b}\,C_S^\ell \iff \exists_{x\in\dot{K}}\forall_{\mathfrak{p}\notin S}\ \operatorname{ord}_{\mathfrak{p}}x + \operatorname{ord}_{\mathfrak{p}}\mathfrak{a} \equiv \operatorname{ord}_{\mathfrak{p}}\mathfrak{b} \pmod{\ell}.$$

From the condition

$$\operatorname{ord}_{\mathfrak{p}}x \equiv \operatorname{ord}_{T\mathfrak{p}}tx \pmod{\ell} \quad \text{for every} \quad x\in\dot{K}/\dot{K}^\ell\ \mathfrak{p}\in\Omega(K)\setminus S$$

we get a sequence of equivalences

$$\operatorname{cl}\mathfrak{a}\,C_S^\ell = \operatorname{cl}\mathfrak{b}\,C_S^\ell \iff \exists_{x\in\dot{K}}\forall_{\mathfrak{p}\notin S}\ \operatorname{ord}_{\mathfrak{p}}x + \operatorname{ord}_{\mathfrak{p}}\mathfrak{a} \equiv \operatorname{ord}_{\mathfrak{p}}\mathfrak{b} \pmod{\ell}$$

$$\iff t(x) + \operatorname{ord}_{T\mathfrak{p}}\widetilde{T}(\mathfrak{a}) \equiv \operatorname{ord}_{T\mathfrak{p}}\widetilde{T}(\mathfrak{b}) \pmod{\ell} \iff \operatorname{cl}\widetilde{T}(\mathfrak{a})C_S^\ell = \operatorname{cl}\widetilde{T}(\mathfrak{b})C_S^\ell,$$

which assures us that $\widehat{T}$ is a well defined injection. It is easy to check that $\widehat{T}$ is a group automorphism of C_S/C_S^ℓ. □

Let S and S' be sufficiently large Hasse sets of K. A triple $\big(T_S, t_S, \prod_{\mathfrak{p}\in S}t_{\mathfrak{p}}\big)$ is called a *small S-equivalence* of K if

- $T_S\colon S\to S'$ is a bijection,
- $t_S\colon E_S/\dot{K}^\ell \to E_{S'}/\dot{K}^\ell$ is a group isomorphism,
- for any $\mathfrak{p}\in S$ the mapping $t_{\mathfrak{p}}\colon \dot{K}_{\mathfrak{p}}/\dot{K}_{\mathfrak{p}}^\ell \to \dot{K}_{T\mathfrak{p}}/\dot{K}_{T\mathfrak{p}}^\ell$ is a group isomorphism preserving the symbol $(\,\cdot\,,\,\cdot\,)_{\mathfrak{p}}$ of degree ℓ, i.e.

$$(x,y)_{\mathfrak{p}} = (t_{\mathfrak{p}}x, t_{\mathfrak{p}}y)_{T\mathfrak{p}} \quad \text{for all } x,y\in\dot{K}_{\mathfrak{p}}/\dot{K}_{\mathfrak{p}}^\ell,$$

- the following diagram

$$\begin{array}{ccc} E_S/\dot{K}^\ell & \xrightarrow{\ i_S\ } & \prod_{\mathfrak{p}\in S}\dot{K}_{\mathfrak{p}}/\dot{K}_{\mathfrak{p}}^\ell \\ \big\downarrow{\scriptstyle t_S} & & \big\downarrow{\scriptstyle \prod_{\mathfrak{p}\in S}t_{\mathfrak{p}}} \\ E_{S'}/\dot{K}^\ell & \xrightarrow{\ i_{S'}\ } & \prod_{\mathfrak{p}\in S}\dot{K}_{T\mathfrak{p}}/\dot{K}_{T\mathfrak{p}}^\ell \end{array}$$

commutes, i.e. $t_S(x\dot{K}^\ell) \equiv t_{\mathfrak{p}}(x\dot{K}_{\mathfrak{p}}^\ell) \bmod \dot{K}_{T\mathfrak{p}}^\ell$ for every $\mathfrak{p}\in S$, $x\in E_S$.

The following theorem is a special case of [3, Theorem 3.4].

THEOREM 3.12. *Any small S-equivalence $\left(T_S, t_S, \prod_{\mathfrak{p}\in S} t_{\mathfrak{p}}\right)$ of degree ℓ of the field K can be extended to a self-equivalence (T,t) of K, which is tame outside S, i.e.*

$$\mathfrak{p} \notin \mathcal{W}(T,t) \quad \textit{for every } \mathfrak{p} \in \Omega(K) \setminus S.$$

4. Hilbert self-equivalence with one wild prime

In this section for a field satisfying (C1), (C2) and a finite prime $\mathfrak{p}$, the class of which is an ℓth power in the ideal class group, we shall construct a Hilbert self-equivalence with $\mathfrak{p}$ as the unique wild prime.

As we have already noticed, in the case of a field satisfying (C1) and (C2) every infinite prime is complex. Therefore its Hilbert symbol is trivial and in constructions of self-equivalences we may ignore infinite primes.

THEOREM 4.1. *If K is a number field satisfying* (C1), (C2) *and $\mathfrak{p}$ is a finite prime of K such that $\operatorname{cl}\mathfrak{p} \in C_K^\ell$, then there exists a Hilbert self-equivalence (T,t) of degree ℓ of K such that $\mathcal{W}(T,t) = \{\mathfrak{p}\}$.*

PROOF. Let $\zeta \in K$ be a fixed primitive ℓth root of unity and let $n = \operatorname{rk}_\ell C_K$.

Denote by S_0 the subset of $\Omega(K)$ consisting of all infinite primes and the unique ℓ-adic prime $\mathfrak{r}$. Thus $C_K/C_K^\ell = C_{S_0}/C_{S_0}^\ell$, because $\operatorname{cl}\mathfrak{r} \in C_K^\ell$. Moreover, there exists $x_{\mathfrak{r}} \in \dot{K}$ and a fractional ideal $\mathfrak{J}$ of K such that $(x_{\mathfrak{r}}) = \mathfrak{r}\cdot\mathfrak{J}^\ell$. Of course, $x_{\mathfrak{r}} \in E_{S_0}$ and $\operatorname{rk}_\ell E_{S_0}/\dot{K}^\ell = \operatorname{rk}_\ell E_K/\dot{K}^\ell + 1$.

Choose primes $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ of K such that the classes $\operatorname{cl}\mathfrak{p}_1, \dots, \operatorname{cl}\mathfrak{p}_n$ form a basis of C_K/C_K^ℓ. For $S = S_0 \cup \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ we have $\operatorname{rk}_\ell C_S = 0$ (i.e. S is sufficiently large) and $E_S/\dot{K}^\ell = E_{S_0}/\dot{K}^\ell$.

By Lemma 2.4 there exist $b_1, \dots, b_n \in \Delta_{S_0}$ such that

$$\left(\frac{b_i}{\mathfrak{p}_i}\right)_\ell = \zeta, \quad \left(\frac{b_i}{\mathfrak{p}_j}\right)_\ell = 1 \quad \text{for all } i,j \in \{1,\dots,n\},\ i \neq j.$$

Multiplying $x_{\mathfrak{r}}$, if necessary, by appropriate powers of b_i, $i \in \{1,\dots,n\}$, we may assume that

$$\left(\frac{x_{\mathfrak{r}}}{\mathfrak{p}_i}\right)_\ell = 1 \quad \text{for all } i \in \{1,\dots,n\},$$

which gives $x_{\mathfrak{r}} \in \dot{K}^{\ell}_{\mathfrak{p}_i}$ for $i = 1, \ldots, n$. By Corollary 2.5 we obtain

$$\begin{aligned} E_S/\dot{K}^{\ell} &= E_{S_0}/\dot{K}^{\ell} = U_{S_0}\dot{K}^{\ell}/\dot{K}^{\ell} \oplus \Delta_{S_0}/\dot{K}^{\ell} \\ &= U_K\dot{K}^{\ell}/\dot{K}^{\ell} \oplus \langle x_{\mathfrak{r}}\dot{K}^{\ell}\rangle \oplus \langle b_1\dot{K}^{\ell}, \ldots, b_n\dot{K}^{\ell}\rangle. \end{aligned}$$

We shall proceed dividing the proof into two cases.

(I) Assume that $\mathfrak{p}$ is ℓ-adic, that is, $\mathfrak{p} = \mathfrak{r}$.

Notice that $(y, x_{\mathfrak{r}})_{\mathfrak{v}} = 1$ for every $y \in E_S$ and every prime $\mathfrak{v} \neq \mathfrak{r}$, because y and $x_{\mathfrak{r}}$ are $\mathfrak{v}$-adic units modulo $\dot{K}^{\ell}_{\mathfrak{v}}$. Hence, by Hilbert reciprocity law, $(y, x_{\mathfrak{r}})_{\mathfrak{r}} = 1$ for every $y \in E_S$. In particular the class $u_{\mathfrak{r}}\dot{K}^{\ell}_{\mathfrak{r}}$ of the primary $\mathfrak{r}$-adic unit $u_{\mathfrak{r}}$ does not belong to the group $E_S\dot{K}^{\ell}_{\mathfrak{r}}/\dot{K}^{\ell}_{\mathfrak{r}}$. Thus we clearly have the inclusion

$$E_S\dot{K}^{\ell}_{\mathfrak{r}}/\dot{K}^{\ell}_{\mathfrak{r}} \subseteq \langle \pi_{\mathfrak{r}}\dot{K}^{\ell}_{\mathfrak{r}}\rangle \oplus \langle u_{\mathfrak{r}}\dot{K}^{\ell}_{\mathfrak{r}}, \pi_{\mathfrak{r}}\dot{K}^{\ell}_{\mathfrak{r}}\rangle^{\perp} \tag{4.1}$$

in the bilinear space $(\dot{K}_{\mathfrak{r}}/\dot{K}^{\ell}_{\mathfrak{r}}, \beta_{\mathfrak{r}})$, where $\pi_{\mathfrak{r}} \in \dot{K}_{\mathfrak{r}}$ is a uniformizer such that $\pi_{\mathfrak{r}} \equiv x_{\mathfrak{r}} \mod \dot{K}^{\ell}_{\mathfrak{r}}$.

Let us define the triple $(T_S, t_S, \prod_{\mathfrak{v}\in S} t_{\mathfrak{v}})$ as follows

$$\begin{aligned} &T_S\colon S \to S, && T_S = \mathrm{id}_S, \\ &t_S\colon E_S/\dot{K}^{\ell} \to E_S/\dot{K}^{\ell}, && t_S = \mathrm{id}_{E_S/\dot{K}^{\ell}} \\ &t_{\mathfrak{v}}\colon \dot{K}_{\mathfrak{v}}/\dot{K}^{\ell}_{\mathfrak{v}} \to \dot{K}_{\mathfrak{v}}/\dot{K}^{\ell}_{\mathfrak{v}}, && t_{\mathfrak{v}} = \mathrm{id}_{\dot{K}_{\mathfrak{v}}/\dot{K}^{\ell}_{\mathfrak{v}}} \quad \text{for every } \mathfrak{v} \in S \setminus \{\mathfrak{r}\}, \\ &t_{\mathfrak{r}}\colon \dot{K}_{\mathfrak{r}}/\dot{K}^{\ell}_{\mathfrak{r}} \to \dot{K}_{\mathfrak{r}}/\dot{K}^{\ell}_{\mathfrak{r}}, && t_{\mathfrak{r}}(u_{\mathfrak{r}}\dot{K}^{\ell}_{\mathfrak{r}}) = u_{\mathfrak{r}}\pi_{\mathfrak{r}}\dot{K}^{\ell}_{\mathfrak{r}},\ t_{\mathfrak{r}}(\pi_{\mathfrak{r}}\dot{K}^{\ell}_{\mathfrak{r}}) = \pi_{\mathfrak{r}}\dot{K}^{\ell}_{\mathfrak{r}}, \text{ and} \\ & && t_{\mathfrak{r}}(v\dot{K}^{\ell}_{\mathfrak{r}}) = v\dot{K}^{\ell}_{\mathfrak{r}} \text{ for every } \ v\dot{K}^{\ell}_{\mathfrak{r}} \in \langle u_{\mathfrak{r}}\dot{K}^{\ell}_{\mathfrak{r}}, \pi_{\mathfrak{r}}\dot{K}^{\ell}_{\mathfrak{r}}\rangle^{\perp}. \end{aligned}$$

Every isomorphism $t_{\mathfrak{v}}$ for $\mathfrak{v} \in S$ as well as $t_{\mathfrak{r}}$, by Lemma 3.8, preserve Hilbert symbols.

We have the following equivalences

$$t_S(y\dot{K}^{\ell}) \equiv t_{\mathfrak{v}}(y\dot{K}^{\ell}_{\mathfrak{v}}) \bmod \dot{K}^{\ell}_{\mathfrak{v}} \quad \text{for all } y \in E_S,\ \mathfrak{v} \in S.$$

Indeed, for $\mathfrak{v} \neq \mathfrak{r}$ it follows directly from the definitions of t_S and local isomorphisms, whereas for $\mathfrak{v} = \mathfrak{r}$, by (4.1), we have $t_{\mathfrak{r}}(z\dot{K}^{\ell}_{\mathfrak{v}}) = z\dot{K}^{\ell}_{\mathfrak{v}} \equiv t_S(z\dot{K}^{\ell}) \bmod \dot{K}^{\ell}_{\mathfrak{v}}$ for every $z \in E_S$.

Thus, we showed that

$$\prod_{\mathfrak{v}\in S} t_{\mathfrak{v}} \circ i_S = i_S \circ t_S$$

and $(T_S, t_S, \prod_{\mathfrak{v}\in S} t_{\mathfrak{v}})$ is a small S-equivalence. By Theorem 3.12, it can be extended to a Hilbert self-equivalence (T,t), which is tame outside S. Notice that the prime $\mathfrak{r}$ is the only prime in S, for which the local isomorphism $t_{\mathfrak{r}}$ is not tame. Hence $\mathcal{W}(T,t) = \{\mathfrak{r}\}$.

(II) Suppose $\mathfrak{p} \neq \mathfrak{r}$.

Since the class $\operatorname{cl}\mathfrak{p}$ is an ℓth power in the group C_K, there exist a fractional ideal $\mathfrak{J}_1$ and $x_{\mathfrak{p}} \in \dot{K}$ such that $(x_{\mathfrak{p}}) = \mathfrak{p}\cdot\mathfrak{J}_1^{\ell}$. Let $S_1 = S \cup \{\mathfrak{p}\}$. Obviously, $x_{\mathfrak{p}} \in E_{S_1}$, $\operatorname{rk}_\ell C_{S_1} = 0$ and $\operatorname{rk}_\ell E_{S_1}/\dot{K}^\ell = \operatorname{rk}_\ell E_S/\dot{K}^\ell + 1$. This gives the following decomposition

$$\begin{aligned} E_{S_1}/\dot{K}^\ell &= E_S/\dot{K}^\ell \oplus \langle x_{\mathfrak{p}}\dot{K}^\ell\rangle \\ &= U_K\dot{K}^\ell/\dot{K}^\ell \oplus \langle b_1\dot{K}^\ell, \dots, b_n\dot{K}^\ell\rangle \oplus \langle x_{\mathfrak{r}}\dot{K}^\ell\rangle \oplus \langle x_{\mathfrak{p}}\dot{K}^\ell\rangle. \end{aligned}$$

Now consider two subcases.

(II.1) Assume $\left(\dfrac{y}{\mathfrak{p}}\right)_\ell = 1$ for every $y \in E_S$.

It follows that

$$E_S\dot{K}_{\mathfrak{p}}^{\ell}/\dot{K}_{\mathfrak{p}}^{\ell} = \{\dot{K}_{\mathfrak{p}}^{\ell}\}. \tag{4.2}$$

Let us take a $\mathfrak{p}$-adic uniformizer $\pi_{\mathfrak{p}}$ such that $\pi_{\mathfrak{p}} \equiv x_{\mathfrak{p}} \bmod \dot{K}_{\mathfrak{p}}^{\ell}$ and define the triple $(T_{S_1}, t_{S_1}, \prod_{\mathfrak{v}\in S_1} t_{\mathfrak{v}})$ as follows

$$\begin{aligned} &T_{S_1}\colon S_1 \to S_1, && T_{S_1} = \mathrm{id}_{S_1}, \\ &t_{S_1}\colon E_{S_1}/\dot{K}^\ell \to E_{S_1}/\dot{K}^\ell, && t_{S_1} = \mathrm{id}_{E_{S_1}/\dot{K}^\ell}, \\ &t_{\mathfrak{v}}\colon \dot{K}_{\mathfrak{v}}/\dot{K}_{\mathfrak{v}}^{\ell} \to \dot{K}_{\mathfrak{v}}/\dot{K}_{\mathfrak{v}}^{\ell}, && t_{\mathfrak{v}} = \mathrm{id}_{\dot{K}_{\mathfrak{v}}/\dot{K}_{\mathfrak{v}}^{\ell}} \quad \text{for every } \mathfrak{v} \in S, \\ &t_{\mathfrak{p}}\colon \dot{K}_{\mathfrak{p}}/\dot{K}_{\mathfrak{p}}^{\ell} \to \dot{K}_{\mathfrak{p}}/\dot{K}_{\mathfrak{p}}^{\ell}, && t_{\mathfrak{p}}(u_{\mathfrak{p}}\dot{K}_{\mathfrak{p}}^{\ell}) = u_{\mathfrak{p}}\pi_{\mathfrak{p}}\dot{K}_{\mathfrak{p}}^{\ell}, \quad t_{\mathfrak{p}}(\pi_{\mathfrak{p}}\dot{K}_{\mathfrak{p}}^{\ell}) = \pi_{\mathfrak{p}}\dot{K}_{\mathfrak{p}}^{\ell}. \end{aligned}$$

The isomorphism $t_{\mathfrak{v}}$ for $\mathfrak{v} \in S$ as well as $t_{\mathfrak{p}}$, by Lemma 3.9, preserve Hilbert symbols. By (4.2), $y \equiv 1 \bmod \dot{K}_{\mathfrak{p}}^{\ell}$ for every $y \in E_S$, so

$$t_{\mathfrak{p}}(y\dot{K}_{\mathfrak{p}}^{\ell}) = 1\dot{K}_{\mathfrak{p}}^{\ell} \equiv y\dot{K}_{\mathfrak{p}}^{\ell} \equiv t_{S_1}(y\dot{K}^\ell) \bmod \dot{K}_{\mathfrak{p}}^{\ell}.$$

Moreover,

$$t_{\mathfrak{p}}(x_{\mathfrak{p}}\dot{K}_{\mathfrak{p}}^{\ell}) = t_{\mathfrak{p}}(\pi_{\mathfrak{p}}\dot{K}_{\mathfrak{p}}^{\ell}) = \pi_{\mathfrak{p}}\dot{K}_{\mathfrak{p}}^{\ell} \equiv x_{\mathfrak{p}}\dot{K}^\ell \equiv t_{S_1}(x_{\mathfrak{p}}\dot{K}^\ell) \bmod \dot{K}_{\mathfrak{p}}^{\ell}.$$

Directly from the definition

$$t_{\mathfrak{v}}(y\dot{K}_{\mathfrak{v}}^{\ell}) = y\dot{K}_{\mathfrak{v}}^{\ell} \equiv t_{S_1}(y\dot{K}^\ell) \bmod \dot{K}_{\mathfrak{v}}^{\ell}$$

for every $y \in E_{S_1}$ and $\mathfrak{v} \in S$. Thus $\left(T_{S_1}, t_{S_1}, \prod_{\mathfrak{v}\in S_1} t_{\mathfrak{v}}\right)$ is a small S_1-equivalence of K. In the same way as in the previous case we conclude that there exists a self-equivalence (T,t), for which $\mathfrak{p}$ is the only wild prime.

(II.2) Assume $\left(\frac{c}{\mathfrak{p}}\right)_\ell \neq 1$ for some $c \in E_S$.

Replacing c by its power, we may assume $\left(\frac{c}{\mathfrak{p}}\right)_\ell = \zeta$, which follows $c \equiv u_{\mathfrak{p}} \bmod \dot{K}_{\mathfrak{p}}^\ell$ and $(c, x_{\mathfrak{p}})_{\mathfrak{p}} = \zeta$. Let V be a direct complement of the subgroup $\langle c\dot{K}^\ell\rangle$ to the group $E_S/\dot{K}^\ell$, i.e.

$$E_S/\dot{K}^\ell = V \oplus \langle c\dot{K}^\ell\rangle.$$

Multiplying, if needed, the elements of the group V by appropriate powers of c we may assume that $\left(\frac{z}{\mathfrak{p}}\right)_\ell = 1$ for every $z \in V$, and then $V \subseteq \dot{K}_{\mathfrak{p}}^\ell$. Thus

$$E_{S_1}/\dot{K}^\ell = V \oplus \langle c\dot{K}^\ell\rangle \oplus \langle x_{\mathfrak{p}}\dot{K}^\ell\rangle.$$

By [8, Lemma 2.1], there exists $x_{\mathfrak{q}} \in \dot{K}$ and $\mathfrak{q} \notin S$ such that

$$\begin{aligned}
&x_{\mathfrak{q}} \in \dot{K}_{\mathfrak{v}}^\ell && \text{for every } \mathfrak{v} \in S \setminus \{\mathfrak{r}\},\\
&x_{\mathfrak{q}} \equiv x_{\mathfrak{p}} \bmod \dot{K}_{\mathfrak{r}}^\ell,\\
&\operatorname{ord}_{\mathfrak{q}} x_{\mathfrak{q}} = 1,\\
&\operatorname{ord}_{\mathfrak{v}} x_{\mathfrak{q}} \equiv 0 \pmod{\ell} && \text{for every } \mathfrak{v} \in \Omega(K) \setminus (S \cup \{\mathfrak{q}\}).
\end{aligned}$$

Fix a $\mathfrak{q}$-adic uniformizer $\pi_{\mathfrak{q}} \equiv x_{\mathfrak{q}} \bmod \dot{K}_{\mathfrak{q}}^\ell$. The set $S_1' = S \cup \{\mathfrak{q}\}$ is sufficiently large and we have

$$E_{S_1'}/\dot{K}^\ell = E_S/\dot{K}^\ell \oplus \langle x_{\mathfrak{q}}\dot{K}^\ell\rangle = V \oplus \langle c\dot{K}^\ell\rangle \oplus \langle x_{\mathfrak{q}}\dot{K}^\ell\rangle.$$

Define $(T_{S_1}, t_{S_1}, \prod_{\mathfrak{v}\in S_1} t_{\mathfrak{v}})$ as follows

$$\begin{aligned}
&T_{S_1}\colon S_1 \to S_1', && T_{S_1}|_S = \mathrm{id}_S,\ T_{S_1}(\mathfrak{p}) = \mathfrak{q},\\
&t_{S_1}\colon E_{S_1}/\dot{K}^\ell \to E_{S_1'}/\dot{K}^\ell, && t_{S_1}|_V = \mathrm{id}_V,\ t_{S_1}(c\dot{K}^\ell) = cx_{\mathfrak{q}}\dot{K}^\ell,\\
&\qquad && t_{S_1}(x_{\mathfrak{p}}\dot{K}^\ell) = x_{\mathfrak{q}}\dot{K}^\ell,\\
&t_{\mathfrak{v}}\colon \dot{K}_{\mathfrak{v}}/\dot{K}_{\mathfrak{v}}^\ell \to \dot{K}_{\mathfrak{v}}/\dot{K}_{\mathfrak{v}}^\ell, && t_{\mathfrak{v}} = \mathrm{id}_{\dot{K}_{\mathfrak{v}}/\dot{K}_{\mathfrak{v}}^\ell} \quad \text{for every } \mathfrak{v} \in S_1 \setminus \{\mathfrak{r},\mathfrak{p}\},\\
&t_{\mathfrak{p}}\colon \dot{K}_{\mathfrak{p}}/\dot{K}_{\mathfrak{p}}^\ell \to \dot{K}_{\mathfrak{q}}/\dot{K}_{\mathfrak{q}}^\ell, && t_{\mathfrak{p}}(u_{\mathfrak{p}}\dot{K}_{\mathfrak{p}}^\ell) = u_{\mathfrak{q}}\pi_{\mathfrak{q}}\dot{K}_{\mathfrak{q}}^\ell, \quad t_{\mathfrak{p}}(\pi_{\mathfrak{p}}\dot{K}_{\mathfrak{p}}^\ell) = \pi_{\mathfrak{q}}\dot{K}_{\mathfrak{q}}^\ell.
\end{aligned} \tag{4.3}$$

As in the previous cases we see that all defined above local isomorphisms preserve Hilbert symbols.

Definition of a local isomorphism $t_{\mathfrak{r}}\colon \dot{K}_{\mathfrak{r}}/\dot{K}_{\mathfrak{r}}^{\ell} \to \dot{K}_{\mathfrak{r}}/\dot{K}_{\mathfrak{r}}^{\ell}$ left. To define it we use Lemma 3.7. Consider subgroups $H = E_{S_1}\dot{K}_{\mathfrak{r}}^{\ell}/\dot{K}_{\mathfrak{r}}^{\ell}$ and $H' = E_{S_1'}\dot{K}_{\mathfrak{r}}^{\ell}/\dot{K}_{\mathfrak{r}}^{\ell}$ of $\dot{K}_{\mathfrak{r}}/\dot{K}_{\mathfrak{r}}^{\ell}$. We shall show that t_{S_1} induces a tame isomorphism $H \to H'$ preserving Hilbert symbols.

We start with checking that t_{S_1} preserves the $\mathfrak{r}$-adic Hilbert symbol. It is obvious that $(t_{S_1}(y),\, t_{S_1}(z))_{\mathfrak{r}} = (y,\, z)_{\mathfrak{r}}$ for $y, z \in V$. By the choice of $x_{\mathfrak{q}}$ we have $x_{\mathfrak{p}} = x_{\mathfrak{q}} \bmod \dot{K}_{\mathfrak{r}}^{\ell}$, so

$$(y, x_{\mathfrak{p}})_{\mathfrak{r}} = (y, x_{\mathfrak{q}})_{\mathfrak{r}} \text{ for every } y \in V$$

and

$$(c, x_{\mathfrak{p}})_{\mathfrak{r}} = (c, x_{\mathfrak{q}})_{\mathfrak{r}} = (cx_{\mathfrak{q}},\, x_{\mathfrak{q}})_{\mathfrak{r}}.$$

Now notice that $(y, c)_{\mathfrak{v}} = 1$ for every prime $\mathfrak{v} \neq \mathfrak{r}$, and then $(y, c)_{\mathfrak{r}} = 1$ by Hilbert reciprocity law. By the choice of $x_{\mathfrak{p}}$ and $x_{\mathfrak{q}}$,

$$(y, cx_{\mathfrak{q}})_{\mathfrak{r}} = (y, c)_{\mathfrak{r}} \cdot (y, x_{\mathfrak{q}})_{\mathfrak{r}} = (y, c)_{\mathfrak{r}} \cdot (y, x_{\mathfrak{p}})_{\mathfrak{r}} = 1 = (y, c)_{\mathfrak{r}} \text{ for } y \in V.$$

Thus t_{S_1} preserves the $\mathfrak{r}$-adic Hilbert symbol.

Now we shall show that t_{S_1} induces a group isomorphism $\tau\colon H \to H'$. To do this it suffices to explain that

$$H = V\dot{K}_{\mathfrak{r}}^{\ell} \oplus \langle x_{\mathfrak{p}}\dot{K}_{\mathfrak{r}}^{\ell}\rangle \oplus \langle c\dot{K}_{\mathfrak{r}}^{\ell}\rangle, \quad H' = V\dot{K}_{\mathfrak{r}}^{\ell} \oplus \langle x_{\mathfrak{q}}\dot{K}_{\mathfrak{r}}^{\ell}\rangle \oplus \langle cx_{\mathfrak{q}}\dot{K}_{\mathfrak{r}}^{\ell}\rangle. \tag{4.4}$$

By the definition of H and H',

$$H = V\dot{K}_{\mathfrak{r}}^{\ell} + \langle x_{\mathfrak{p}}\dot{K}_{\mathfrak{r}}^{\ell}\rangle + \langle c\dot{K}_{\mathfrak{r}}^{\ell}\rangle, \quad H' = V\dot{K}_{\mathfrak{r}}^{\ell} + \langle x_{\mathfrak{q}}\dot{K}_{\mathfrak{r}}^{\ell}\rangle + \langle cx_{\mathfrak{q}}\dot{K}_{\mathfrak{r}}^{\ell}\rangle,$$

whereas by the choice of $x_{\mathfrak{q}}$, $V\dot{K}_{\mathfrak{r}}^{\ell} \oplus \langle x_{\mathfrak{p}}\dot{K}_{\mathfrak{r}}^{\ell}\rangle = V\dot{K}_{\mathfrak{r}}^{\ell} \oplus \langle x_{\mathfrak{q}}\dot{K}_{\mathfrak{r}}^{\ell}\rangle$.
On the one hand for every prime $\mathfrak{v} \notin \{\mathfrak{p}, \mathfrak{r}\}$ elements of $V \oplus \langle x_{\mathfrak{p}}\dot{K}^{\ell}\rangle$ are (modulo ℓth powers) $\mathfrak{v}$-adic units, so $(y, x_{\mathfrak{p}})_{\mathfrak{v}} = 1$. Moreover, for $y = zx_{\mathfrak{p}}$, $z \in V \subseteq \dot{K}_{\mathfrak{p}}^{\ell}$ we have $(y, x_{\mathfrak{p}})_{\mathfrak{p}} = (z, x_{\mathfrak{p}})_{\mathfrak{p}}(x_{\mathfrak{p}}, x_{\mathfrak{p}})_{\mathfrak{p}} = 1$. Thus $(y, x_{\mathfrak{p}})_{\mathfrak{r}} = 1$ by Hilbert reciprocity law and

$$(y, x_{\mathfrak{p}})_{\mathfrak{r}} = 1,\ \forall y \in V\dot{K}_{\mathfrak{r}}^{\ell} \oplus \langle x_{\mathfrak{p}}\dot{K}_{\mathfrak{r}}^{\ell}\rangle. \tag{4.5}$$

On the other hand $(c, x_{\mathfrak{p}})_{\mathfrak{v}} = 1$ for every $\mathfrak{v} \notin \{\mathfrak{p}, \mathfrak{r}\}$, because c and $x_{\mathfrak{p}}$ are (modulo ℓth powers) $\mathfrak{v}$-adic units. Thus, by Hilbert reciprocity law, $(c, x_{\mathfrak{p}})_{\mathfrak{r}} = (c, x_{\mathfrak{p}})_{\mathfrak{p}}^{-1} = \zeta^{-1} \neq 1$. This fact together with (4.5) gives $c\dot{K}_{\mathfrak{r}}^{\ell} \notin V\dot{K}_{\mathfrak{r}}^{\ell} \oplus \langle x_{\mathfrak{p}}\dot{K}_{\mathfrak{r}}^{\ell}\rangle$ and explain the first part of (4.4).

To explain the second part of (4.4) observe that

$$(cx_{\mathfrak{p}}, x_{\mathfrak{p}})_{\mathfrak{r}} = (c, x_{\mathfrak{p}})_{\mathfrak{r}} \cdot (x_{\mathfrak{p}}, x_{\mathfrak{p}})_{\mathfrak{r}} = \zeta^{-1} \neq 1,$$

which with (4.5) gives $cx_{\mathfrak{p}}\dot{K}_{\mathfrak{r}}^{\ell} \notin V\dot{K}_{\mathfrak{r}}^{\ell} \oplus \langle x_{\mathfrak{q}}\dot{K}_{\mathfrak{r}}^{\ell}\rangle$.

As we notice above, the isomorphism τ preserves the $\mathfrak{r}$-adic Hilbert symbol. Notice else that

$$\operatorname{ord}_{\mathfrak{r}} x_{\mathfrak{p}} \equiv 0 \equiv \operatorname{ord}_{\mathfrak{r}} x_{\mathfrak{q}} \pmod{\ell} \quad \text{and} \quad \operatorname{ord}_{\mathfrak{r}} c \equiv \operatorname{ord}_{\mathfrak{r}} cx_{\mathfrak{q}} \pmod{\ell}.$$

Hence τ is tame on H.

Now assume that the group H contains the $\mathfrak{r}$-adic primary unit $u_{\mathfrak{r}}$. Then $u_{\mathfrak{r}}\dot{K}_{\mathfrak{r}}^{\ell} = v^{\epsilon_1}c^{\epsilon_2}x_{\mathfrak{p}}^{\epsilon_3}\dot{K}_{\mathfrak{r}}^{\ell}$ for some $\epsilon_1, \epsilon_2, \epsilon_3 \in \{0, 1, \ldots, \ell-1\}$. Since $x_{\mathfrak{p}}$ is an $\mathfrak{r}$-adic unit modulo $\dot{K}_{\mathfrak{r}}^{\ell}$, $(u_{\mathfrak{r}}, x_{\mathfrak{p}})_{\mathfrak{r}} = 1$. Thus

$$1 = (v^{\epsilon_1}c^{\epsilon_2}x_{\mathfrak{p}}^{\epsilon_3},\, x_{\mathfrak{p}})_{\mathfrak{r}} = (v,\, x_{\mathfrak{p}})_{\mathfrak{r}}^{\epsilon_1}(c,\, x_{\mathfrak{p}})_{\mathfrak{r}}^{\epsilon_2}(x_{\mathfrak{p}},\, x_{\mathfrak{p}})_{\mathfrak{r}}^{\epsilon_3} = (c, x_{\mathfrak{p}})_{\mathfrak{r}}^{\epsilon_2} = \zeta^{-\epsilon_2}.$$

Hence $\epsilon_2 = 0$ and $u_{\mathfrak{r}}\dot{K}_{\mathfrak{r}}^{\ell} \in V\dot{K}_{\mathfrak{r}}^{\ell} \oplus \langle x_{\mathfrak{p}}\dot{K}_{\mathfrak{r}}^{\ell}\rangle$. By the definition of t_{S_1} we get $\tau(u_{\mathfrak{r}}\dot{K}_{\mathfrak{r}}^{\ell}) = u_{\mathfrak{r}}\dot{K}_{\mathfrak{r}}^{\ell}$.

Finally, all assumptions of Lemma 3.7 hold, so there exists a tame isomorphism $t_{\mathfrak{r}}\colon \dot{K}_{\mathfrak{r}}/\dot{K}_{\mathfrak{r}}^{\ell} \to \dot{K}_{\mathfrak{r}}/\dot{K}_{\mathfrak{r}}^{\ell}$, which is an extension of τ and preserves Hilbert symbols of degree ℓ.

We shall show that $(T_{S_1}, t_{S_1}, \prod_{\mathfrak{v}\in S_1} t_{\mathfrak{v}})$ is a small S_1-equivalence. To do this we should prove that the diagram

$$\begin{array}{ccc} E_{S_1}/\dot{K}^{\ell} & \xrightarrow{i_{S_1}} & \prod_{\mathfrak{v}\in S_1} \dot{K}_{\mathfrak{v}}/\dot{K}_{\mathfrak{v}}^{\ell} \\ \big\downarrow t_{S_1} & & \big\downarrow \prod_{\mathfrak{v}\in S_1} t_{\mathfrak{v}} \\ E_{S_1'}/\dot{K}^{\ell} & \xrightarrow{i_{S_1'}} & \prod_{\mathfrak{v}\in S_1} \dot{K}_{T\mathfrak{v}}/\dot{K}_{T\mathfrak{v}}^{\ell} \end{array}$$

commutes, i.e.

$$t_{S_1}(x\dot{K}^{\ell}) \equiv t_{\mathfrak{v}}(x\dot{K}_{\mathfrak{v}}^{\ell}) \bmod \dot{K}_{T\mathfrak{v}}^{\ell}, \quad \text{for every } \mathfrak{v} \in S_1,\ x \in E_{S_1}. \tag{4.6}$$

Of course, $t_{S_1}(z\dot{K}^{\ell}) \equiv t_{\mathfrak{r}}(z\dot{K}_{\mathfrak{r}}^{\ell}) \bmod \dot{K}_{\mathfrak{r}}^{\ell}$ for every $z \in E_{S_1}$, by the definition of $t_{\mathfrak{r}}$.

Now concentrate on proving $t_{S_1}(z\dot{K}^{\ell}) \equiv t_{\mathfrak{p}}(z\dot{K}_{\mathfrak{p}}^{\ell}) \bmod \dot{K}_{\mathfrak{q}}^{\ell}$ for every $z \in E_{S_1}$ It suffices to show this congruence for $x_{\mathfrak{p}}, c$ and $z \in V$.

By the definitions (4.3),

$$t_{S_1}(x_{\mathfrak{p}}\dot{K}^{\ell}) = x_{\mathfrak{q}}\dot{K}^{\ell} \equiv \pi_{\mathfrak{q}}\dot{K}_{\mathfrak{q}}^{\ell} = t_{\mathfrak{p}}(\pi_{\mathfrak{p}}\dot{K}_{\mathfrak{p}}^{\ell}) \equiv t_{\mathfrak{p}}(x_{\mathfrak{p}}\dot{K}_{\mathfrak{p}}^{\ell}) \bmod \dot{K}_{\mathfrak{q}}^{\ell}.$$

Since the local isomorphisms preserve Hilbert symbols,

$$(c, x_{\mathfrak{q}})_{\mathfrak{q}} = (c, x_{\mathfrak{q}})_{\mathfrak{q}}(x_{\mathfrak{q}}, x_{\mathfrak{q}})_{\mathfrak{q}} = (cx_{\mathfrak{q}}, x_{\mathfrak{q}})_{\mathfrak{q}} = (c, x_{\mathfrak{p}})_{\mathfrak{p}} = \zeta,$$

which means that $c = u_{\mathfrak{q}}$ is the primary $\mathfrak{q}$-adic unit. Thus

$$t_{S_1}(c\dot{K}^{\ell}) = cx_{\mathfrak{q}}\dot{K}^{\ell} \equiv u_{\mathfrak{q}}\pi_{\mathfrak{q}}\dot{K}_{\mathfrak{q}}^{\ell} = t_{\mathfrak{p}}(u_{\mathfrak{p}}\dot{K}_{\mathfrak{p}}^{\ell}) \equiv t_{\mathfrak{p}}(c\dot{K}_{\mathfrak{p}}^{\ell}) \bmod \dot{K}_{\mathfrak{q}}^{\ell}.$$

If $z \in V$, then $z \in \dot{K}_{\mathfrak{p}}^{\ell}$ and from $(z, x_{\mathfrak{q}})_{\mathfrak{q}} = (z, x_{\mathfrak{p}})_{\mathfrak{p}} = 1$ it follows $z \in \dot{K}_{\mathfrak{q}}^{\ell}$. Hence

$$t_{S_1}(z\dot{K}^{\ell}) = z\dot{K}^{\ell} \equiv 1\dot{K}_{\mathfrak{q}}^{\ell} = t_{\mathfrak{p}}(1\dot{K}_{\mathfrak{p}}^{\ell}) \equiv t_{\mathfrak{p}}(z\dot{K}_{\mathfrak{p}}^{\ell}) \bmod \dot{K}_{\mathfrak{q}}^{\ell}.$$

Now suppose $\mathfrak{v} \in S_1 \setminus \{\mathfrak{r}, \mathfrak{p}\}$. Obviously, for $z \in V$ the congruence in (4.6) holds. By the choice, the elements $x_{\mathfrak{p}}$, $x_{\mathfrak{q}}$ are local ℓth powers in $\dot{K}_{\mathfrak{v}}$ and then

$$t_{S_1}(x_{\mathfrak{p}}\dot{K}^{\ell}) = x_{\mathfrak{q}}\dot{K}^{\ell} \equiv 1\dot{K}_{\mathfrak{v}}^{\ell} = t_{\mathfrak{v}}(1\dot{K}_{\mathfrak{v}}^{\ell}) \equiv t_{\mathfrak{v}}(x_{\mathfrak{p}}\dot{K}_{\mathfrak{p}}^{\ell}) \bmod \dot{K}_{\mathfrak{v}}^{\ell},$$

$$t_{S_1}(c\dot{K}^{\ell}) = cx_{\mathfrak{q}}\dot{K}^{\ell} \equiv c\dot{K}_{\mathfrak{v}}^{\ell} = t_{\mathfrak{v}}(c\dot{K}_{\mathfrak{v}}^{\ell}) \bmod \dot{K}_{\mathfrak{v}}^{\ell}.$$

We have just completed proving (4.6). Thus the triple $(T_{S_1}, t_{S_1}, \prod_{\mathfrak{v}\in S_1} t_{\mathfrak{v}})$ is a small S_1-equivalence of degree ℓ, for which $t_{\mathfrak{p}}$ is the only one wild isomorphism.

Analogously as in the case (I) one shows that this equivalence has an extension to a self-equivalence (T, t) of degree ℓ of the field K such that $\mathcal{W}(T, t) = \{\mathfrak{p}\}$. □

5. Summary

In this section we present the proof of the main Theorem 1.1. We precede this proof with some remarks on the wild sets of a composition of two self-equivalences.

Suppose K is a fixed number field containing a primitive ℓth root of unity and (T_1, t_1), (T_2, t_2) are fixed Hilbert self-equivalences of degree ℓ of K. To simplify the notation denote the wild sets $\mathcal{W}(T_1, t_1)$, $\mathcal{W}(T_2, t_2)$ by $\mathcal{W}_{T_1}$, $\mathcal{W}_{T_2}$, respectively.

Of course, the pair $(T_2 \circ T_1,\ t_2 \circ t_1)$ is a Hilbert self-equivalence of degree ℓ of K. We shall describe the wild set $\mathcal{W}_{T_2\circ T_1} = \mathcal{W}(T_2 \circ T_1,\ t_2 \circ t_1)$ of the self-equivalence $(T_2 \circ T_1,\ t_2 \circ t_1)$.

Suppose $\mathfrak{p}$ is a finite prime such that $\mathfrak{p} \notin \mathcal{W}_{T_1}$ and $T_1\mathfrak{p} \notin \mathcal{W}_{T_2}$. Then

$$\operatorname{ord}_{T_2T_1\mathfrak{p}} t_2t_1x \equiv \operatorname{ord}_{T_1\mathfrak{p}} t_1x \equiv \operatorname{ord}_{\mathfrak{p}} x \pmod{\ell} \quad \text{for every } x \in \dot{K}.$$

Thus we clearly have the inclusion

$$\mathcal{W}_{T_2\circ T_1} \subseteq T_1^{-1}(\mathcal{W}_{T_2}) \cup \mathcal{W}_{T_1}.$$

Now let $\mathfrak{p}$ be a finite prime such that $\mathfrak{p} \in \mathcal{W}_{T_1}$, $T_1\mathfrak{p} \notin \mathcal{W}_{T_2}$ and let $a \in \dot{K}$ be such that $\operatorname{ord}_{T_1\mathfrak{p}} t_1a \not\equiv \operatorname{ord}_{\mathfrak{p}} a \pmod{\ell}$. Then

$$\operatorname{ord}_{T_2T_1\mathfrak{p}} t_2t_1a \equiv \operatorname{ord}_{T_1\mathfrak{p}} t_1a \not\equiv \operatorname{ord}_{\mathfrak{p}} a \pmod{\ell},$$

i.e. $\mathfrak{p}$ is a wild prime of the self-equivalence $(T_2 \circ T_1,\ t_2 \circ t_1)$.

Next assume $\mathfrak{p}$ is a finite prime such that $\mathfrak{p} \notin \mathcal{W}_{T_1}$ and $T_1\mathfrak{p} \in \mathcal{W}_{T_2}$, but $b = t_1(a) \in \dot{K}$ is such that $\operatorname{ord}_{T_2T_1\mathfrak{p}} t_2b \not\equiv \operatorname{ord}_{T_1\mathfrak{p}} b \pmod{\ell}$. Then

$$\operatorname{ord}_{T_2T_1\mathfrak{p}} t_2t_1a \not\equiv \operatorname{ord}_{T_1\mathfrak{p}} t_1a \equiv \operatorname{ord}_{\mathfrak{p}} a \pmod{\ell},$$

i.e. $\mathfrak{p}$ is a wild prime of the self-equivalence $(T_2 \circ T_1,\ t_2 \circ t_1)$.

As a result we get the inclusion

$$(\mathcal{W}_{T_1} \setminus T_1^{-1}(\mathcal{W}_{T_2})) \cup (T_1^{-1}(\mathcal{W}_{T_2}) \setminus \mathcal{W}_{T_1}) \subseteq \mathcal{W}_{T_2\circ T_1},$$

that is

$$(\mathcal{W}_{T_1} \cup T_1^{-1}(\mathcal{W}_{T_2})) \setminus (\mathcal{W}_{T_1} \cap T_1^{-1}(\mathcal{W}_{T_2})) \subseteq \mathcal{W}_{T_2\circ T_1}.$$

COROLLARY 5.1. *Under the above assumptions and notations, if*

$$\mathcal{W}_{T_1} \cap T_1^{-1}(\mathcal{W}_{T_2}) = \emptyset$$

then

$$\mathcal{W}_{T_2\circ T_1} = \mathcal{W}_{T_1} \cup T_1^{-1}(\mathcal{W}_{T_2}).$$

It is better to formulate this corollary in a more useful way.

COROLLARY 5.2. *Assume A, B are disjoint sets of finite primes of the field K. If (T_1, t_1), (T_2, t_2) are Hilbert self-equivalences of degree ℓ of K such that $\mathcal{W}(T_1, t_1) = A$ and $\mathcal{W}(T_2, t_2) = T_1(B)$, then $\mathcal{W}(T_2 \circ T_1,\ t_2 \circ t_1) = A \cup B$.*

We use this corollary to prove the main theorem.

PROOF OF THEOREM 1.1. We use induction on k. For $k = 1$ the statement follows from Theorem 4.1. Now suppose $k > 1$. By Theorem 4.1, there exists a Hilbert self-eqivalence (T_1, t_1) of degree ℓ of K such that $\mathcal{W}(T_1, t_1) = \{\mathfrak{p}_1\}$. Denote $\mathfrak{q}_i = T_1\mathfrak{p}_i$ for $i = 2, \dots, k$. By Proposition 3.11, the classes $\operatorname{cl}\mathfrak{q}_2, \dots, \operatorname{cl}\mathfrak{q}_k$ are ℓth powers in the group C_K. Thus the inductive assumption gives us a Hilbert self-equivalence (T_2, t_2) of degree ℓ of K such that $\mathcal{W}(T_2, t_2) = \{\mathfrak{q}_2, \dots, \mathfrak{q}_k\}$. By Corollary 5.2, we obtain $\mathcal{W}(T_2 \circ T_1,\ t_2 \circ t_1) = \{\mathfrak{p}_1\} \cup \{\mathfrak{p}_2, \dots, \mathfrak{p}_k\}$. □

6. Final remarks

Our main result required assumptions both on the field K and the primes playing a role in the final statement. The fact that K should contain a primitive ℓth root of unity is unquestionable, but the necessity of the assumptions on the primes might be discussed. It is obvious that changing part of them will influence the final result. Below we give an example.

Let K be a number field containing a primitive ℓth root of unity (we do not assume (C2)). Notice that for any finite prime $\mathfrak{v}$ of K the mapping $\varphi\colon \dot{K}_\mathfrak{v}/\dot{K}_\mathfrak{v}^\ell \to \dot{K}_\mathfrak{v}/\dot{K}_\mathfrak{v}^\ell$, $\varphi(x) = x^{\ell-1}$, is a group automorphism of $\dot{K}_\mathfrak{v}/\dot{K}_\mathfrak{v}^\ell$ which preserves Hilbert symbol of degree ℓ. Indeed, for all $x, y \in \dot{K}_\mathfrak{v}/\dot{K}_\mathfrak{v}^\ell$ we have

$$(\varphi(x), \varphi(y))_\mathfrak{v} = (x^{\ell-1}, y^{\ell-1})_\mathfrak{v} = (x, y)_\mathfrak{v}^{(\ell-1)^2} = (x, y)_\mathfrak{v}.$$

Unfortunately, this automorphism is not tame.

Using the automorphisms defined above it is easy to define a Hilbert self-equivalence of degree ℓ for which the wild set is "relatively big". The following theorem shows such a case.

THEOREM 6.1. *Let K be a number field containing a primitive ℓth root of unity. If the set $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ of finite primes of K contains all ℓ-adic primes and the ideal classes $\operatorname{cl}\mathfrak{p}_1, \dots, \operatorname{cl}\mathfrak{p}_m$ generate the group C_K/C_K^ℓ, then there exists a Hilbert self-equivalence (T, t) of degree ℓ of K such that $\mathcal{W}(T, t) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$.*

PROOF. Let S_∞ be the set of all infinite primes of K and let $S = S_\infty \cup \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$. By Remark 2.2, $\operatorname{rk}_\ell C_S = 0$, i.e. the set S is sufficiently large.

Let us define the triple $(T_S, t_S, \prod_{\mathfrak{v}\in S} t_{\mathfrak{v}})$ in the following way

$$\begin{aligned} &T_S\colon S \to S, && T_S = \mathrm{id}_S,\\ &t_S\colon E_S/\dot{K}^{\ell} \to E_S/\dot{K}^{\ell}, && t_S(x) = x^{\ell-1}\\ &t_{\mathfrak{v}}\colon \dot{K}_{\mathfrak{v}}/\dot{K}^{\ell}_{\mathfrak{v}} \to \dot{K}_{\mathfrak{v}}/\dot{K}^{\ell}_{\mathfrak{v}}, && t_{\mathfrak{v}}(x) = x^{\ell-1} \quad \text{for every } \mathfrak{v}\in S. \end{aligned}$$

Obviously, $(T_S, t_S, \prod_{\mathfrak{v}\in S} t_{\mathfrak{v}})$ is a small S-equivalence of degree ℓ of K, which, by Theorem 3.12, can be extended to a self-equivalence (T,t) of degree ℓ of K and tame outside S. Every finite prime $\mathfrak{v}$ in S is wild, so $\mathcal{W}(T,t) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$. □

The notion of a tame prime of a Hilbert semi-equivalence of degree ℓ (ℓ-prime number) came into existence as a natural generalization of the same notion for a Hilbert self-equivalence of degree 2. In the case of $\ell > 2$ it seems interesting to consider also another generalization which we shall describe below.

Suppose (T,t) is a Hilbert self-equivalence of degree ℓ of K. A finite prime $\mathfrak{p}$ of K is called semi-tame if

$$\forall_{a\in\dot{K}/\dot{K}^{\ell}} (\mathrm{ord}_{\mathfrak{p}}\, a \equiv 0 \pmod{\ell} \iff \mathrm{ord}_{T\mathfrak{p}}\, ta \equiv 0 \pmod{\ell}).$$

A finite prime $\mathfrak{p}$ is called semi-wild if it is not semi-tame. The set of all semi-tame primes of a Hilbert self-equivalence (T,t) is called semi-wild and denoted by $\mathcal{SW}(T,t)$.

Of course, every tame prime is semi-tame, but the opposite implication is not true. Thus the following inclusion:

$$\mathcal{SW}(T,t) \subseteq \mathcal{W}(T,t).$$

In the same way as in the case of a tame isomorphism one can define the notion of a semi-tame local isomorphism. For finite primes $\mathfrak{p}$ and $\mathfrak{p}'$ of K a local isomorphism $\varphi\colon \dot{K}_{\mathfrak{p}}/\dot{K}^{\ell}_{\mathfrak{p}} \to \dot{K}_{\mathfrak{p}'}/\dot{K}^{\ell}_{\mathfrak{p}'}$ preserving Hilbert symbols of degree ℓ is said to be semi-tame, if

$$\forall_{x\in\dot{K}_{\mathfrak{p}}/\dot{K}^{\ell}_{\mathfrak{p}}} (\mathrm{ord}_{\mathfrak{p}}\, x \equiv 0 \pmod{\ell} \iff \mathrm{ord}_{\mathfrak{p}'}\, \varphi(x) \equiv 0 \pmod{\ell}).$$

and semi-wild, otherwise.

It is a routine matter to check that if (T,t) is a Hilbert self-equivalence of degree ℓ of K, then a finite prime $\mathfrak{p}$ is semi-tame if and only if the local isomorphism $t_{\mathfrak{p}}\colon \dot{K}_{\mathfrak{p}}/\dot{K}^{\ell}_{\mathfrak{p}} \to \dot{K}_{T\mathfrak{p}}/\dot{K}^{\ell}_{T\mathfrak{p}}$ induced by t is semi-tame.

Analyzing the proof of Theorem 1.1 we notice that the local isomorphisms $t_{\mathfrak{p}}$ constructed above are semi-tame. This leads us to the following theorem, which is a stronger version of Theorem 1.1.

THEOREM 6.2. *Let $\ell > 2$ be a prime number and let K be a number field satisfying* (C1) *and* (C2). *If $W = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$ is a set of finite primes of K and the classes $\operatorname{cl}\mathfrak{p}_1, \dots \operatorname{cl}\mathfrak{p}_k$ are ℓth powers in C_K, then there exists a Hilbert self-equivalence (T,t) of degree ℓ of K such that $\mathcal{SW}(T,t) = W$.*

In the proof of Theorem 6.1 all local isomorphisms $t_{\mathfrak{v}}$ for $\mathfrak{v} \in S$ constructed there are semi-tame, so similar strengthening of Theorem 6.1 is not possible.

The problem of describing all finite sets of primes which are wild sets or semi-wild sets of Hilbert self-equivalences of degree $\ell > 2$ is open.

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