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EXPONENTIAL CONVERGENCE FOR MARKOV SYSTEMS

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Abstract. Markov operators arising from graph directed constructions of iterated function systems are considered. Exponential convergence to an invariant measure is proved.

1. Introduction

We are concerned with Markov operators corresponding to Markov systems introduced by Werner ([12], [11]) and independently by Mauldin and Urbański ([8]). They are graph directed constructions generalizing iterated function systems with place dependent probabilities (see [1], [7]). The action of a Markov system can be roughly described as follows. Consider a metric space X partitioned into finite number of subsets $X = X_1 \cup X_2 \cup \ldots \cup X_N$. Every subset X_i is placed at the vertex of a directed multigraph. Edges of a multigraph are identified with transformations which are chosen at random with place dependent probabilities. The existence of an attractive invariant measure for Markov systems was proved by Werner and, in more general setting, by Horbacz and Szarek [4].

In the present paper we prove the exponential rate of convergence to an invariant measure for such systems. We use the coupling method developed by Hairer in [2], [3] and adapted to random iteration of functions in [10], [5] and [13]. Our main tool is a general criterion for the existence of an exponentially attractive invariant measure established in [6].

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The paper is organized as follows. Section 2 introduces basic definitions needed throughout the paper. Markov systems are described in Section 3. The main theorem of this paper is formulated in Section 4 and proved in Section 5.

2. Notation and basic definitions

Let (X, d) be a *Polish* space, i.e. a complete and separable metric space and denote by $\mathcal{B}_X$ the σ-algebra of Borel subsets of X. By $B_b(X)$ we denote the space of bounded Borel-measurable functions equipped with the supremum norm, $C_b(X)$ stands for the subspace of bounded continuous functions. By $\mathcal{M}_{fin}(X)$ and $\mathcal{M}_1(X)$ we denote the sets of nonnegative Borel measures on X such that $\mu(X) < \infty$ for $\mu \in \mathcal{M}_{fin}(X)$ and $\mu(X) = 1$ for $\mu \in \mathcal{M}_1(X)$. Elements of $\mathcal{M}_1(X)$ are called *probability* measures. Elements of $\mathcal{M}_{fin}(X)$ for which $\mu(X) \leq 1$ are called *subprobability measures.* By $\operatorname{supp} \mu$ we denote the support of the measure μ. We also define

$$\mathcal{M}_1^L(X) = \Big\{\mu \in \mathcal{M}_1(X) : \int L(x)\mu(dx) < \infty\Big\}$$

where $L\colon X \to [0, \infty)$ is an arbitrary Borel measurable function and

$$\mathcal{M}_1^1(X) = \Big\{\mu \in \mathcal{M}_1(X) : \int d(\bar{x}, x)\mu(dx) < \infty\Big\},$$

where $\bar{x} \in X$ is fixed. The definition of $\mathcal{M}_1^1(X)$ is independent of the choice of $\bar{x}$.

The space $\mathcal{M}_1(X)$ is equipped with the *Fourtet–Mourier metric*:

$$\|\mu_1 - \mu_2\|_{FM} = \sup\Big\{\Big|\int_X f(x)(\mu_1 - \mu_2)(dx)\Big| : f \in \mathcal{F}\Big\},$$

where

$$\mathcal{F} = \{f \in C_b(X) : |f(x) - f(y)| \leq 1 \text{ and } |f(x)| \leq 1 \text{ for } x, y \in X\}.$$

The space $(\mathcal{M}_1(X), \|\cdot\|_{FM})$ is complete (see [9]). By $\|\cdot\|$ we denote the total variation norm. If a measure μ is nonnegative then $\|\mu\|$ is simply the total mass of μ.

Let $P\colon B_b(X) \to B_b(X)$ be a *Markov operator*, i.e. a linear operator satisfying $P\mathbf{1}_X = \mathbf{1}_X$ and $Pf(x) \geq 0$ if $f \geq 0$. Denote by P^* the dual operator, i.e operator $P^*\colon \mathcal{M}_{fin}(X) \to \mathcal{M}_{fin}(X)$ defined as follows

$$P^*\mu(A) := \int_X P\mathbf{1}_A(x)\mu(dx) \quad \text{for } A \in \mathcal{B}_X.$$

We say that $\mu_* \in \mathcal{M}_1(X)$ is *invariant* for P if

$$\int_X Pf(x)\mu_*(dx) = \int_X f(x)\mu_*(dx) \quad \text{for every } f \in B_b(X)$$

or, alternatively, we have $P^*\mu_* = \mu_*$.

By $\{\mathbf{P}_x : x \in X\}$ we denote the *transition probability function* for P, i.e. the family of measures $\mathbf{P}_x \in \mathcal{M}_1(X)$, $x \in X$, such that maps $x \mapsto \mathbf{P}_x(A)$ are measurable for every $A \in \mathcal{B}_X$ and

$$Pf(x) = \int_X f(y)\mathbf{P}_x(dy) \quad \text{for } x \in X \text{ and } f \in B_b(X),$$

or equivalently $P^*\mu(A) = \int_x \mathbf{P}_x(A)\mu(dx)$ for $A \in \mathcal{B}_X$ and $\mu \in \mathcal{M}_{fin}(X)$.

3. Markov systems

Let (X, d) be a Polish space of the form $X = \bigcup_{j=1}^N X_j$, where X_j are nonempty Borel subsets such that $\sup\{d(x, y) : x \in X_i, y \in X_j\} > 0$ for $i \neq j$. Assume that for each $j \in \{1, \dots, N\}$ there exists a finite subset $N_j \subset \mathbb{N}$ and Borel measurable maps

$$w_{jn}\colon X_j \to X, \quad n \in N_j,$$

such that

$$\forall_{j\in\{1,\dots,N\}}\forall_{n\in N_j}\exists_{m\in\{1,\dots,N\}} \quad w_{jn}(X_j) \subset X_m.$$

Furthermore, for each $j \in \{1, \dots, N\}$ and $n \in N_j$ there exist Borel measurable functions $p_{jn}\colon X_j \to [0, 1]$ such that $\sum_{n\in N_j} p_{jn}(x) = 1$ for $x \in X_j$, $j \in \{1, \dots, N\}$. Following Werner (see [12]), we call $V = \{1, \dots, N\}$ the *set of vertices*, and the subsets $X_1, \dots, X_N$ the *vertex sets*. Further, we call

$$E = \{(j, n) : j \in \{1, \dots, N\}, n \in N_j\}$$

the *set of edges* and write

$$p_e := p_{jn} \quad \text{and} \quad w_e := w_{jn} \quad \text{for} \quad e = (j, n) \in E.$$

For an edge $e = (j, n) \in E$ we denote by $i(e) := j$ the *initial vertex* of e, the *terminal vertex* $t(e)$ of e is equal to m if and only if $w_e(X_j) \subset X_m$. The quadruple (V, E, i, t) a *directed multigraph*. We have $E = \bigcup_{i=1}^{N} E_j$ where $E_j = \{e \in E : i(e) = j\}$. A sequence $(e_1, \dots, e_r)$ of edges is called a *path* if $t(e_k) = i(e_{k+1})$ for $k = 1, \dots, r-1$.

We call the family $(X_{i(e)}, w_e, p_e)_{e \in E}$ a *Markov system*. A Markov system is *irreducible* if and only if its directed multigraph is irreducible, that is, there is a path from any vertex to any other. An irreducible Markov system has *period* p if the set of vertices can be partitioned into p nonempty subsets $V_1, \dots, V_p$ such that for all $e \in E$

$$i(e) \in V_i \Rightarrow t(e) \in V_{i+1} \mod p$$

and p is the largest number with this property. A Markov system is *aperiodic* if it has period 1.

We define Markov operator on $B_b(X)$ by

$$Pf(x) = \sum_{e \in E} p_e(x) f(w_e(x)) \quad \text{for } x \in X,\ f \in B_b(X). \tag{3.1}$$

an its dual operator acting on measures by

$$P^* \mu(A) = \sum_{e \in E} \int_{w_e^{-1}(A)} p_e(x) \mu(dx) \quad \text{for } A \in \mathcal{B}_X,\ \mu \in \mathcal{M}_1(X).$$

4. Main result

We will show that operator (3.1) has an exponentially attractive invariant measure, provided the following conditions hold:

B1 There exists $\alpha \in (0, 1)$ such that for $j \in \{1, \dots, N\}$ and $x, y \in X_j$

$$\sum_{e \in E_j} p_e(x) d(w_e(x), w_e(y)) < \alpha d(x, y).$$

B2 There exists $l > 0$ such that for $j \in \{1, \dots, N\}$ and $x, y \in X_j$

$$\sum_{e \in E_j} |p_e(x) - p_e(y)| \leq l d(x, y).$$

B3 There exist $M > 0$ such that for $j \in \{1, \dots, N\}$, $e \in E_j$ and $x, y \in X_j$

$$d(w_e(x), w_e(y)) \leq M d(x, y).$$

B4 There exist $\delta > 0$ such that for $e \in E$

$$p_e|_{X_{i(e)}} > \delta.$$

B5 For each $j \in \{1, \dots, N\}$ there exists $\bar{x}_j \in X_j$ such that

$$\sup_{e \in E_j} d(w_e(\bar{x}_j), \bar{x}_j) < \infty.$$

B6 The Markov system $(X_{i(e)}, w_e, p_e)_{e \in E}$ is aperiodic.

THEOREM 4.1. *If Markov system $(X_{i(e)}, w_e, p_e)_{e \in E}$ satisfies assumptions* **B1**–**B6** *then its Markov operator P possesses a unique invariant measure $\mu_* \in \mathcal{M}_1^1(X)$, moreover, there exists $q \in (0, 1)$ and $C > 0$ such that*

$$\|P^{*n}\mu - \mu_*\|_{FM} \leq q^n C(1 + \int_X L(x)\mu(dx))$$

for $\mu \in \mathcal{M}_1^1(X)$, $n \in \mathbb{N}$, where $L(x) = d(x, \bar{x}_j)$ for $x \in X_j$, $j \in \{1, \dots, N\}$.

EXAMPLE. Let the set V of vertices consists of two elements a and b and let $E = \{(a, a), (a, b), (b, a), (b, b)\}$ be the set of edges. The multigraph (E, V) is aperiodic. Put $X_a = [2, 4] \times [1, 3] \subset \mathbb{R}^2$, $X_b = [0, 2] \times [3, 5] \subset \mathbb{R}^2$ and define maps $w_e, e \in E$ as follows

$$\begin{aligned}
w_{ab}(x, y) &= \Big(\frac{1}{2}x + \frac{1}{2}y, \frac{1}{2}x + \frac{1}{2}y + \frac{7}{2}\Big),\\
w_{aa}(x, y) &= \Big(\frac{1}{2}x + \frac{1}{2}y + 2, \frac{1}{2}x + \frac{1}{2}y + \frac{3}{2}\Big),\\
w_{ba}(x, y) &= \Big(\frac{1}{2}x + \frac{1}{2}y + 2, \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}\Big),\\
w_{bb}(x, y) &= \Big(\frac{1}{2}x + \frac{1}{2}y + 1, \frac{1}{2}x + \frac{1}{2}y + \frac{3}{2}\Big).
\end{aligned}$$

Let $p_{ab}(x) = p_{aa}(x) = \frac{1}{2}$ for $x \in X_a$ and $p_{ab}(x) = p_{aa}(x) = 0$ for $x \in X_b$. Similarly, $p_{ba}(x) = p_{bb}(x) = \frac{1}{2}$ for $x \in X_b$ and $p_{ba}(x) = p_{bb}(x) = 0$ for $x \in X_a$. Then $w_{ab}(X_a) = [1,2] \times [4,5] \subset X_b$, $w_{aa}(X_a) = [3,4] \times [2,3] \subset X_a$, $w_{bb}(X_b) = [1,2] \times [3,4] \subset X_b$, $w_{ba}(X_b) = [2,3] \times [2,3] \subset X_a$, so $(X_a, X_b, (w_e)_{e\in E}, (p_e)_{e\in E})$ define the Markov system. Conditions **B1**–**B6** are fulfilled (with $\alpha = \frac{1}{2}$ in **B1**), so Theorem 4.1 gives the existence of an exponentially attractive invariant measure μ_*. It can be shown (see [8, Example 5.2.1]) that the support of this measure $[2,4] \times \{3\} \cup \{2\} \times [3,5]$ cannot be obtained as the limit set for any conformal iterated function system (i.e. not the graph directed one).

5. Proof of the main result

5.1. An exponential convergence theorem

Let $T\colon B_b(X) \to B_b(X)$ be a Markov operator with transition probability function $\{\mathbf{P}_x : x \in X\}$. We assume that there exists the family $\{\mathbf{Q}_{x,y} : x, y \in X\}$ of sub-probabilistic measures on X^2 such that maps $(x,y) \mapsto \mathbf{Q}_{x,y}(B)$ are measurable for every Borel $B \subset X^2$ and

$$\mathbf{Q}_{x,y}(A \times X) \le \mathbf{P}_x(A) \quad \text{and} \quad \mathbf{Q}_{x,y}(X \times A) \le \mathbf{P}_y(A)$$

for every $x, y \in X$ and Borel $A \subset X$.

Define on X^2 the family of measures $\{\mathbf{R}_{x,y} : x, y \in X\}$ which on rectangles $A \times B$ are given by

$$\mathbf{R}_{x,y}(A \times B) = \frac{1}{1 - \mathbf{Q}_{x,y}(X^2)}(\mathbf{P}_x(A) - \mathbf{Q}_{x,y}(A \times X))(\mathbf{P}_y(B) - \mathbf{Q}_{x,y}(X \times B)),$$

when $\mathbf{Q}_{x,y}(X^2) < 1$ and $\mathbf{R}_{x,y}(A \times B) = 0$ otherwise. The family $\{\mathbf{B}_{x,y} : x, y \in X\}$ of measures on X^2 defined by

$$\mathbf{B}_{x,y} = \mathbf{Q}_{x,y} + \mathbf{R}_{x,y} \quad \text{for} \quad x, y \in X \tag{5.1}$$

is *a coupling* (see [2], [3]) for $\{\mathbf{P}_x : x \in X\}$, i.e. for every $B \in \mathcal{B}_{X^2}$ the map $X^2 \ni (x,y) \mapsto \mathbf{B}_{x,y}(B)$ is measurable and

$$\mathbf{B}_{x,y}(A \times X) = \mathbf{P}_x(A), \quad \mathbf{B}_{x,y}(X \times A) = \mathbf{P}_y(A)$$

for every $x, y \in X$ and $A \in \mathcal{B}_X$.

Now we list assumptions on Markov operator T and transition probabilities $\{\mathbf{Q}_{x,y} : x, y \in X\}$.

A0 T is a *Feller operator*, i.e. $T(C_b(X)) \subset C_b(X)$.

A1 There exists a *Liapunov function* for T, i.e. a continuous function $L\colon X \to [0,\infty)$ such that L is bounded on bounded sets, $\lim_{x\to\infty} L(x) = +\infty$ (for bounded X this condition is omitted) and for some $\lambda \in (0,1)$, $c > 0$

$$TL(x) \le \lambda L(x) + c \quad \text{for } x \in X.$$

A2 There exist $F \subset X^2$ and $\alpha \in (0,1)$ such that $\operatorname{supp} \mathbf{Q}_{x,y} \subset F$ and

$$\int_{X^2} d(u,v)\mathbf{Q}_{x,y}(du,dv) \le \alpha d(x,y) \quad \text{for } (x,y) \in F. \tag{5.2}$$

A3 There exist $\delta > 0$, $l > 0$, and $\nu \in (0,1]$ such that

$$1 - \|\mathbf{Q}_{x,y}\| \le l d(x,y)^{\nu} \quad \text{and} \quad \mathbf{Q}_{x,y}(D(\alpha d(x,y))) \ge \delta \tag{5.3}$$

for $(x,y) \in F$, where $D(r) = \{(x,y) \in X^2 : d(x,y) < r\}$ for $r > 0$.

A4 There exist $\beta \in (0,1)$, $\tilde{C} > 0$ and $R > 0$ such that for

$$\kappa(\,(x_n,y_n)_{n\in\mathbb{N}_0}\,) = \inf\{n \in \mathbb{N}_0 : (x_n,y_n) \in F \text{ and } L(x_n) + L(y_n) < R\}$$

we have

$$\mathbb{E}_{x,y}\beta^{-\kappa} \le \tilde{C} \quad \text{whenever} \quad L(x) + L(y) < \frac{4c}{1-\lambda},$$

where $\mathbb{E}_{x,y}$ denotes here the expectation with respect to the Markov chain starting from (x,y) and with transition function $\{\mathbf{B}_{x,y} : x, y \in X\}$.

The next theorem (see [6]) is the essential tool in proving Theorem 4.1.

THEOREM 5.1. *Assume* **A0**–**A4**. *Then operator T possesses a unique invariant measure $\mu_* \in \mathcal{M}_1^L(X)$ and there exist $q \in (0,1)$ and $C > 0$ such that*

$$\|T^{*n}\mu - \mu_*\|_{FM} \le q^n C\Big(1 + \int_X L(x)\mu(dx)\Big)$$

for $\mu \in \mathcal{M}_1^L(X)$ and $n \in \mathbb{N}_0$.

The proof of the following lemma may be found in [6].

Lemma 5.1. *Let $(Y_n^y)_{n\in\mathbb{N}_0}$ with $y\in Y$ be a family of Markov chains on a metric space Y. Suppose that $V\colon Y\to[0,\infty)$ is a Liapunov function for their transition function $\{\pi_y : y\in Y\}$, i.e. there exist $a\in(0,1)$ and $b>0$ such that*

$$\int_Y V(x)\pi_y(dx)\le aV(y)+b \quad \textit{for} \quad y\in Y.$$

Then there exist $\lambda\in(0,1)$ and $C_0>0$ such that for

$$\rho((y_k)_{k\in\mathbb{N}_0})=\inf\Big\{k\ge 1 : V(y_k)<\frac{2b}{1-a}\Big\}$$

we have

$$\mathbb{E}_y\lambda^{-\rho}\le C_0(V(y_0)+1) \quad \textit{for } y\in Y.$$

5.2. Proof of Theorem 4.1

We are going to verify assumptions of Theorem 5.1. The family $\{\mathbf{P}_x : x\in X\}$ of probability measures on X is defined by

$$\mathbf{P}_x=\sum_{e\in E}p_e(x)\delta_{w_e(x)} \quad \text{for } x\in X,$$

where δ_x is the Dirac measure at x, is the transition probability function for P. Define the family $\{\mathbf{Q}_{x,y} : x,y\in X\}$ of subprobability measures on X^2 by

$$\mathbf{Q}_{x,y}=\sum_{e\in E}\min\{p_e(x),p_e(y)\}\delta_{(w_e(x),w_e(y))} \quad \text{for } x,y\in X_j$$

and $\mathbf{Q}_{x,y}=0$ for $x\in X_i$, $y\in X_j$, $i\ne j$, $i,j\in\{1,\dots,N\}$. It is clear that

$$\mathbf{Q}_{x,y}(A\times X)\le\mathbf{P}_x(A) \quad \text{and} \quad \mathbf{Q}_{x,y}(X\times A)\le\mathbf{P}_y(A)$$

for every $x,y\in X$ and $A\subset X$. Let $\{\mathbf{B}_{x,y} : x,y\in X\}$ be as in (5.1).

Conditions **B2** and **B3** imply that Markov operator P satisfies **A0**. Observe, that for $x\in X_j$

$$PL(x)=\sum_{e\in E_j}p_e(x)d(w_e(x),\bar{x}_{t(e)})\le\alpha d(x,\bar{x}_j)+c,$$

where $c = \sup_{j\in\{1,\dots,N\}} \sup_{e\in E_j} d(w_e(\bar{x}_j, \bar{x}_j) + \sup_{i,j\in\{1\dots,N\}} d(\bar{x}_i, \bar{x}_j) < \infty$, by **B5**. This implies that L is a Liapunov function for P and **A1** is fulfilled. Moreover, we have $\mathcal{M}_1^L(X) = \mathcal{M}_1^1(X)$.

Define $F = \bigcup_{i=1}^N X_j \times X_j \subset X \times X$. Assumption **B1** gives **A2**. From **B4** we obtain $\delta > 0$ such that

$$\mathbf{Q}_{x,y}(D(\alpha d(x,y))) \geq \sum_{e\in E_j\colon d(w_e(x),w_e(y))<\alpha d(x,y)} p_e(x)p_e(y) \geq \delta^2$$

for $(x, y) \in F$. Moreover, since

$$\|\mathbf{Q}_{x,y}\| + \sum_{e\in E_j\colon p_e(x)\geq p_e(y)} |p_e(x) - p_e(y)| = 1$$

for $x, y \in E_j$, $j \in \{1, \dots, N\}$, assumption **B2** implies **A3**.

Observe that for $e \in E$, $x \in X_{i(e)}$, **B3** gives

$$L(w_e(x)) = d(w_e(x), \bar{x}_{t(e)}) \leq ML(x) + c. \tag{5.4}$$

By Lemma 2.5 in [4] assumption **B6** implies that for every $j, k \in V$ there exist $s \in \mathbb{N}$ and paths $(e_1, \dots, e_s)$, $(\tilde{e}_1, \dots, \tilde{e}_s)$ such that

$$i(e_1) = j, \quad i(\tilde{e}_1) = k \quad \text{and} \quad t(e_s) = t(\tilde{e}_s).$$

For $r > 0$ define $\tilde{D}(r) = \{(x, y) \in X^2 : L(x) + L(y) < r\}$. For every $(x, y) \in \tilde{D}(\frac{4c}{1-\alpha})$ inequality (5.4) gives

$$(w_{e_s} \circ \dots \circ w_{e_1}(x), w_{\tilde{e}_s} \circ \dots \circ w_{\tilde{e}_1}(y)) \in \tilde{D}(R) \cap F \tag{5.5}$$

with $R = M^s \frac{4c}{1-\alpha} + 2c\frac{M^s-1}{M-1}$.

Fix $(x_0, y_0) \in \tilde{D}(\frac{4c}{1-\alpha})$. Let $(X_n, Y_n)_{n\in\mathbb{N}_0}$ be the Markov chain starting at (x_0, y_0) and with transition probability $\{\mathbf{B}_{x,y} : x, y \in X\}$. Let $\mathbb{P}_{x_0,y_0}$ be the probability measure on $(X^2)^\infty$ induced by $(X_n, Y_n)_{n\in\mathbb{N}_0}$ and let $\mathbb{E}_{x_0,y_0}$ be the expectation with respect to $\mathbb{P}_{x_0,y_0}$. Define the time $\rho\colon (X^2)^\infty \to \mathbb{N}_0$ of the first visit in $\tilde{D}(\frac{4c}{1-\alpha})$

$$\rho((x_n, y_n)_{n\in\mathbb{N}_0}) = \inf\{n \in \mathbb{N}_0 : (x_n, y_n) \in \tilde{D}(\tfrac{4c}{1-\alpha})\}$$

and the time of the n-th visit in $\tilde{D}(\frac{4c}{1-\alpha})$

$$\begin{aligned} \rho_1 &= \rho, \\ \rho_{n+1} &= \rho_n + \rho \circ T_{\rho_n} \quad \text{for } n > 1, \end{aligned}$$

where $T_n((y_k)_{k\in\mathbb{N}_0}) = (y_{k+n})_{k\in\mathbb{N}_0}$. The strong Markov property implies that

$$\mathbb{E}_{x_0,y_0}(\lambda^{-\rho}\circ T_{\rho_n}|\mathcal{F}_{\rho_n}) = \mathbb{E}_{X_{\rho_n},Y_{\rho_n}}(\lambda^{-\rho}) \quad \text{for } n\in\mathbb{N},$$

where $\mathcal{F}_{\rho_n}$ is σ -algebra in $(X^2)^\infty$ generated by ρ_n. Since $(X_{\rho_n}, Y_{\rho_n}) \in \tilde{D}(\frac{4c}{1-\alpha})$, Lemma 5.1 gives

$$\begin{aligned}\mathbb{E}_{x_0,y_0}(\lambda^{-\rho_{n+1}}) &= \mathbb{E}_{x_0,y_0}(\lambda^{-\rho_n}\mathbb{E}_{x_0,y_0}(\lambda^{-\rho}\circ T_{\rho_n}|\mathcal{F}_{\rho_n}))\\ &= \mathbb{E}_{x_0,y_0}(\lambda^{-\rho_n}\mathbb{E}_{Y_{\rho_n}}(\lambda^{-\rho}))\\ &\le \mathbb{E}_{x_0,y_0}(\lambda^{-\rho_n})[C_0(\tfrac{4c}{1-\alpha}+1)].\end{aligned}$$

Taking $a = C_0(\frac{4c}{1-\alpha}+1)$ we obtain

$$\mathbb{E}_{x_0,y_0}(\lambda^{-\rho_{n+1}}) \le a^{n+1}.$$

Define

$$\widehat{\kappa}((x_n,y_n)_{n\in\mathbb{N}_0}) = \inf\{n\in\mathbb{N}_0 :\ (x_n,y_n)\in\tilde{D}(\tfrac{4c}{1-\alpha}) \text{ and } (x_{n+s},y_{n+s})\in F\},$$

and $\sigma = \inf\{n\ge 1 :\ \widehat{\kappa} = \rho_n\}$, where s is as in (5.5). For $x\in X_i$, $y\in X_j$, where $i\ne j$, we have $\mathbf{B}_{x,y} = \mathbf{P}_x\otimes\mathbf{P}_y$, so **B4** together with (5.5) give $\mathbb{P}_{x_0,y_0}(\sigma = k) \le (1-p)^{k-1}$ for $k\ge 1$, where $p = (\delta)^{2s}$. Let $\beta > 1$. Hölder inequality implies that

$$\begin{aligned}\mathbb{E}_{x_0,y_0}(\lambda^{-\frac{\widehat{\kappa}}{\beta}}) &\le \sum_{k=1}^{\infty}\mathbb{E}_{x_0,y_0}(\lambda^{-\frac{\rho_k}{\beta}}\mathbf{1}_{\{\sigma=k\}})\\ &\le \sum_{k=1}^{\infty}[\mathbb{E}_{x_0,y_0}(\lambda^{-\rho_k})]^{\frac{1}{\beta}}\mathbb{P}_{x_0,y_0}(\sigma=k)^{1-\frac{1}{\beta}}\\ &\le \sum_{k=1}^{\infty}a^{\frac{k}{\beta}}(1-p)^{(k-1)(1-\frac{1}{\beta})}\\ &= (1-p)^{(\frac{1}{\beta}-1)}\sum_{k=1}^{\infty}\Big[\Big(\frac{a}{1-p}\Big)^{\frac{1}{\beta}}(1-p)\Big]^k.\end{aligned}$$

Choosing sufficiently large β and setting $\gamma = \lambda^{\frac{1}{\beta}}$ we obtain

$$\mathbb{E}_{x_0,y_0}(\gamma^{-\widehat{\kappa}}) \le \tilde{C}$$

for some $\tilde{C} > 0$. The observation that $\kappa \le \widehat{\kappa} + s$ gives **A4** and completes the proof.

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