ON COMPUTER-ASSISTED PROVING THE EXISTENCE OF PERIODIC AND BOUNDED ORBITS

ROMAN SRZEDNICKI

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Abstract. We announce a new result on determining the Conley index of the Poincaré map for a time-periodic non-autonomous ordinary differential equation. The index is computed using some singular cycles related to an index pair of a small-step discretization of the equation. We indicate how the result can be applied to computer-assisted proofs of the existence of bounded and periodic solutions. We provide also some comments on computer-assisted proving in dynamics.

1. Remarks on the history of computer assisted proofs

1.1. Computer assistance in mathematics

The assistance of computers in developing mathematics is probably as old as computers themselves. Computers are applied in numerical simulations leading to formulating and testing new theorems or in rigorous proving. The latter applications consist, in particular, in complete formalizations of proofs

(like the system Mizar initiated by Andrzej Trybulec in 1973, c.f. [32]), in proving complicated formulas (requiring, for example, long symbolic calculations with *Maple* or *Mathematica* packages), or in providing rigorous assessments using interval arithmetic (c.f. [17, 24]). Relying in computers within rigorous mathematics is sometimes contested; due to possible errors in the processor or in the program code, one cannot be sure whether a computer performs exactly the same calculations as one expects (recall the Pentium processor bug discovered in 1994, c.f. [44]).

It seems the four-color theorem was the first famous result which was proved with the aid of a computer (in 1976 by Kenneth Appel and Wolfgang Haken; c.f. [1, 2]; compare also [41] for a simpler proof). Other important results proved with the computer assistance include the solution of the Kepler conjecture about dense sphere packing, announced in 1998 (c.f. [12]) and completed only recently in [13], and the proof of the non-existence of finite projective planes of order 10 (c.f. [21, 22]). It should be noted that sometimes computer-assisted proofs of important results predated the 'classical' proofs; that was, for example, in the case of the solution of the Robbins problem in Boolean algebras (c.f. [27] and [26]) and of the isoperimetric problem on two pieces of spheres separated by a flat disc (the 'double-bubble conjecture', c.f. [14] and [18]).

1.2. Computer assisted proofs in dynamical systems

Perhaps the first essential result related to dynamics, proved with the computer assistance, was the solution of the Feigenbaum conjecture given in 1982 (c.f. [23, 24]). It refers to the existence of a solution of a certain functional equation arising in period-doubling bifurcations of iterations of unimodular maps. A related important result obtained at that time with the computer's aid was given in [8]. Computer-assisted proofs of results on the Lorenz system of equations

$$(1) \qquad \dot{x} = \sigma(y - x), \quad \dot{y} = \rho x - xz - y, \quad \rho z = xy - \beta z,$$

where σ, ρ, and β are scalar parameters, started to appear in the 90s of the last century. The numerical evidence of a chaotic behavior of the system with $\sigma = 10$, $\rho = 28$, and $\beta = \frac{8}{3}$ was announced in 1963 by Edward Lorenz in [25]. Since then, hundreds of papers devoted to explanation of the dynamics of the system were published. In [15, 16], the shooting method based on the intermediate value property was applied in order to prove results on homoclinic orbits of (1). The first rigorous result which explained the chaotic nature of (1) was announced by Konstantin Mischaikow and Marian Mrozek in [28] in 1995 and was explained in details in the consecutive papers [29, 30].

Proofs of all basic result related to the Lorenz system, predicted by numerical evidence, were given by Warwick Tucker at the turn of the century (c.f. [42, 43]). The publication of papers mentioned above inspired research on the computer-assisted proofs in dynamics. In particular, Hubbard's and Saari's conjectures in mechanics were proved in, respectively, [3] and [33]. At present several groups of mathematicians specializes in that kind of research; among them there are cooperating teams CAPA (Computer-aided proofs in analysis) at Uppsala, CAPD (Computer assisted proofs in dynamics) at Kraków, and CHomP (Computational homology project) at Rutgers University; see the corresponding web-pages [4, 5, 7]. The achievements of CAPD include results on problems of celestial mechanics (e.g. [6, 19, 20, 47]), on Kuramoto-Sivashinsky equations (e.g. [31, 40, 45, 51]), and on complicated behavior of Hénon, Rössler, and other dynamical systems (e.g. [9, 10, 46, 48, 49, 50]).

Up to now the vast majority of proofs of essential results on the system (1) and similar ones, are computer-assisted. The difficulty in 'classical' proving qualitative results for those systems lies mainly in the lack of analytic formulas for solutions; the required estimates are given numerically and the errors of these estimates grow exponentially as time goes to infinity. Usually it is impossible to perform all such calculations 'by hand' and therefore an aid of computers is necessary.

Computer assisted proofs in dynamical systems are frequently based on topological tools being far reaching improvements of the intermediate value property. They include the Brouwer degree (on which the method of covering relations introduced by Piotr Zgliczyński is based, c.f. [11]), the fixed point index, and the Conley index. The papers [35, 36] explain how the latter tool can be applied. In the remaining part of the present text we describe a recent research on that subject.

2. On the Conley index of Poincaré maps

2.1. Dynamical systems

We begin with a recollection of some basic definitions of the dynamical systems theory. Let $\mathbb{T}$ denote either the set of real numbers $\mathbb{R}$ or the set of integer numbers $\mathbb{Z}$. Let X be a topological space. A continuous map $\phi\colon X \times \mathbb{T} \to X$ is called a *dynamical system* on X if it satisfies

$$(2) \qquad \forall x \in X\colon \phi(x, 0) = x,$$

$$(3) \qquad \forall x \in X,\ s, t \in \mathbb{T}\colon \phi(x, s + t) = \phi(\phi(x, s), t).$$

The set $\mathbb{T}$ represents *time*. We call X the *phase space* of ϕ and we denote by ϕ_t the map $x \mapsto \phi(x,t)$. For $x \in X$ the set $\{\phi_t(x) \colon t \in \mathbb{T}\}$ is called the *orbit* of x. An orbit is called *bounded* provided its closure is compact. If an orbit consists of exactly one point, it is called *stationary*. A non-stationary orbit is called *periodic* if $\phi_T(x) = x$ for some $T > 0$ (such a T is called a *period*). Clearly, each periodic orbit is bounded. The problem of existence of periodic orbits belongs to the most important ones in the theory of dynamical systems. It is frequently solved by topological methods applied to the estimates obtained by numerical approximations, like in several computer-assisted results on chaotic behavior of the Lorenz and Rössler systems, mentioned in Subsection 1.2. In fact, chaotic dynamics is often indicated by the existence of an infinite set of periodic orbits.

Let $A \subset X$. The *invariant part* of A (with respect to ϕ) is defined as

$$\mathrm{Inv}(A) := \{x \in A \colon \phi_t(x) \in A \ \forall t \in \mathbb{T}\}.$$

A set $A \subset X$ is called *invariant* provided $A = \mathrm{Inv}(A)$. A compact invariant set S is called *isolated* if it is maximal invariant set in some its neighborhood.

In the case $\mathbb{T} = \mathbb{Z}$, ϕ is called a *discrete-time* dynamical system; the properties (2) and (3) imply ϕ can be regarded as the family of iterations of the homeomorphism ϕ_1. On the other hand, if $g \colon X \to X$ is a homeomorphism then the formula $\phi^g(x,n) := g^n(x)$ for $x \in X$ and $n \in \mathbb{Z}$ defines a discrete-time dynamical system, i.e. ϕ^g is the family of iterations of g (hence $\phi_1^g = g$). By abuse of terminology, we frequently write g to mean the dynamical system ϕ^g.

If $\mathbb{T} = \mathbb{R}$, ϕ is called a *continuous-time* dynamical system. Usually such a system is generated by the solutions of an autonomous ordinary differential equation

$$\dot{x} = v(x),$$

where $v \colon \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector-field. Actually, not every autonomous equation generates a dynamical system since, in general, solutions are not defined in the whole $\mathbb{R}$. That problem can be easily overcome by a suitable modification of v, if we are interested in qualitative properties of solutions in a bounded region of the phase space. Here we deal with bounded orbits mainly, hence in the sequel we assume all considered equations generate dynamical systems.

2.2. Discrete-time Conley index

Assume X is a metrizable locally compact space. Let $g \colon X \to X$ be a homeomorphism and let N and L be compact subsets of X, $L \subset N$. The pair

(N, L) is called a *weak index pair* (for g) if

(4)
$$S := \mathrm{Inv}\left(\overline{N \setminus L}\right) \subset \mathrm{int}(N \setminus L),$$

(5)
$$g(L) \cap N \subset L,$$

(6)
$$\overline{g(N) \setminus N} \cap N \subset L.$$

It follows S is isolated invariant set; (N, L) is therefore also called a weak index pair for S. (A more restrictive notion of *index pair* is not used in the present text.) It follows by (5) and (6) the map

$$g_{(N,L)}\colon N/L \to N/L, \quad g_{(N,L)}(x) := \begin{cases} g(x), & \text{if } x, g(x) \in N \setminus L, \\ *, & \text{otherwise}, \end{cases}$$

is continuous; it is called the *index map*. (Here N/L denotes $(N \setminus L) \cup \{*\}$ endowed with the quotient topology, where $*$ is a point outside of N.) Let H denote the singular homology functor with coefficients in the field of rational numbers $\mathbb{Q}$. The index map induces a graded endomorphism

$$H(g_{(N,L)})\colon H(N/L, *) \to H(N/L, *).$$

Denote by $R\phi$ the Leray reduction of an endomorphism $\phi\colon V \to V$ of a vector space V over $\mathbb{Q}$. It is defined as follows. Put

$$KV := V \Big/ \bigcup_k \ker \phi^k,$$

the quotient of V by the generalized kernel of ϕ. Let $K\phi\colon KV \to KV$ be the induced monomorphism. If V is finite dimensional, $R\phi := K\phi$. In the general case put

$$RV := \bigcap_k \mathrm{Im}(K\phi)^k$$

and define $R\phi\colon RV \to RV$ as the restriction of $K\phi$. $R\phi$ is thus an automorphism of RV. In an obvious way one extends the definition of R to the graded endomorphisms.

The *Conley index* of S (denoted $\mathrm{CH}(S, g)$) is defined as the conjugacy class of the graded automorphism $RH(g_{(N,L)})$. It follows by [34, 37], each isolated invariant set S has at least one week index pair (N, L) satisfying (4) and the definition is independent of the choice of (N, L). The name of the index commemorates Charles Conley (1933–1984), who invented an analogous

tool for continuous-time dynamical systems in the 70s of the 20th century, improving the retract method of Tadeusz Ważewski.

The basic property of the index is the following one: if $\mathrm{CH}(S, g) \neq 0$ then S is nonempty; in particuar there exists a bounded orbit. Moreover, if X is an ENR (Euclidean neighborhood retract; for example $\mathbb{R}^n$, a manifold, a polyhedron or a cubical set) and the Lefschetz number $\Lambda(\mathrm{CH}(S, g))$ is nonzero then g has a fixed point. Recall that the Lefschetz number of a graded endomorphism $\psi := \{\psi_n\}_n$ of finite dimensional spaces, such that $\psi_n = 0$ for almost all n, is defined as

$$\Lambda(\psi) := \sum_{n=0}^{\infty} (-1)^n \mathrm{trace}\, \psi_n.$$

Actually, the Conley index has other properties; in particular it is additive and multiplicative (in some sense), hence it provides an 'algebraic' information on the 'size' of the isolated invariant set, in a slightly similar way like the fixed point index provides an information on the number of fixed points of a map.

2.3. The Poincaré map

Roughly speaking, a Poincaré map is the first-return map to a section of a dynamical system. Usually, after a change of variables such a map can be reduced to the translation operator of a time-periodic non-autonomous equation. Below we state the formal definition of that reduced case only.

Let $T > 0$ and let $f \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a smooth and bounded time-dependent vector-field such that the map $t \mapsto f(t, x)$ is T-periodic for each x. The system of equations

$$\dot{t} = 1, \quad \dot{x} = f(t, x)$$

generates two dynamical systems:

$$\phi \text{ on } \mathbb{R}/T\mathbb{Z} \times \mathbb{R}^n,$$
$$\psi \text{ on } \mathbb{R} \times \mathbb{R}^n.$$

The Poincaré map $P \colon \mathbb{R}^n \to \mathbb{R}^n$ is defined for $x \in \mathbb{R}^n$ by the formula

$$\psi_T(0, x) = (T, P(x)).$$

It follows, in particular, the dynamical system ϕ has a T-periodic orbit if and only if P has a fixed point.

2.4. A theorem on the Conley index of the Poincaré map

We follow the notation from subsection 2.3. Moreover, for $s \in \mathbb{R}/T\mathbb{Z}$ and a set $Z \subset \mathbb{R}/T\mathbb{Z} \times \mathbb{R}^n$ we define the s-section of Z as

$$Z_s := \{x \in \mathbb{R}^n \colon (s, x) \in Z\}.$$

Assume $h > 0$ and assume (N, L) is a weak index pair for ϕ_h. Let S be equal to the invariant part of $\overline{N \setminus L}$ with respect to ϕ_h. One can prove that S is also an isolated invariant set for the continuous-time system ϕ, and S_0, the 0-section of S, is an isolated invariant set for P. Consequently, if $\mathrm{CH}(S_0, P) \neq 0$ (hence S_0 is nonempty) then ϕ has a bounded orbit and if $\Lambda(\mathrm{CH}(S_0, P)) \neq 0$ (hence S_0 contains a fixed point of P) then ϕ has a T-periodic orbit. Therefore our aim is to find a method of determining the Conley index of S_0.

By $(\widetilde{N}, \widetilde{L})$ we denote the lift of (N, L) to $\mathbb{R} \times \mathbb{R}^n$, i.e.

$$\widetilde{N} := \{(t, x) \in \mathbb{R} \times \mathbb{R}^n \colon (t \bmod T, x) \in N\},$$

$$\widetilde{L} := \{(t, x) \in \mathbb{R} \times \mathbb{R}^n \colon (t \bmod T, x) \in L\}.$$

Let u and v be singular cycles on (N_0, L_0). (u, v) is called a *pair of contiguous cycles over* $[0, T]$ provided there exist singular chains w on $\widetilde{N} \cap ([0, T] \times \mathbb{R}^n)$ and z on $\widetilde{L} \cap ([0, T] \times \mathbb{R}^n)$ such that

$$\partial w = 0 \times u - T \times v + z,$$

where $0 \times u$ and $T \times v$ denote the images of the cycles u and v under the chain maps induced by the embeddings $x \mapsto (0, x)$ and, respectively, $x \mapsto (T, x)$.

By $|c|$ we denote the support of a q-dimensional singular cycle $c = \sum_i \alpha_i \sigma_i$, i.e.

$$|c| = \bigcup_i \sigma(\Delta_q),$$

where Δ_q is the standard q-dimensional simplex. The pair of contiguous cycles (u, v) is called h-movable if

$$\psi_t(|w|) \subset \widetilde{N}, \quad \psi_t(|z|) \subset \widetilde{L}$$

for each $t \in [0, h]$.

THEOREM 1 (compare [39, Theorem 4.7]). *Assume $\frac{T}{h} \in \mathbb{Q}$ and both N_0 and L_0 be ENRs. Let $k := \dim H(N_0, L_0)$, let $A = [a_{ij}]$ be a graded $(k \times k)$-matrix over $\mathbb{Q}$, and let*

$$\left(u_j, \sum_{i=1}^{k} a_{ij} u_i \right), \quad j = 1, \ldots, k,$$

be h-movable pairs of contiguous cycles over $[0, T]$ such that the homology classes of u_j form a basis of $H(N_0, L_0)$. Then the Conley index $\mathrm{CH}(S_0, P)$ is equal to the conjugacy class of RA.

A proof of Theorem 1 can be found in [39]. A similar theorem, less practical in applications, appeared already in [38]. As a consequence of Theorem 1 one gets the following result.

COROLLARY 1. *Under assumptions of Theorem 1, if $RA \neq 0$ then ϕ has a bounded orbit and if $\Lambda(A) \neq 0$ then ϕ has a T-periodic orbit.*

A simple case of Theorem 1 implies another corollary.

COROLLARY 2. *If N_0 and L_0 are ENRs, $H(N_0, L_0)$ is 1-dimensional and there exists a pair of h-movable cycles (u, v) such that the homology class of u is a generator of $H(N_0, L_0)$, and*

$$v = u \quad or \quad v = -u$$

then the equation $\dot{x} = f(t, x)$ has a T-periodic solution.

2.5. On application of Theorem 1 to computer-assisted proving

As it was already pointed out, usually it is difficult (or even impossible, due to limitations of computers) to find a satisfactory approximation of the map ϕ_T if T is large. On the other hand, for relatively small values of h, the h-discretization ϕ_h can be computed with a good precision and its essential properties can be rigorously proved. Contrary to the other computer-assisted results on Poincaré maps, Theorem 1 allows to avoid any computations related to the values of solutions at times greater then h.

Theorem 1 requires an index pair of ENRs for ϕ_h. For a given differential equation, such a pair consisting of cubical sets, can be found by programs taken from the libraries of CAPD (accessible in the web-page [5]). Let (N, L) be such a pair. For a given basis $[u_j]$ of homology classes in $H(N_0, L_0)$, one should push forward each cycle u_j in order to obtain corresponding chains

w and z, and the cycle v_j. Up to some technical details that is enough: the homology classes $[v_j]$ are represented in the basis by some matrix A and Theorem 1 asserts the Conley index is equal to the conjugacy class of the Leray reduction of A. As it was indicated in Corollary 1, the reduction provides an information on the existence of bounded and periodic orbits. Based on Theorem 1, an algorithm for computation of the index for some planar equation is constructed in [39].

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Institute of Mathematics
Faculty of Mathematics and Computer Science
Jagiellonian University
Łojasiewicza 6
30–348 Kraków
Poland
e-mail: srzednicki@im.uj.edu.pl