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# FUNCTIONAL ANALYSIS AND NONLINEAR BOUNDARY VALUE PROBLEMS: THE LEGACY OF ANDRZEJ LASOTA

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*2012 Annual Lecture dedicated to the memory of Professor Andrzej Lasota*

## 1. Introduction

After a few papers on hyperbolic partial differential equations, the first part of the research career of Andrzej Lasota was devoted to various problems on ordinary differential equations and systems, with a special emphasis upon multi-point boundary value problems and periodic solutions. Many of those contributions are joint papers with Zdzisław Opial. The first one, published in 1961, was devoted to *de La Vallée Poussin's interpolation boundary value problem*

$$x^{(n)} = f(t, x, x', \dots, x^{(n-1)}), \quad x(t_k) = a_k \quad (k = 1, 2, \dots, n),$$

where $a = t_1 < t_2 < \ldots < t_n = b$ and $a_1, \dots, a_n \in \mathbb{R}$ are given. When $f \equiv 0$, this corresponds to interpolation by a polynomial of degree $n - 1$.

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The proof of this result motivated Lasota to consider the fixed point problem in a Banach space $B$ of the form

$$x = A(x)x + b(x)$$

when, for each $x \in B$, $A(x)$ belongs to a suitable class of linear operators on $B$ and $b$ is completely continuous and sublinear at infinity. His results can be seen as extensions of Ivar Fredholm's first theorem for linear integral equations.

In 1964, Lasota and Opial considered the existence of $\omega$-periodic solutions for systems of the form

$$x' = A(t,x)x + b(t,x)$$

where $A = (a_{ij})$, and $a_{ij}, b_i \colon \mathbb{R}^{m+1} \to \mathbb{R}$ satisfy Carathéodory conditions and are $\omega$-periodic with respect to $t$.

Lasota and coworkers have also considered second order differential equations or systems

$$x'' = f(t,x,x')$$

with various linear two-point boundary conditions, and first order systems

$$x' = f(t,x)$$

with fairly general linear boundary conditions.

About 40 papers on ordinary differential equations have been written by Lasota, between 1961 and 1980, the last one dealing with the so-called 'uniqueness implies existence' methodology. Most of those papers are a beautiful blend of linear functional analysis, fixed point theory (essentially Schauder's theorem) and inequalities, namely ingredients which are still basic in the present day studies of nonlinear boundary value problems for ordinary differential equations. Many papers are joint work, first with Opial, and later with several young collaborators. They are listed in the bibliography, but only the ones dealing with boundary value problems and periodic solutions are described here. Furthermore, for the sake of brevity, we have not considered, when analyzing Lasota's legacy, the extensions of his results to difference equations, functional differential equations and differential relations.

## 2. Ordinary differential equations with interpolation conditions

### 2.1. Early history

Polynomial interpolation seems to have been the motivation of a paper of Onorato Niccoletti of 1898 [82], devoted to nonlinear ordinary differential equations with some linear boundary conditions.

The interpolation by a polynomial of degree $n-1$ ($n \geq 2$ an integer) of the values $a_1, a_2, \ldots, a_n$ of a real function given at $n$ points

$$a = t_1 < t_2 < \ldots < t_n = b,$$

can of course be written in the form of a 'boundary value problem' on $[a,b] = [t_1, t_n]$

$$x^{(n)} = 0, \quad x(t_k) = a_k \quad (k = 1, 2, \ldots, n). \tag{2.1}$$

Its unique solution (a polynomial of degree $n-1$ which vanishes at $n$ points is identically zero), is given by the *Lagrange interpolation polynomial*

$$x(t) = \sum_{k=1}^{n} \frac{(t-t_1)\ldots\widehat{(t-t_k)}\ldots(t-t_n)}{(t_k-t_1)\ldots\widehat{(t_k-t_k)}\ldots(t_k-t_n)}$$

where $\widehat{\cdot}$ means that the corresponding factor is missing. The quotations marks are used because the data are not only given at the boundary of $[a,b]$ when $n \geq 3$. But the terminology *multipoint boundary value problem* is standard.

Natural generalizations are the *linear non-homogeneous multipoint boundary value problem*

$$x^{(n)} = h(t), \quad x(t_k) = a_k \quad (k = 1, 2, \ldots, n), \tag{2.2}$$

where the integrable function $h\colon [a,b] \to \mathbb{R}$ is given, and the *nonlinear non-homogeneous multipoint boundary value problem*

$$x^{(n)} = f(t, x, x', \ldots, x^{(n-1)}), \quad x(t_k) = a_k \quad (k = 1, 2, \ldots, n), \tag{2.3}$$

where the nonlinear Carathéodory function $f\colon [a,b] \times \mathbb{R}^n \to \mathbb{R}$ is given.

Such problems are special cases of those considered already by Niccoletti in 1898, where a more general class of boundary conditions involving also the values of some derivatives at some points is considered. Niccoletti also treated the case of systems of such equations, which contains in particular boundary value problems for *systems* of first order equations of the form

$$x_1' = f_1(t, x_1, \ldots, x_n), \ldots, x_n' = f_n(t, x_1, \ldots, x_n), \tag{2.4}$$
$$x_1(t_1) = a_1, \ldots x_n(t_n) = a_n,$$

now usually referred as *Niccoletti's problem.* Using Émile Picard's method of successive approximations, Niccoletti proved existence and uniqueness, for globally Lipschitzian functions $f$, when $b - a$ is sufficiently small.

The special case where $n = 2$ is the well known *Dirichlet* or *Picard boundary-value problem*

$$x'' = f(t, x, x'), \quad x(a) = a_1,\ x(b) = a_2, \tag{2.5}$$

already considered by Picard [83] in 1893 by the same method.

In contrast to problems (2.1) and (2.2) which are always uniquely solvable, either existence or uniqueness may fail for problem (2.3). For example, the special case of (2.5)

$$x'' = -x, \quad x(0) = 0,\ x(\pi) = 1,$$

has no solution, and the other special case

$$x'' = -x, \quad x(0) = 0,\ x(\pi) = 0,$$

has infinitely many solutions. Indeed, the problem

$$x'' = -x, \quad x(a) = a_1,\ x(b) = a_2, \tag{2.6}$$

has a solution if and only if one can find real numbers $A$ and $B$ such that

$$A \cos a + B \sin a = a_1, \quad A \cos b + B \sin b = a_2,$$

which requires that $\cos a \sin b - \cos b \sin a \neq 0$, i.e., that $b - a \neq 0 \pmod{\pi}$. In particular, existence (and indeed uniqueness) is insured if $b - a < \pi$.

The question of finding estimates for $b - a$ insuring the existence of a solution for the two-point boundary value problem

$$x'' = f(t, x, x'), \quad x(a) = a_1,\ x(b) = a_2,$$

was considered by Picard [84] in 1896 when $f$ is Lipschitzian with respect to the last two variables. As a special case, he showed that the linear homogeneous problem

$$x'' + p_1(t)x' + p_2(t)x = 0, \quad x(a) = 0 = x(b), \tag{2.7}$$

only has the trivial solution when

$$\|p_2\|_\infty \frac{(b-a)^2}{2} + \|p_1\|_\infty (b-a) < 1.$$

In 1929, Charles J. de La Vallée Poussin [15] generalized Picard's uniqueness result for (2.7) by showing that the problem

$$\begin{gathered} x^{(n)} + p_1(t)x^{(n-1)} + \ldots + p_{n-1}(t)x' + p_n(t)x = 0, \\ x(t_0) = x(t_1) = \ldots = x(t_n) = 0, \end{gathered} \tag{2.8}$$

only has the trivial solution if

$$\sum_{j=1}^{n} \|p_j\|_\infty \frac{(b-a)^j}{j!} < 1,$$

and extended the result to problem (2.3) with $f$ such that

$$|f(t, x_1, \ldots, x_n) - f(t, y_1, \ldots, y_n)| \le \sum_{j=1}^{n} L_j |x_j - y_j| \tag{2.9}$$

and

$$\sum_{j=1}^{n} L_j \frac{(b-a)^j}{j!} < 1.$$

In Poland, linear and nonlinear boundary value of interpolation type had been considered in 1946–47 by Jan Mikusiński [80] and Mieczysław Biernacki [5],

## 2.2. The introduction of functional analysis

The development of linear functional analysis in the first quarter of the XX$^{\text{th}}$ century as well as Stefan Banach's fixed point theorem of 1922 [2] – an abstract version of the method of successive approximations – made possible to express the above results in a functional analytic way. But a more essential step was made the same year 1922 by George D. Birkhoff and Oliver D. Kellogg [6], when they extended Brouwer's fixed point theorem (any continuous self map of a closed $n$-ball has at least one fixed point) to continuous self maps of a convex compact set of the function space $C^n([a,b])$ or $L^2([a,b])$. Their motivation was the obtention of existence of solutions of the boundary value problem

$$x^{(n)} = f(t, x, x', \ldots, x^{(n-1)}), \tag{2.10}$$

$$\int_a^b \sum_{j=0}^{n-1} p_{ij}(t) x^{(j)}(t)\, dt + \sum_{j=0}^{n-1} \sum_{k=1}^{m} q_{ijk} x(j)(t_k) = a_i$$

$(i = 1, 2, \ldots, n;\ a \le t_1 < t_2 < \ldots t_m \le b)$. They proved in particular that (2.10) has at least one solution when $b - a$ is sufficiently small.

In a series of beautiful papers written between 1927 and 1930 [88]–[90], Juliusz Schauder extended Birkhoff–Kellogg's fixed point theorem to continuous self maps of a convex compact set of an arbitrary Banach space, a result referred as *Schauder fixed point theorem*. His motivation and applications were essentially partial differential equations and his unique example for ordinary differential equations was an alternative proof of Giuseppe Peano's existence result for Cauchy problem (for which indeed no fixed point technique is really needed).

A paper of 1930 of Renato Caccioppoli reproduced Birkhoff–Kellogg's fixed point theorem for $C^n([a,b]$ without any reference to Birkhoff, Kellogg or Schauder. One year later, another one [10] acknowledged the work of those authors and gave applications to problem (2.10) with $b - a$ is arbitrary and $f$ is bounded everywhere or sublinear at infinity in $x_1, \ldots, x_n$. Those results stimulated a lot of activity in Italy. They were improved by Giuseppe Scorza-Dragoni and its school at Roma and Padova, through Caccioppoli's functional analytic approach, and by Silvio Cinquini at Pisa, who used the shooting approach based upon the solution of the associated Cauchy problem. A side aspect was a strong fight between Scorza-Dragoni and Cinquini lasting for more than ten years (including the Second World War), about the "topological or not" or "elementary or not" character of their respective approaches.

## 2.3. Existence results for multi-point boundary value problems (1961–62)

The first paper of Lasota devoted to a multipoint boundary value problem, written with Opial and published in 1961 [53], was directly motivated by de La Vallée Poussin's one:

In the case where the function $f(t, x_0, \dots, x_{n-1})$ satisfies the Lipschitz condition (2.9) one knows very well the following uniqueness theorem: in order that problem (2.3) has at most one solution, it suffices that any function $x(t)$ satisfying the differential inequality

$$|x^{(n)}(t)| \leq \sum_{j=1}^{n} L_j |x^{(j)}(t)| \tag{2.11}$$

and such that $x(t_i) = 0$ $(i = 1, \dots, n)$ is identically zero.

Its aim was to obtain a more general sufficient condition for the existence of at least one solution of (2.3). The main theorem goes as follows.

THEOREM 2.1. *Assume that the following conditions hold.*
(1) *$f$ is continuous and satisfies the inequality*

$$|f(t, x_0, \dots, x_{n_1})| \leq M + \sum_{j=0}^{n-1} L_i |x_i| \quad (M \geq 0,\ L_j > 0). \tag{2.12}$$

(2) *For every $t_1 < t_2 < \dots < t_n$ in $[a, b]$, the only function verifying the differential inequality* (2.11) *and*

$$x(t_j) = 0 \quad (j = 1, \dots, n) \tag{2.13}$$

*is $x(t) \equiv 0$.*
*Then problem* (2.3) *has at least one solution.*

The result was first proved for the linear case

$$x^{(n)} = \sum_{j=0}^{n-1} p_j(t) x^{(j)} + q(t). \tag{2.14}$$

LEMMA 2.2. *If the $p_j$ and $q$ are continuous and $|p_j(t)| \leq L_j$ $(t \in [a, b],\ 1 \leq j \leq n - 1)$ and if condition* (2) *of Theorem 2.1 holds, then equation* (2.14) *with boundary conditions* (2.13) *has a unique solution.*

To deduce the general case (and it is easy to reduce the problem to homogeneous boundary conditions (2.13)), the equation was written in the form

$$x^{(n)} = \sum_{j=0}^{n-1} p_j(t, x, x', \dots, x^{(n-1)}) x^{(j)} + q(t, x, x', \dots, x^{(n-1)}), \tag{2.15}$$

where, for $i = 0, \dots, n-1$,

$$p_i(t, x_0, \dots, x_{n-1}) = \frac{f(t, x_0, \dots, x_{n-1})}{M + \sum_{j=0}^{n-1} L_j |x_j|} L_i \varepsilon(x_i),$$

$$q(t, x_0, \dots, x_{n-1}) = \frac{f(t, x_0, \dots, x_{n-1})}{M + \sum_{j=0}^{n-1} L_j |x_j|} \Big( M + \sum_{j=0}^{n-1} (\eta(x_j) - \varepsilon(x_j)) L_j x_j \Big),$$

with

$$\varepsilon(s) = \begin{cases} s & \text{if } s \in [-1, 1], \\ 1 & \text{if } s > 1, \\ -1 & \text{if } s < -1, \end{cases} \qquad \text{and} \qquad \eta(s) = \begin{cases} 1 & \text{if } s \geq 0, \\ -1 & \text{if } s < 0. \end{cases}$$

Notice that $\eta$ is not continuous at 0 but it only occurs as multiplied by $s$, so that the product has a continuous extension. Indeed, the given definition of $q$ is slightly incorrect and should be replaced by

$$q(t, x_0, \dots, x_{n-1}) = f(t, x_0, \dots, x_{n-1}) - \sum_{i=0}^{n-1} \frac{f(t, x_0, \dots, x_{n-1})}{M + \sum_{j=0}^{n-1} L_j |x_j|} L_i \varepsilon(x_i).$$

Therefore

$$|q(t, x_0, \dots, x_{n-1})| \leq \frac{|f(t, x_0, \dots, x_{n-1})|}{M + \sum_{j=0}^{n-1} L_j |x_j|} \Big| M + \sum_{j=0}^{n-1} (L_j |x_j| - L_j \varepsilon(x_j) x_j) \Big|.$$

The function $\xi(s) = |s| - \varepsilon(s) s$ given by

$$\xi(s) = \begin{cases} 0 & \text{if } s > 1, \\ s - s^2 & \text{if } s \in [0, 1], \\ -s - s^2 & \text{if } s \in [-1, 0), \\ 0 & \text{if } s < -1, \end{cases}$$

is continuous, nonnegative and bounded (by 1) on $\mathbb{R}$ and hence $q$ is continuous and bounded by $M + \sum_{j=0}^{n-1}$ on $[a, b] \times \mathbb{R}^n$. On the other hand $|p_j| \leq L_j$ $(j = 0, \dots, n-1)$.

The next step consisted in introducing the Banach space $E$ of functions $x \in C^{n-1}([a, b])$ with the usual norm $\|x\| = \sup_{t \in [a,b]} \sum_{j=0}^{n-1} |x^{(j)}(t)|$ and in introducing the mapping $T\colon E \to E$ which to each $x \in E$ associates the (unique) solution $y$ of the linear multipoint boundary value problem

$$y^{(n)} = \sum_{j=0}^{n-1} p_j(t, x(t), \ldots, x^{(n-1)}) y^{(j)} + q(t, x(t), \ldots, x^{(n-1)}),$$
$$y(t_0) = \ldots = y(t_n) = 0,$$

as it follows from Lemma 2.2.

Lasota and Opial then showed that $T(E)$ is a compact subset of $E$, and used Schauder's fixed point theorem to obtain a fixed point $z$ of $T$, i.e., of a function $z \in C^{(n)}([a,b])$ satisfying the boundary conditions (2.13) and the differential equation

$$\begin{aligned} z^{(n)} &= \sum_{j=0}^{n-1} p(t, z, \ldots, z^{(n-1)}) z^{(j)} + q(t, z, \ldots, z^{(n-1)}) \\ &= f(t, z, \ldots, z^{(n-1)}). \end{aligned}$$

This methodology, widely used by Lasota (alone or in collaboration) in subsequent papers on similar or other problems, is reminiscent of a technique used by Juliusz Schauder and Jean Leray in dealing with quasilinear elliptic Dirichlet problems of the form

$$\sum_{i,j=1}^{N} a_{ij}(x, u, \nabla u) u_{ij}(x) = b(x, u, \nabla u) \text{ in } \Omega, \quad u(x) = \varphi(x) \text{ on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. First, for $v$ in a suitable Hölder space, the linear Dirichlet problem

$$\sum_{i,j=1}^{N} a_{ij}(x, v, \nabla v) u_{ij}(x) = b(x, v, \nabla v) \text{ in } \Omega, \quad u(x) = \varphi(x) \text{ on } \partial\Omega,$$

is uniquely solved, namely $u = T(v)$, and then the solution of the quasilinear problem is reduced to finding a fixed point of $T$ using some topological fixed point theorem. As we shall see, a similar approach, sometimes called *Schauder's linearization*, had been used in 1956 by Mario Volpato [98] in a problem of periodic solutions of second order differential equations. This reference is not quoted in [53], whose bibliography is restricted to two books and de La Vallée Poussin's paper [15]. In 1953, the same Volpato [97] had obtained sufficient conditions for the existence of a solution of (2.3) under some complicated conditions upon $f$ listed on one page and half. In 1974, Anna Krakowiak [30] has obtained similar results for first order systems of differential equations.

The same problem (2.3) was considered one year later by Lasota in [34], where the richer bibliography refers to the earlier contributions of Caccioppoli [10], Cinquini [12], Levine [70], Volpato [97] and Zwirner [100]. Instead of the linear growth condition (2.12), Lasota considered the nonlinear one

$$|f(t, x_0, \ldots, x_{n-1})| \leq A + B \sum_{j=0}^{n-1} |x_i|^{\alpha}. \tag{2.16}$$

The sublinear case where $0 < \alpha < 1$ had been treated by Cinquini [12]. For $\alpha \geq 1$, Lasota proved the existence of a solution when $b - a \leq c$ with $c$ sufficiently small, and gave estimates for $c$. His approach was still based upon Schauder's fixed point theorem. A map $S\colon C([a,b]) \to C^n([a,b])$ was defined which, to any $h \in E$ associates the unique solution $x_h$ of the linear interpolation problem (2.2). Now, if

$$N_f\colon E \to C([a,b]), \quad x \mapsto f(\cdot, x(\cdot), \ldots, x^{(n-1)}(\cdot)),$$

denotes the Nemitsky operator associated to $f$, the solutions of (2.3) are the fixed points in $E$ of the mapping $T = S \circ N_f$. Lasota showed the existence of some closed convex set $\Omega_f$ of $E$ mapped by $T$ into a compact subset of $\Omega_f$. The definition of $\Omega_f$ is rather general but somewhat cumbersome.

The same year, Lasota [35] considered the more general problem

$$\begin{gathered} x_i^{(n_i)} = f_i(t, x_1, \ldots, x_1^{(n_i-1)}, \ldots, x_m, \ldots, x_m^{(n_m-1)}), \quad i = 1, \ldots, m, \\ \sum_{i=1}^{m} L_{\nu i} x_i = r_\nu, \quad \nu = 1, \ldots, N, \quad N = \sum_{i=1}^{m} n_i, \end{gathered} \tag{2.17}$$

where $L_{\nu i} : C^{n_i - 1}([a,b]) \to \mathbb{R}$ denotes a linear continuous functional. He introduced the corresponding linear homogeneous system

$$\begin{gathered} x_i^{(n_i)} = \sum_{k=1}^{m} \sum_{j=0}^{n_k-1} p_{ijk}(t) x_k^{(j)}, \quad i = 1, \ldots, m, \\ \sum_{i=1}^{m} L_{\nu i} x_i = 0, \quad \nu = 1, \ldots, N, \quad N = \sum_{i=1}^{m} n_i, \end{gathered} \tag{2.18}$$

and proved the following generalization of the result of [53].

THEOREM 2.3. *Assume that there exist continuous functions $\overline{p}_{ijk} < \widehat{p}_{ijk}$ such that the following conditions hold*
(1) $|f_i(t, x_1^0, \dots, x_m^{n_m-1})| \le M_i + \sum_{k=1}^m \sum_{j=0}^{n_k-1} \overline{p}_{ijk}(t)|x_l^j|, \quad i = 1, \dots, m$
(2) *for any system $p_{ijk}$ of continuous functions over $[a,b]$ such that $|p_{ijk}| < \widehat{p}_{ijk}$, system* (2.17) *only has the trivial solution.*

*Then problem* (2.18) *has at least one solution.*

No proof was given, but it would clearly follow the same line as the one given in [53]. Using results of Anatoly Yu. Levin [70, 71] for the linear interpolation problem, Lasota deduced from Theorem 2.3 that *when condition* (2.12) *holds and*

$$\sum_{j=1}^{n} \frac{L_{n-j}(b-a)^j}{j\left[\frac{j-1}{2}\right]!\left[\frac{j}{2}\right]!} < 1$$

*the problem*

$$x^{(n)} = f(t, x, \dots, x^{(n-1)}),$$
$$x(t_i) = r_i \quad (i = 1, \dots, l), \quad x^{(j)}(t_{l+j}) = s_j \quad (j = 1, \dots, n-l),$$

*has at least one solution.*

From results of Marko Švec [93] for the linear problem, Lasota also deduced from Theorem 2.3 that *when*

$$|f_i(t, x_1, \dots, x_m)| \le M_i + \sum_{k=1}^{m} a_{ik}|x_k| \quad (i = 1, \dots, m)$$

*and $b - a < \frac{1}{r_0}$ with $r_0$ the largest eigenvalue of the matrix $(a_{jk})$, the problem*

$$x_i' = f_i(t, x_1, \dots, x_m), \quad x_i(t_i) = r_i \quad (i = 1, \dots, m),$$

*has at least one solution.*

Further results about the estimation of $b-a$ for existence and uniqueness to equations of order two and four have been given later by Zdzisław Denkowski [16–18], Franciszek Hugon Szafraniec [94], Janusz Traple [95, 96] and Józef Myjak [81].

## 3. Using Pontryagin's maximum principle (1963)

In a joint paper with Opial [36], Lasota has used Pontryagin's maximum principle of control theory to evaluate the maximal length $h(P_1,\dots,P_n)$ of the interval $[a,b]$ on which the linear boundary value problem

$$x^{(n)} + p_1(t)x^{(n-1)} + \ldots + p_{n-1}(t)x' + p_n(t)x = 0, \tag{3.1}$$
$$x(a) = x'(a) = \ldots = x^{(n-2)}(a) = x(b) = 0,$$

has only the trivial solution. Here $P_j$ denotes an upper bound of the function $|p_j|$ on $[a,b]$ $(j = 1,\dots,n)$. They first proved the following

Theorem 3.1. *Let $u$ be the solution of the initial value problem*

$$u^{(n)} + P_1|u^{(n-1)}| + \ldots + P_{n-1}|u'| + P_n|u| = 0,$$
$$u(a) = \ldots = u^{(n-2)}(a) = 0, \quad u^{(n-1)}(a) = 1.$$

*Denote by $h = h(P_1,\dots,P_n)$ the smallest positive root of equation $u(t) = 0$. Then, if $b - a < h(P_1,\dots,P_n)$, problem (3.1) has only the trivial solution. Furthermore, if the continuous function $f\colon [a,b] \times \mathbb{R}^n \to \mathbb{R}$ is such that*

$$|f(t,x_0,\dots,x_{n-1})| \le M + \sum_{j=1}^{n} P_j|x_{n-j}|$$

*then the nonlinear two-point boundary value problem*

$$x^{(n)} = f(t,x,x',\dots,x^{(n-1)}),$$
$$x(a) = r_0,\dots,x^{(n-2)}(a) = r_{n-2},\ x(b) = c,$$

*has at least one solution when $r_0, r_1,\dots,r_{n-2}, c \in [0,d]$ and*

$$b - a < \min\{h(P_1,\dots,P_n), a\}.$$

Subsequent uses of this approach have been made by Yu.A. Melentsova [76, 77], Yu.A. Melentsova and G.N. Mil'shtein [78, 79], Lloyd Jackson [24, 25], Johnny Henderson [21], Johnny Henderson and Robert W. McGwier Jr. [22], and Marc Henrard [23].

## 4. Fixed point theory and applications

### 4.1. A first nonlinear version of Fredholm's first theorem for integral equations (1963)

Lasota's paper [37] provided the fixed point version of the method used in [53] for boundary value problems of interpolation type. Given a Banach space $E$ and the set $L_s(E,E)$ of linear mappings with pointwise convergence, Lasota assumed the existence of a subset $Q \subset L_s(E,E)$ such that

1. *Any sequence* $(A_n)$ *in* $Q$ *contains a convergent subsequence converging pointwise to some* $A \in Q$.
2. *The set* $\bigcup_{A\in Q;\|x\|=1} Ax$ *is relatively compact in* $E$.

If now $A\colon E \to Q$ and $b\colon E \to E$ are mappings, he proved the following result for the fixed point problem in $E$

$$x = A(x)x + b(x). \tag{4.1}$$

Theorem 4.1. *If the following conditions hold:*

(a) *For any* $A \in Q$, *equation* $x = Ax$ *only has the trivial solution*
(b) $x_n \to x \quad \Rightarrow \quad A(x_n) \to A(x)$ *pointwise*
(c) $b$ *is totally continuous*
(d) $\lim_{\|x\|\to\infty} \frac{\|b(x)\|}{\|x\|} = 0$.

*Then problem* (4.1) *has at least one solution.*

Notice that when $A$ and $B$ are constant mappings, Theorem 4.1 corresponds to Fredholm's first theorem for linear integral equations.

As mentioned earlier, the idea of the proof is an abstraction of the one used in [53]. Given any $y \in E$, the problem

$$x = A(y)x + b(y)$$

has a unique solution $x = Ty$. The assumptions imply that $T$ maps $E$ into a compact subset of $E$ and hence has a fixed point, by Schauder's theorem, which is a solution of (4.1). As applications, one gets all the results given (without proof) in [35] for multipoint boundary value problems associated to systems of ordinary differential equations, new existence results for nonlinear Hammerstein equations and a better evaluation of an upper bound for $b - a$ in the case of the planar Niccoletti problem

$$x_i' = f_i(t, x_1, x_2), \quad x_1(a) = r_1, \quad x_2(b) = r_2,$$

when

$$|f_i(t, x_1, x_2)| \leq M_i(t) + k_{i1}|x_1| + k_{i2}|x_2| \quad (i = 1, 2).$$

Another application of Theorem 4.1 was given by Lasota in [38] to existence results for some boundary value problems on a bounded sufficiently regular domain $D \subset \mathbb{R}^n$ for semilinear elliptic equations.

### 4.2. A second nonlinear version of Fredholm's first theorem for linear integral equations (1966)

In [41], Lasota gave a new generalization of Fredholm's first theorem for linear integral equations, stated in terms of multivalued mappings. In a Banach space $E$, let $c(E)$ denote the family of nonempty convex subsets of $E$. A mapping $H\colon E \mapsto c(E)$ is called *completely continuous* if $\cup_{u \in B} H(u)$ is relatively compact whenever $B \subset E$ is bounded and if the conditions $\lim u_n = u_0, \lim v_n = v_0$ and $v_n \in H(u_n)$ for all $n$, entails $v_0 \in H(u_0)$. A mapping $h\colon E \mapsto E$ is called *completely continuous* if the mapping $u \mapsto \{h(u)\}$ of $E$ into $c(E)$ is completely continuous in the above sense.

THEOREM 4.2. *Let $H$ be a homogeneous and completely continuous mapping of $E$ into $c(E)$, and let $h$ be a completely continuous mapping of $E$ into itself such that* $\lim\{\rho(h(u), H(u))/\|u\|\} = 0$ *as* $\|u\| \to \infty$. *If $u \in H(u)$ is satisfied only for $u = 0$, then there exists at least one solution to the equation $u = h(u)$.*

Applications were made to general existence theorem for boundary-value problems for systems of first order differential equations with boundary conditions of the form $N(x) = r$, where $N$ is a continuous and homogeneous mapping of $C^n(I)$ into $\mathbb{R}^m$.

Generalizations and further applications of Theorem 4.2 have been given by Klaus Schmitt and Hal Smith [91]–[92], and by Mawhin and S.B. Tshinanga [73].

## 5. Second order nonlinear differential equations with Sturm–Liouville boundary conditions

### 5.1. Bounds on the length of solubility interval (1963)

In paper [39], Lasota has considered the two-point boundary value problem

$$x'' = f(t, x, x'), \quad x'(a) = r, \quad x(b) + hx'(b) = s, \tag{5.1}$$

assuming that $f(t, x, y)$ is Lipschitz continuous in $x$ and $y$ for fixed $t$, and measurable in $t$ for fixed $x, y$. He proved the following existence condition.

THEOREM 5.1. *Assume that the following conditions hold.*
(1) $|f(t, x, y)| \le L(t) + M|x| + K|y|$, *where* $L(t)$ *is non-negative and summable in* $[a, b]$ *and* $M, K \ge 0$.
(2) *The solution of* $w' = w^2 + Kw + M, \quad w(0) = 0$ *is such that* $|h|w(t) < 1$ *in* $[a, b]$.

*Then problem* (5.1) *has a solution.*

This result generalized earlier work with $f(t, x, y)$ continuous in $t$. Some mild extensions and generalizations were also stated and proved. Notice that the paper [28] of James L. Kaplan, Lasota and James A. Yorke used Wazewski's method to prove some standard results on lower and upper solutions for second order scalar equations with Sturm–Liouville boundary conditions.

### 5.2. Extensions of Hartman's theorems (1972)

The joint paper [68] with Yorke is devoted to the following boundary value problems for second order systems

$$x'' = f(t, x, x'), \quad x(0) - A_0 x'(0) = 0, \quad x(1) + A_1 x'(1) = 0, \tag{5.2}$$

where $A_0, A_1$ are semi-positive definite matrices and $f$ is continuous.

For the first time, Lasota used Leray–Schauder continuation theorem [69] to prove his existence results, motivated by earlier ones of Philip Hartman [20]. The first theorem goes as follows.

THEOREM 5.2. *Assume that there exists $\sigma > 0$ and $K \geq 0$ such that for $(t, x, y) \in [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n$,*

$$|y|^2 + x \cdot f(t, x, y) \geq -K(1 + |x| + |x \cdot y|) + \sigma |f(t, x, y)|. \tag{5.3}$$

*Then* (5.2) *has at least one solution when $A_0, A_1 \geq 0$.*

Condition (5.3) can be weakened at the expense of another assumption.

THEOREM 5.3. *Assume that there exists $\sigma > 0$ and $K \geq 0$ such that, for $(t, x, y) \in [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n$,*

$$|y|^2 + x \cdot f(t, x, y) \geq -K(1 + |x| + |x \cdot y|) + \sigma |y|. \tag{5.4}$$

*Assume furthermore that*

$$|f(t, x, y)| \leq \phi(|y|) \quad \text{with} \quad \int_0^\infty \frac{s\, ds}{\phi(s)} = \infty. \tag{5.5}$$

*Then* (5.2) *has at least one solution when $A_0, A_1 \geq 0$.*

Variants of this theorem have been given by Robert E. Gaines and Mawhin in [19].

## 6. Periodic solutions

### 6.1. Extension of Volpato's method (1964)

In this important paper [56], Lasota and Opial proved various existence theorems for $\omega$-periodic solutions of differential systems of the form

$$x' = A(t, x)x + b(t, x), \tag{6.1}$$

where $A\colon \mathbb{R}^{m+1} \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$, $b\colon \mathbb{R}^{m+1} \to \mathbb{R}^m$ satisfy Carathéodory conditions and are $\omega$-periodic with respect to $t$.

Such problems had already been considered in 1958 by Ioan Barbalat and Aristide Halanay [3], who mentioned the special case of a second order equation

$$u'' + p(t, u, u')u = q(t, u, u')$$

already treated by Volpato [97] in 1956. Barbalat and Halanay assumed that the matrix function $A$ is such that, for any $\omega$-periodic function $y(t)$, the linear equation

$$x' = A(t, y(t))x$$

has only the trivial $\omega$-periodic solution, and that the corresponding Green matrix $G_y(t,s)$ satisfies the inequality

$$\|G_y(t,s)\| \leq L$$

for all $t, s \in [0, \omega]$, all $\omega$-periodic continuous functions $y$ and some $L > 0$. An application was given to a perturbation of a Hamiltonian system in $\mathbb{R}^{2m}$ of the form

$$z' = JH(t,z)z + f(t,z),$$

where $J$ is the symplectic matrix, $f$ has at most a linear growth with sufficiently small slope and $H(t,z)$ is a symmetric matrix for each $t$ and $z$ and its smallest and largest eigenvalues satisfy suitable conditions. The approach, inspired by Volpato's special case, consisted in denoting by $U(y)$, for any $y \in C_\omega$, the unique $\omega$-periodic solution of the linear problem

$$x' = A(t,y)x + b(t,y),$$

so that the $\omega$-periodic solutions of (6.1) are the fixed points of $U$ in $C_\omega$, whose existence follows from Schauder's fixed point theorem.

Barbalat and Halanay's paper was not quoted in [56] but was mentioned in [57], which summarized and commented the methods and results of [56] and related papers on interpolation problems:

> see also Barbalat–Halanay [3] where one can find many ideas in close relation with our general method.

On the other hand, the introduction of [57] nicely commented the underlying philosophy:

> The method that we use in the problem of the existence of periodic solutions of ordinary differential equations is quite general. It applies as well to this particular problem as to many other problems of this type (existence of solutions of the interpolation problem for nonlinear differential equations [53, 55], the existence of solutions to general boundary value problems for systems of differential equations [35], the existence of solutions of integral equations [37], the existence of solutions of partial differential equations [38] etc.) This is for this reason that the method finds its best place in the frame

of functional analysis. But, due to lack of time, it will not be question here of functional analysis – we restrict ourselves to present the essential of the methof on the example of the chosen special example.

Let $L_{mm}$ be the space of all $m \times m$ matrices $(a_{ij}(t))$ with all entries in $L(\omega)$, and let $C_\omega(\mathbb{R})$ be the Banach space of continuous $\omega$-periodic functions of $\mathbb{R}$ into $\mathbb{R}^m$. The main result is the following one

THEOREM 6.1. *Assume that the following conditions hold:*

(1) *there exists a subset $\mathcal{A} \subset L_{mm}(\omega)$ bounded, weakly closed and such that for any matrix $(a_{ij}) \in \mathcal{A}$ the linear system*

$$x' = A(t)x \tag{6.2}$$

*has only the trivial $\omega$-periodic solution*

(2) *for every continuous $\omega$-periodic function $x(t)$, the matrices $A(\cdot, x(\cdot)) \in \mathcal{A}$.*

(3) $\lim_{n\to\infty} \frac{1}{n} \int_0^\omega \sup_{|y|\le n} |b(t,y)|\, dt = 0$, *where* $|y| = \sum_{i=1}^m |y_i|$, $|b| = \sum_{i=1}^m |b_i|$.

*Then there exists at least one $\omega$-periodic solution of system* (6.1).

The idea of the proof is reminiscent from the technique used for interpolation boundary conditions. From the assumptions, it follows that for every continuous $\omega$-periodic function $y$, the linear system

$$x_i' = \sum_{j=1}^m a_i j(t, y(t))x + b(t, y(t))$$

has a unique $\omega$-periodic solution $x = T(y)$ and $T\colon C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$, completely continuous,maps some closed ball $B \in C_\omega(\mathbb{R})$ into itself. The existence of a fixed point of $T$, and hence of a $\omega$-periodic solution of (6.1) follows from Schauder's theorem.

The authors then specialized this theorem to a number of particular nonlinear differential equations. For example, the version of the result for a scalar equation of order $m$ goes as follows.

THEOREM 6.2. *Assume the following conditions hold.*

(1) *There exists a bounded, weakly closed subset $\mathcal{A} \subset L_{1m}(\omega)$ such that for any $(a_1, \dots, a_m) \in \mathcal{A}$ the linear equation*

$$x^{(m)} + \sum_{i=1}^{m} a_i(t) x^{(m-i)} = 0$$

*only has the trivial solution.*

(2) *For every continuous $\omega$-periodic function $(x_1, \ldots, x_m)$, $(a_i(\cdot, x(\cdot))) \in \mathcal{A}$.*

(3) $\lim_{n\to\infty} \frac{1}{n} \int_0^\omega \sup_{|x_i|\le n} |b(t, x_1, \ldots, x_m)|\, dt = 0.$

*Then equation*

$$x^{(m)} + \sum_{i=1}^{m} a_i(t, x, \ldots, x^{(m-1)}) x_i = b(t, x, x', \ldots, x^{(m-1)})$$

*has at least one $T$-periodic solution.*

The still more special case of a second order equation

$$x'' + P(t, x, x')x = Q(t, x, x') \tag{6.3}$$

had already been considered in 1956 by Volpato [98] who, using (for the first time maybe) a methodology similar to that exploited and generalized by Lasota and Opial, had proved the *existence of at least one $\omega$-periodic solution of* (6.3) *when*

1. $0 \le p(t) \le P(t, x, y) \le P(t)$ *for all* $(t, x, y) \in \mathbb{R}^3$, *where* $0 \le p(t) \le P(t)$ *are continuous, $\omega$-periodic and such that*

$$0 < \int_0^\omega p(t)\, dt, \quad \omega \int_0^\omega P(t)\, dt \le 4, \tag{6.4}$$

2. $Q$ *is bounded on* $\mathbb{R}^3$.

The second condition (6.4) had already been introduced by Alexandr M. Lyapunov in 1892 [72], who also assumed $p > 0$. It was improved by Göran Borg [7] in 1944. Those conditions imply that whenever $q$ is continuous, $\omega$-periodic and such that $p(t) \le q(t) \le P(t)$ for all $t \in \mathbb{R}$, the linear equation

$$x'' + q(t)x = 0 \tag{6.5}$$

has all its solutions either unbounded, or satisfying the conditions

$$x(\omega) = e^{\pm i\theta} x(0), \quad x'(\omega) = e^{\pm i\theta} x'(0)$$

for some $\theta > 0$. Consequently (6.5) only has the trivial $\omega$-periodic solution. Lasota and Opial showed the existence of at least one $\omega$-periodic solution to (6.3) when

1. *$p(t) \leq P(t,x,y) \leq P(t)$ for all $(t,x,y) \in \mathbb{R}^3$ where $p$ and $q$ are $\omega$-periodic, continuous and such that*

$$0 \leq p(t) \leq P(t), \quad 0 < \omega \int_0^\omega p(t)\,dt \leq \omega \int_0^\omega P(t)\,dt \leq 16, \tag{6.6}$$

2. $\displaystyle\lim_{n\to\infty} \frac{1}{n} \int_0^\omega \sup_{|x|+|y|\leq n} |Q(t,x,y)|\,dt = 0,$

and that 16 is the best possible constant in (6.6). Other conditions upon $p$ and $P$ were given as well, and have been followed by other ones insuring the existence of an $\omega$-periodic solution for (6.3).

One of them, essentially due to Mawhin and James R. Ward Jr. [74], requires that

$$\int_0^\omega p(t)\,dt > 0, \quad P(t) \leq \frac{4\pi^2}{\omega^2} \tag{6.7}$$

with strict inequality on a subset of $(0,\omega)$ of positive measure, what will be written

$$P(t) \lesssim \frac{4\pi^2}{\omega^2}$$

and can be stated equivalently as

$$\omega \int_0^\omega P(t)\,dt < 4\pi^2.$$

Another one requires the existence of some integer $k \geq 1$ such that

$$\frac{4\pi^2 k^2}{\omega^2} \lesssim p(t) \leq P(t) \lesssim \frac{4\pi^2(k+1)^2}{\omega^2}. \tag{6.8}$$

Inequality (6.6) can be seen as a condition upon $P$ in $L^1$-norm and inequality (6.7) as a condition upon $P$ in $L^\infty$-norm. One can therefore think about similar inequalities in any $L^p$-norm. The literature on variants and extensions of Lyapunov inequalities, involving other norms and other boundary conditions is very wide. One can consult the recent interesting survey of Antonio Cañada and Salvador Villegas [11].

Notice that, as shown by Mawhin and Ward in [74] in the case of elliptic partial differential equations, Schauder's linearization technique followed by Schauder's fixed point theorem can be replaced by a direct application of

Leray–Schauder's method [69]. If one assumes that *there exists* $R > 0$ *such that for all* $\lambda \in [0,1]$ *and all possible* $\omega$*-periodic solution of the system*

$$x' = A(t, \lambda x)x + \lambda b(t,x)$$

*satisfies the inequality* $\|x\|_\infty < R$, *and if the system* $x' = A(0)x$ *has only the trivial* $\omega$*-periodic solution, then* by Leray–Schauder's theorem, *system* (6.1) *has at least one* $\omega$*-periodic solution such that* $\max_{t\in\mathbb{R}} |x| < R$. This allows generalizations of the condition upon $b(t,x)$. The approach can also be applied to other boundary conditions.

## 6.2. Periodic solutions of higher order equations (1966)

In the paper [52], Lasota and Szafraniec have considered the existence of $\omega$-periodic solutions of higher order differential equations of the form

$$Lx + a(t, x^{(n-1)}, \cdots, x)x = b(t, x^{(n-1)}, \cdots, x) \tag{6.9}$$

where

$$Lx := x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_{n-1}(t)x',$$

and have found existence conditions in terms of various norms of the coefficients $a_j$. Letting for $p \geq 1$,

$$\|f\|_p := \left(\frac{1}{\omega}\int_0^\omega |f(t)|^p\,dt\right)^{1/p}, \quad \|f\|_\infty := \sup_{t\in[0,\omega]} |f(t)|,$$

they first showed that *if one of the following conditions*

$$(\infty)\quad \sum_{i=1}^{n-1} \|a_i\|_\infty \left(\frac{\omega}{2\pi}\right)^i + 2\|a_n\|_\infty \left(\frac{\omega}{2\pi}\right)^n < 1$$

$$(2)\quad \sum_{i=1}^{n} \|a_i\|_2 \left(\frac{\omega}{2\pi}\right)^i < \frac{1}{\pi}$$

$$(1)\quad \frac{\omega}{2}\|a_1\|_1 + \pi^2 \sum_{i=1}^{n} \|a_i\|_1 \left(\frac{\omega}{2\pi}\right)^i < 1$$

*is satisfied, then the linear differential equation*

$$Lx + a_n(t)x = 0$$

*only has the trivial $\omega$-periodic solution.* Then, using Lasota–Opial's methodology, they deduced from this result and Schauder's fixed point theorem that *if*

1. *the $a_i$ satisfy condition* $(\infty)$ *or* $(2)$ *or* $(1)$ *above,*
2. $\lim\limits_{k\to\infty} \frac{1}{k} \int_0^\omega b(t, x_1, \ldots, x_n)\, dt = 0,$
3. $a(t, x_1, \ldots, x_n)x^+(t) \geq \varphi(t) \geq 0,$
4. $\int_0^\omega \varphi(t)\, dt > 0,$

*then equation* (6.9) *has at least one $\omega$-periodic solution.*

More recent contributions in this directions are due to Małgorzata Wypich [99] and Monika Kubicová [32].

## 7. First order systems with linear boundary conditions

### 7.1. Reduction to a nonlinear integral equation (1965)

In the paper [58], Lasota and Opial first showed that *for any continuous linear mapping of $C^n$ onto $\mathbb{R}^n$*, where $C^n$ denotes the space of continuous functions $x(t)$ from a compact real interval $\Delta$ into $\mathbb{R}^n$, with usual norm $\|x\|_\infty$,*there exists a continuous $n \times n$ matrix $A(t)$ defined on $\Delta$ such that the restriction $L_A$ of $L$ to the subspace $C^n_A$ of $C^n$ of all solutions of the linear differential equation*

$$y' = A(t)y$$

*also maps $C^n_A$ onto $\mathbb{R}^n$.*

This result can be applied to the solution of boundary value problems of the type

$$x' = f(t, x), \quad Lx = r,$$

where $f$ is an $n$-vector Carathéodory function on $\Delta \times \mathbb{R}^n$ and $r \in \mathbb{R}^n$. In fact, such a problem is equivalent to the solution of a Hammerstein integral equation

$$x(t) = \int_\Delta G(t, s)[f(s, x(s)) - A(s)x(s)]\, ds + H(t)r,$$

where the Green matrix $G$ and the matrix $H$ depend only on $A, L$.

For example, for the periodic problem

$$x' = f(t, x), \quad x(\omega) - x(0) = 0,$$

if we take $A(t) = Id$, the system $y' = y$ has the set of solutions $\{ce^t : c \in \mathbb{R}^n\}$ and, with $Lx = x(\omega) - x(0)$, we have $L(ce^\omega - c) = (e^\omega - 1)c$ which is onto $\mathbb{R}^n$. This periodic problem is therefore equivalent to the Hammerstein equation

$$x(t) = \int_0^\omega G(t, s)[f(s, x(s)) - x(s)]\, ds,$$

where $G$ is the Green matrix associated to the linear $\omega$-periodic problem

$$y' = y + h(t), \quad y(0) = y(\omega).$$

### 7.2. Existence conditions for Niccoletti's problem (1966)

In paper [52], Lasota and Czesław Olech have considered Niccoletti's problem for a first order system

$$\text{(B)} \qquad x' = f(t, x), \quad x_i(t_i) = c_i, \quad i = 1, \cdots, n,$$

where the $c_i$ are given and the functions $f(t, x)$ satisfy Carathéodory conditions.

They proved *uniqueness for all* $(c_1, \cdots, c_n)$ *and* $(t_1, \cdots, t_n)$ *if conditions*

$$|f(t, u) - f(t, v)| \le p(t)|u - v|$$

*and*

$$\text{(P)} \qquad \int_0^h p(t)\, dt < \pi/2$$

*hold.* They obtained *existence if*

$$|f(t, x)| \le p(t)|x| + g(t, x),$$

*condition (P) holds, and*

$$\lim_{k\to\infty} \frac{1}{k} \int_0^h \sup\{g(t, x) \colon |x| \le k\}\, dt = 0.$$

Actually, condition (P) was shown to imply uniqueness for the problem

$$\text{(C)} \qquad |dx/dt| \leq p(t)|x|, \quad x_i(t_i) = 0 \text{ for } i = 1, \cdots, n.$$

This uniqueness assertion was obtained as a consequence of the following geometrical lemma: *if* $S = \{x \colon |x| = 1\}$*;* $\rho(x,y)$ *is the spherical distance between* $x, y \in S$*, and* $x^i = ({x_1}^i, \cdots, {x_n}^i) \in S$ *with* ${x_i}^i = 0$ *for* $i = 1, \cdots, n$*, then* $\sum \rho(x^i, x^{i+1}) \geq \pi/2$*.*

The existence was proved using an auxiliary differential inclusion, and an example showed that the constant $\pi/2$ in condition (P) was optimal.

### 7.3. The case of differential inclusions (1965)

This paper [59] starts as follows:

> The role played in the existence problems of the theory of differential equations by the general topological and functional fixed point theorems of various types is well known. Theorems of Banach, Schauder, Leray and Schauder, Tikhonov and others are frequently used in the proofs of the existence of solutions of initial Cauchy problems, boundary value problems, general linear problems and, in particular, in the proofs of the existence of periodic solutions.

In that paper, Lasota and Opial have shown that replacing Schauder's fixed point theorem by its Ky Fan's extension to multi-valued mappings allowed the study of some boundary value problems for differential inclusions

$$x' \in F(t,x)$$

when $F$ takes values in non-empty closed convex subsets of $\mathbb{R}^n$. This is a pioneering paper in set-valued analysis.

### 7.4. Existence conditions in terms of multi-valued mappings (1966)

In paper [60], Lasota and Opial proved the *existence of a unique solution to* the boundary value problem associated to the first order system

$$x' = f(t,x), \quad L[x] = r,$$

where $L \in \mathcal{L}(C(\Delta, \mathbb{R}^n), \mathbb{R}^n)$, *when there exists a Carathéodory upper semi continuous map* $F \colon \Delta \times \mathbb{R}^n \to cf(\mathbb{R}^n)$ *(the closed convex subsets of* $\mathbb{R}^n$*) such that*

$$f(t,x) - f(t,y) \in F(t, x-y)$$

*and* $x \equiv 0$ *is the unique absolutely continuous solution of the differential inclusion*

$$x' \in F(t,x), \quad L[x] = 0.$$

The proof uses Schauder's theorem on the invariance of domain under completely continuous perturbations of identity.

An interesting special case is that of

$$|f(t,x) - f(t,y)| \leq \omega(t, |x-y|)$$

and the problem

$$|x'| \leq \omega(t, |x|), \quad L[x] = 0,$$

only has the trivial solution, and its consequence: *if*

$$|f(t,x) - f(t,y)| \leq \varphi(t)|x-y|, \quad \textit{with} \int_0^b \varphi(t)\, dt < \pi,$$

*then for any* $\lambda > 0$ *the problem*

$$x' = f(t,x), \quad x(a) + \lambda x(b) = r,$$

*has a unique solution.*

Those results will be generalized by Stanisław Kasprzyk and Józef Myjak [29], Lasota and Shui-Nee Chow [13], Lasota [47, 49], and Krakowiak [31].

## 8. Shooting methods for second order differential equations

### 8.1. 'Uniqueness implies existence' theorem (1967)

In paper [63], Lasota and Opial studied equation

$$x'' = f(t, x, x')$$

under the assumption of global existence and uniqueness for the Cauchy problem on $[a, b]$, and proved the following 'uniqueness implies existence result'.

THEOREM 8.1. *Assume that for any pairs* $(t_1, r_1)$ *and* $(t_2, r_2)$ *with* $a < t_1 < t_2 < b$ *and* $r_1, r_2 \in \mathbb{R}$ *the boundary value problem*

$$x'' = f(t, x, x'), \quad x(t_1) = r_1, \quad x(t_2) = r_2,$$

*has at most one solution. Then, for each such pair, a solution exists.*

They also have observed that the result does not hold for the closed interval $[a, b]$. More general two-point boundary conditions can be considered. The proof uses shooting arguments. This seminal paper has inspired generalizations to differential equations of order 3 by Lloyd Jackson and Keith Schrader [26] and of order $n$ by Philip Hartman [20].

### 8.2. Sturm–Liouville boundary conditions (1968–1969)

In paper [50], Lasota and Marian Luczyński have considered the nonlinear Sturm–Liouville boundary value problem

$$x'' = f(t, x, x'), \quad \alpha x(c) + \beta x'(c) = p, \quad \gamma x(d) + \delta x'(d) = q, \tag{8.1}$$

where $\alpha\delta - \beta\gamma \neq 0$, when $f$ *is such that the local Cauchy problem has a unique solution, and such that problem* (8.1) *has at most one solution for any* $a < c < d < b$ *and* $p, q \in \mathbb{R}$. The authors have proved that, *under those conditions, there is at most one solution for the problem*

$$x'' = f(t, x, x'), \quad x(c) = p, \quad x(d) = q. \tag{8.2}$$

Furthermore, *if all solutions exists on* $[a, b]$, *problem* (8.2) *has exactly one solution.*

In [51] the same authors have proved that *problem*

$$x'' = f(t, x, x'), \quad x(\alpha) = p, \quad x(\beta) - \delta x'(\beta)/h = q,$$

*has a unique solution when all solutions exist on* $[a, b]$*,*

$$|f_x| \leq M, \quad f_{x'} \leq K, \quad b - a \leq \int_{-h}^{+\infty} \frac{du}{u^2 + K|u| + M},$$

*and* $a \leq \alpha < \beta \leq b$*,* $p, q \in \mathbb{R}$*,* $\delta = 0$ *or* $1$*.*

## 8.3. Unification through 'condition C' (1970)

In [45], Lasota has generalized and unified the 'uniqueness implies existence' results above for (8.1), by introducing the following *Condition C:*

1. $f$ *is continuous on* $(a, b) \times \mathbb{R}^2$*.*
2. *For every* $t_0 \in (a, b)$ *and* $p, q \in \mathbb{R}$*, the Cauchy problem* $x(t_0) = p$*,* $x'(t_0) = r$ *has a unique solution defined over* $(a, b)$*.*

Problem (8.1) is called *globally unique* if is has at most one solution for any $a < c < d < b$ and any $p, q \in \mathbb{R}$ and *globally solvable* if it has at least one solution for any $a < c < d < b$ and any $p, q \in \mathbb{R}$. Then the following result was proved.

THEOREM 8.2. *If* $f$ *satisfies Conditon* $C$ *and problem*

$$x'' = f(t, x, x'), \quad x(c) = p, \quad x(d) = q,$$

*is globally unique, then it is globally solvable.*

Applications were given, based upon earlier results of Lees and Levin.

## 9. Conclusions

The main features of Lasota's papers on the functional analysis approach to boundary value problems are characterized by a careful choice of underlying function spaces to obtain maximal generality, an abundant use of Schauder's linearization followed by Schauder's fixed point theorem, an elegant use of various types of inequalities, a special care for getting sharp existence and/or uniqueness conditions, and a clever use of differential inclusions to state existence conditions for differential equations.

Other results of Lasota on boundary value problems are based upon shooting method, Pontryagin's maximum principle, Brouwer's invariance of domain theorem, and Wazewski's method.

Several papers of Lasota have been described or quoted in a large number of monographs on differential equations and nonlinear analysis, like the ones of Bailey, Shampine, Waltman [1], Bernfeld, Lakshmikantham [4], Browder [8], Gaines, Mawhin [19], Kamenskii, Obukhovskii, Zecca [27], Piccinini, Stampacchia, Vidossich [85], Reissig, Sansone, Conti [86] and Rouche, Mawhin [87].

Lasota's contributions to the methods of functional analysis in nonlinear boundary value problems impress by their originality, number and elegance. They fully belong to the rich functional analytic and topological tradition of the Polish mathematical school. They have inspired many further contributions in Poland and abroad, and will continue to do so. They reflect the nice personality of their author.

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