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ON THE CONTINUOUS DEPENDENCE OF SOLUTIONS TO ORTHOGONAL ADDITIVITY PROBLEM ON GIVEN FUNCTIONS

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Abstract. We show that the solution to the orthogonal additivity problem in real inner product spaces depends continuously on the given function and provide an application of this fact.

Let E be a real inner product space of dimension at least 2.

A function f mapping E into an abelian group is called orthogonally additive, if

$$f(x+y)=f(x)+f(y) \quad \text{for all } x,y\in E \text{ with } x\perp y.$$

It is well known, see [3, Corollary 10] and [1, Theorem 1], that every orthogonally additive function f defined on E has the form

$$f(x)=a(\|x\|^2)+b(x) \quad \text{for } x\in E, \tag{1}$$

where a and b are additive functions uniquely determined by f. Consequently, given an abelian group G we have an operator Λ which to any orthogonally

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additive $f\colon E \to G$ assigns a pair (a, b) of additive functions such that (1) holds, i.e.,

$$\Lambda f = (a, b) \tag{2}$$

where

$$a\colon \mathbb{R} \to G, \quad b\colon E \to G \quad \text{are additive and (1) holds.} \tag{3}$$

Putting

$$\mathrm{Hom}_{\perp}(E, G) = \{f\colon E \to G \mid f \text{ is orthogonally additive}\}$$

and

$$\mathrm{Hom}(S, G) = \{f\colon S \to G \mid f \text{ is additive}\}$$

for $S \in \{\mathbb{R}, E\}$ we see that $\Lambda\colon \mathrm{Hom}_{\perp}(E, G) \to \mathrm{Hom}(\mathbb{R}, G) \times \mathrm{Hom}(E, G)$ given by (2) and (3) is an additive bijection.

To consider its continuity assume that G is a topological group and given a non-empty set S consider G^S of all functions from S into G with the usual addition and with the Tychonoff topology; clearly G^S is a topological group. In what follows we consider $\mathrm{Hom}_{\perp}(E, G)$ and $\mathrm{Hom}(S, G)$ for $S \in \{\mathbb{R}, E\}$ with the topology induced by G^E and G^S, respectively.

The main result of this note reads.

Theorem 1. *Isomorphism* $\Lambda\colon \mathrm{Hom}_{\perp}(E, G) \to \mathrm{Hom}(\mathbb{R}, G) \times \mathrm{Hom}(E, G)$ *given by* (2) *and* (3) *is a homeomorphism.*

Proof. To show that Λ is continuous at zero fix neighbourhoods $\mathcal{V} \subset \mathrm{Hom}(\mathbb{R}, G)$ and $\mathcal{W} \subset \mathrm{Hom}(E, G)$ of zeros. We may assume

$$\mathcal{V} = \{a \in \mathrm{Hom}(\mathbb{R}, G) : a(\alpha_n) \in U \text{ for } n \in \{1, \ldots, N\}\}$$

and

$$\mathcal{W} = \{b \in \mathrm{Hom}(E, G) : b(x_n) \in U \text{ for } n \in \{1, \ldots, N\}\}$$

with a neighbourhood U of zero in G and some $\alpha_1, \ldots, \alpha_N \in \mathbb{R}, x_1, \ldots, x_N \in E, N \in \mathbb{N}$. Choose a symmetric neighbourhood U_0 of zero in G such that $U_0 + U_0 \subset U$ and $x_{N+1}, \ldots, x_{2N} \in E$ with

$$2\|x_{N+n}\|^2 = |\alpha_n| \quad \text{for } n \in \{1, \ldots, N\}$$

and put

$$\mathcal{U} = \bigcap_{n=1}^{N} \{f \in \mathrm{Hom}_\perp(E,G) : f(\frac{1}{2}x_n) \in U_0 \text{ and } f(-\frac{1}{2}x_n) \in U_0\}$$
$$\cap \bigcap_{n=N+1}^{2N} \{f \in \mathrm{Hom}_\perp(E,G) : f(x_n) \in U_0 \text{ and } f(-x_n) \in U_0\}.$$

Clearly $\mathcal{U}$ is a neighbourhood of zero in $\mathrm{Hom}_\perp(E,G)$ and to show that $\Lambda(\mathcal{U}) \subset \mathcal{V} \times \mathcal{W}$ fix an $f \in \mathcal{U}$. Then we have (2) and (3) and, by (3),

$$b(x_n) = 2b(\frac{1}{2}x_n) = f(\frac{1}{2}x_n) - f(-\frac{1}{2}x_n) \in U_0 + U_0 \subset U$$

for $n \in \{1, \ldots, N\}$, whence $b \in \mathcal{W}$, and

$$\begin{aligned} a(\alpha_n) &\in \{a(|\alpha_n|), -a(|\alpha_n|)\} \\ &= \{2a(\|x_{N+n}\|^2), -2a(\|x_{N+n}\|^2)\} \\ &= \{f(x_{N+n}) + f(-x_{N+n}), -(f(x_{N+n}) + f(-x_{N+n}))\} \\ &\subset U_0 + U_0 \subset U \end{aligned}$$

for $n \in \{1, \ldots, N\}$, whence $a \in \mathcal{U}$.

To get continuity of Λ^{-1} it is enough to observe that the homomorphism $\Lambda_1 \colon \mathrm{Hom}(\mathbb{R}, G) \to \mathrm{Hom}_\perp(E,G)$ given by

$$(\Lambda_1 a)(x) = a(\|x\|^2) \quad \text{for } x \in E$$

is continuous. □

Corollary 1. *If G is Hausdorff and* $\mathrm{Hom}(\mathbb{R}, G) \neq \{0\}$*, then* $\mathrm{Hom}(E,G)$ *is closed and nowhere dense in* $\mathrm{Hom}_\perp(E,G)$.

For the proof the following lemma will be used.

Lemma 1. *If* $\mathrm{Hom}(\mathbb{R}, G) \neq \{0\}$*, then* $\mathrm{Hom}(\mathbb{R}, G)$ *is not discrete.*

Proof. Fix arbitrarily a positive integer N, reals $\alpha_1, \ldots, \alpha_N$ and a neighbourhood U of zero in G. To show that the set

$$\{a \in \mathrm{Hom}(\mathbb{R}, G) : a(\alpha_n) \in U \text{ for } n \in \{1, \ldots, N\}\} \tag{4}$$

is different from $\{0\}$ let H be a Hamel basis of $\mathbb{R}$ (i.e., a basis of the vector space $\mathbb{R}$ over the field $\mathbb{Q}$ of rationals) and let H_0 be a finite subset of H such that

$$\alpha_n \in \mathrm{Lin}_{\mathbb{Q}} H_0 \quad \text{for} \quad n \in \{1, \ldots, N\}.$$

Since (see [2, Theorem 4.2.3]) $\mathrm{card} H = \mathfrak{c}$, there exists a function $c_0 \colon H \to \mathbb{R}$ such that

$$c_0(H_0) = \{0\} \quad \text{and} \quad c_0(H \setminus H_0) = H.$$

Let $c \colon \mathbb{R} \to \mathbb{R}$ be the additive extension of c_0 and consider an $a \in \mathrm{Hom}(\mathbb{R}, G) \setminus \{0\}$. Clearly $a \circ c$ is additive and

$$a \circ c(\alpha_n) \in a(c(\mathrm{Lin}_{\mathbb{Q}} H_0) = a(\mathrm{Lin}_{\mathbb{Q}} c_0(H_0)) = a(\{0\}) = \{0\}$$

for $n \in \{1, \ldots, N\}$ which proves that $a \circ c$ belongs to set (4). To see that $a \circ c \neq 0$ consider a $\beta \in \mathbb{R}$ with $a(\beta) \neq 0$. Then

$$\beta \in \mathrm{Lin}_{\mathbb{Q}} H = \mathrm{Lin}_{\mathbb{Q}} c(H \setminus H_0) \subset c(\mathrm{Lin}_{\mathbb{Q}} H) = c(\mathbb{R})$$

whence $\beta = c(\alpha)$ for some $\alpha \in \mathbb{R}$ and $a \circ c(\alpha) = a(\beta) \neq 0$. □

Proof of Corollary 1. By the standard argument the set $\mathrm{Hom}(E, G)$ is closed in G^E. Since

$$\mathrm{Hom}(E, G) = \Lambda^{-1}(\{0\} \times \mathrm{Hom}(E, G))$$

and Λ is a homeomorphism, it is enough to observe that according to Lemma 1 the set $\{0\} \times \mathrm{Hom}(E, G)$ is nowhere dense in $\mathrm{Hom}(\mathbb{R}, G) \times \mathrm{Hom}(E, G)$. □

We finish with some remarks.

Remarks.

1. Since projections are open, if $\mathrm{Hom}(\mathbb{R}, G)$ is discrete, then so is also G. The converse is not true as the next remark shows.
2. If G is uniquely divisible and $G \neq \{0\}$, then $\mathrm{Hom}(\mathbb{R}, G) \neq \{0\}$ and, by Lemma 1, $\mathrm{Hom}(\mathbb{R}, G)$ is not discrete.
3. $\mathrm{Hom}(\mathbb{R}, \mathbb{Z}) = \{0\}$.
4. The following three sentences are equivalent:

$$\mathrm{Hom}(\mathbb{R}, G) = \{0\}, \quad \mathrm{Hom}(E, G) = \{0\}, \quad \mathrm{Hom}_{\perp}(E, G) = \{0\}.$$

The reader interested in further problems connected with orthogonal additivity is referred to the survey paper [4] by Justyna Sikorska.

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References

[1] Baron K., Rätz J., *On orthogonally additive mappings on inner product spaces,* Bull. Polish Acad. Sci. Math. **43** (1995), 187–189.

[2] Kuczma M., *An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality*, Second edition (edited by A. Gilányi), Birkhäuser Verlag, Basel, 2009.

[3] Rätz J., *On orthogonally additive mappings,* Aequationes Math. **28** (1985), 35–49.

[4] Sikorska J., *Orthogonalities and functional equations,* Aequationes Math. **89** (2015), 215–277.

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