Annales Mathematicae Silesianae **27** (2013), 59–66
Prace Naukowe Uniwersytetu Śląskiego nr 3106, Katowice

ON ESTIMATES FOR THE BESSEL TRANSFORM

Ahmed Abouelaz, Radouan Daher, Mohamed El Hamma

Abstract. Using a Bessel translation operator, we obtain a generalization of Theorem 2.2 in [3] for the Bessel transform for functions satisfying the (ψ, δ, β)-Bessel Lipschitz condition in the space $L_{2,\alpha}(\mathbb{R}^+)$.

1. Introduction and preliminaries

Integral transforms and their inverses (e.g., the Bessel transform) are widely used to solve various problems in calculus, mechanics, mathematical physics, and computational mathematics (see, e.g., [6, 7, 9, 10]).

Let

$$\mathrm{B} = \frac{d^2}{dt^2} + \frac{(2\alpha+1)}{t}\frac{d}{dt},$$

be the Bessel differential operator. For $\alpha > -\frac{1}{2}$, we introduce the Bessel normalized function of the first kind j_α defined by

$$j_\alpha(x) = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n (x/2)^{2n}}{n!\Gamma(n+\alpha+1)},$$

Received: 25.11.2012. Revised: 22.07.2013.
(2010) Mathematics Subject Classification: 42A38.
Key words and phrases: Bessel operator, Bessel transform, Bessel translation operator.

where $\Gamma(x)$ is the gamma-function (see [5]). The function $y = j_\alpha(x)$ satisfies the differential equation

$$\mathrm{B}y + y = 0$$

with the initial conditions $y(0) = 1$ and $y'(0) = 0$. The function $j_\alpha(x)$ is infinitely differentiable and even.

LEMMA 1.1. *The following inequalities are valid for Bessel function* j_α
(1) $|j_\alpha(x)| \leq 1$
(2) $1 - j_\alpha(x) = O(x^2)$, $0 \leq x \leq 1$.

PROOF. (See [1]) □

LEMMA 1.2. *The following inequality is true*

$$|1 - j_\alpha(x)| \geq c,$$

with $x \geq 1$, *where* $c > 0$ *is a certain constant.*

PROOF. The asymptotic formulas for the Bessel function imply that $j_\alpha(x) \to 0$ as $x \to \infty$. Consequently, a number $x_0 > 0$ exists such that with $x \geq x_0$ the inequality $|j_\alpha(x)| \leq \frac{1}{2}$ is true. Let $m = \min_{x \in [1, x_0]} |1 - j_\alpha(x)|$. With $x \geq 1$ we get the inequality

$$|1 - j_\alpha(x)| \geq c,$$

where $c = \min(\frac{1}{2}, m)$. □

Assume that $\mathrm{L}_{2,\alpha}(\mathbb{R}^+)$, $\alpha > -\frac{1}{2}$, is the Hilbert space of measurable functions $f(x)$ on $\mathbb{R}^+$ with the finite norm

$$\|f\| = \|f\|_{2,\alpha} = \left(\int_0^\infty |f(t)|^2 t^{2\alpha+1} dt \right)^{1/2}.$$

It is well known that the Bessel transform of a function $f \in \mathrm{L}_{2,\alpha}(\mathbb{R}^+)$ is defined (see [4, 5, 8]) by the formula

$$\widehat{f}(\lambda) = \int_0^\infty f(t) j_\alpha(\lambda t) t^{2\alpha+1} dt, \quad \lambda \in \mathbb{R}^+.$$

The inverse Bessel transform is given by the formula

$$f(t) = (2^\alpha \Gamma(\alpha+1))^{-2} \int_0^\infty \widehat{f}(\lambda) j_\alpha(\lambda t) \lambda^{2\alpha+1} d\lambda.$$

THEOREM 1.3 ([4]). *If $f \in \mathrm{L}_{2,\alpha}(\mathbb{R}^+)$ then we have the Parseval's equality*

$$\|\widehat{f}\| = (2^\alpha \Gamma(\alpha+1))\|f\|.$$

In $\mathrm{L}_{2,\alpha}(\mathbb{R}^+)$, consider the Bessel translation operator T_h

$$\mathrm{T}_h f(t) = c_\alpha \int_0^\pi f(\sqrt{t^2+h^2-2th\cos\varphi}) \sin^{2\alpha}\varphi d\varphi,$$

where

$$c_\alpha = \left(\int_0^\pi \sin^{2\alpha}\varphi d\varphi\right)^{-1} = \frac{\Gamma(\alpha+1)}{\Gamma(1/2)\Gamma(\alpha+\frac{1}{2})}.$$

It is easy to see that

$$\mathrm{T}_0 f(x) = f(x).$$

The operator T_h is linear, homogeneous, and continuous. Below are some properties of this operator (see [5]):
(1) $\mathrm{T}_h j_\alpha(\lambda x) = j_\alpha(\lambda h) j_\alpha(\lambda x)$.
(2) T_h is self-adjoint: If $f(x)$ is continuous function such that

$$\int_0^\infty x^{2\alpha+1}|f(x)|dx < \infty$$

and $g(x)$ is continuous and bounded for all $x \geq 0$, then

$$\int_0^\infty (\mathrm{T}_h f(x)) g(x) x^{2\alpha+1} dx = \int_0^\infty f(x)(\mathrm{T}_h g(x)) x^{2\alpha+1} dx.$$

(3) $\mathrm{T}_h f(x) = \mathrm{T}_x f(h)$.
(4) $\|\mathrm{T}_h f - f\| \to 0$ as $h \to 0$.
The following relation connect the Bessel translation operator, in [2], we have

$$\widehat{(\mathrm{T}_h f)}(\lambda) = j_\alpha(\lambda h)\widehat{f}(\lambda). \tag{1.1}$$

For any function $f(x) \in \mathrm{L}_{2,\alpha}(\mathbb{R}^+)$ we define differences of the order m such that $m \in \{1, 2, \ldots\}$ with a step $h > 0$ by

$$\Delta_h^m f(x) = (\mathrm{T}_h - \mathrm{I})^m f(x), \tag{1.2}$$

where I is the unit operator.

LEMMA 1.4. *Let $f \in \mathrm{L}_{2,\alpha}(\mathbb{R}^+)$. Then*

$$\|\Delta_h^m f(x)\|^2 = \frac{1}{(2^\alpha \Gamma(\alpha+1))^2} \int_0^\infty |1 - j_\alpha(\lambda h)|^{2m} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda.$$

PROOF. From formulas (1.1) and (1.2), we have

$$\widehat{(\Delta_h^m f)}(\lambda) = (j_\alpha(\lambda h) - 1)^m \widehat{f}(\lambda).$$

By Parseval's identity, we obtain the result. □

In [3], we have

THEOREM 1.5. *Let $f \in \mathrm{L}_{2,\alpha}(\mathbb{R}^+)$. Then the following are equivalents*
(1) $f \in \mathrm{Lip}(\psi, \alpha, 2)$
(2) $\int_r^\infty |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} s\lambda = O(\psi(r^{-2})$ *as* $h \to +\infty$,
where $\mathrm{Lip}(\psi, \alpha, 2)$ *is the* ψ*-Bessel Lipschitz class.*

The main aim of this paper is to establish a generalization of Theorem 1.5 in the Bessel transform. For this purpose, we use the Bessel translation operator.

2. Main Results

In this section we give the main result of this paper. We need first to define (ψ, δ, β)-Bessel Lipschitz class.

DEFINITION 2.1. *A function $f \in \mathrm{L}_{2,\alpha}(\mathbb{R}^+)$ is said to be in the (ψ, δ, β)-Bessel Lipschitz class, denote by $\mathrm{Lip}^2(\psi, \delta, \beta)$, if*

$$\|\Delta_h^m f(t)\| = O(h^\delta \psi(h^\beta)) \quad \textit{as } h \to 0,$$

where

(1) $\delta > m$, $\beta > 0$ *and* $m \in \{1, 2, \ldots\}$,
(2) ψ *is a continuous increasing function on* $[0, \infty)$,
(3) $\psi(0) = 0$ *and* $\psi(ts) = \psi(t)\psi(s)$ *for all* $t, s \in [0, \infty)$,
(4) *and*

$$\int_0^{1/h} s^{2m-2\delta-1}\psi(s^{-2\beta})ds = O(h^{2\delta-2m}\psi(h^{2\beta})) \quad \textit{as } h \to 0.$$

Theorem 2.2. *Let* $f \in \mathrm{L}_{2,\alpha}(\mathbb{R}^+)$. *Then the following are equivalent*
(1) $f \in \mathrm{Lip}^2(\psi, \delta, \beta)$,
(2) $\int_r^\infty |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O(r^{-2\delta}\psi(r^{-2\beta}))$ *as* $r \to +\infty$.

Proof. (1) $\Longrightarrow$ (2): Assume that $f \in \mathrm{Lip}^2(\psi, \delta, \beta)$. Then

$$\|\Delta_h^m f(t)\| = O(h^\delta \psi(h^\beta)) \quad \text{as } h \to 0.$$

Lemma 1.4 gives

$$\|\Delta_h^m f(x)\|^2 = \frac{1}{(2^\alpha \Gamma(\alpha+1))^2} \int_0^\infty |1 - j_\alpha(\lambda h)|^{2m} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda.$$

If $\lambda \in [\frac{1}{h}, \frac{2}{h}]$ then $\lambda h \geq 1$ and Lemma 1.2 implies that

$$1 \leq \frac{1}{c^{2m}} |1 - j_\alpha(\lambda h)|^{2m}.$$

Then

$$\begin{aligned} \int_{1/h}^{2/h} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda &\leq \frac{1}{c^{2m}} \int_{1/h}^{2/h} |1 - j_\alpha(\lambda h)|^{2m} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \\ &\leq \frac{1}{c^{2m}} \int_0^\infty |1 - j_\alpha(\lambda h)|^{2m} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \end{aligned}$$

and there exists a positive constant C such that

$$\int_{1/h}^{2/h} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \leq C h^{2\delta} \psi(h^{2\beta}).$$

We obtain

$$\int_r^{2r} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \leq C r^{-2\delta} \psi(r^{-2\beta}).$$

So that

$$\begin{aligned}\int_r^{\infty}|\widehat{f}(\lambda)|^2\lambda^{2\alpha+1}d\lambda &= \Big[\int_r^{2r}+\int_{2r}^{4r}+\int_{4r}^{8r}+\ldots\Big]|\widehat{f}(\lambda)|^2\lambda^{2\alpha+1}d\lambda\\ &\leq C\big(r^{-2\delta}\psi(r^{-2\beta})+(2r)^{-2\delta}\psi((2r)^{-2\beta})+\ldots\big)\\ &\leq Cr^{-2\delta}\psi(r^{-2\beta})\big(1+2^{-2\delta}\psi(2^{-2\beta})\\ &\quad +(2^{-2\delta}\psi(2^{-2\beta}))^2+(2^{-2\delta}\psi(2^{-2\beta}))^3+\ldots\big)\\ &\leq CK_{\delta,\beta}r^{-2\delta}\psi(r^{-2\beta}),\end{aligned}$$

where $K_{\delta,\beta}=(1-2^{-2\delta}\psi(2^{-2\beta}))^{-1}$ since $2^{-2\delta}\psi(2^{-2\beta})<1$. This proves that

$$\int_r^{\infty}|\widehat{f}(\lambda)|^2\lambda^{2\alpha+1}d\lambda=O(r^{-2\delta}\psi(r^{-2\beta}))\quad\text{as } r\to+\infty.$$

(2) $\Longrightarrow$ (1): Suppose now that

$$\int_r^{\infty}|\widehat{f}(\lambda)|^2\lambda^{2\alpha+1}d\lambda=O(r^{-2\delta}\psi(r^{-2\beta}))\quad\text{as } r\to+\infty.$$

We have to show that

$$\int_0^{\infty}|1-j_\alpha(\lambda h)|^{2m}|\widehat{f}(\lambda)|^2\lambda^{2\alpha+1}d\lambda=O(h^{2\delta}\psi(h^{2\beta}))\quad\text{as } h\to 0.$$

We write

$$\int_0^{\infty}|1-j_\alpha(\lambda h)|^{2m}|\widehat{f}(\lambda)|^2\lambda^{2\alpha+1}d\lambda=\mathrm{I}_1+\mathrm{I}_2,$$

where

$$\mathrm{I}_1=\int_0^{1/h}|1-j_\alpha(\lambda h)|^{2m}|\widehat{f}(\lambda)|^2\lambda^{2\alpha+1}d\lambda$$

and

$$\mathrm{I}_2=\int_{1/h}^{\infty}|1-j_\alpha(\lambda h)|^{2m}|\widehat{f}(\lambda)|^2\lambda^{2\alpha+1}d\lambda.$$

Firstly, we have from (1) in Lemma 1.1

$$\mathrm{I}_2\leq 4^m\int_{1/h}^{\infty}|\widehat{f}(\lambda)|^2\lambda^{2\alpha+1}d\lambda=O(h^{2\delta}\psi(h^{2\beta})).$$

Set

$$g(x) = \int_x^\infty |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda.$$

From (1) and (2) of Lemma 1.1 and integration by parts, we obtain

$$\begin{aligned}
\mathrm{I}_1 &= \int_0^{1/h} |1 - j_\alpha(\lambda h)|^{2m} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \\
&\leq 2^m \int_0^{1/h} |1 - j_\alpha(\lambda h)|^{m} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \\
&\leq -C_1 h^{2m} \int_0^{1/h} x^{2m} g'(x) dx \\
&\leq -C_1 g(1/h) + 2mC_1 h^{2m} \int_0^{1/h} x^{2m-1} g(x) dx \\
&\leq C_2 h^{2m} \int_0^{1/h} x^{2m-1} x^{-2\delta} \psi(x^{-2\beta}) dx \\
&\leq C_2 h^{2m} \int_0^{1/h} x^{2m-1-2\delta} \psi(x^{-2\beta}) dx \\
&\leq C_3 h^{2m} h^{2\delta-2m} \psi(h^{2\beta}) \\
&\leq C_3 h^{2\delta} \psi(h^{2\beta}),
\end{aligned}$$

where C_1, C_2 and C_3 are a positive constants and this ends the proof. □

References

[1] Abilov V.A., Abilova F.V., *Approximation of functions by Fourier–Bessel sums*, Izv. Vyssh. Uchebn. Zaved. Mat. **8** (2001), 3–9.

[2] Abilov V.A., Abilova F.V., Kerimov M.K., *On estimates for the Fourier–Bessel Integral transform in the* $L_2(\mathbb{R}_+)$, Comp. Math. Math. Phys. **49** (2009), no. 7, 1103–1110.

[3] El Hamma M., Daher R., *Generalization of Titchmarsh's theorem for the Bessel transform*, Rom. J. Math. Comput. Sci. **2** (2012), no. 2, 17–22.

[4] Kipriyanov I.A., *Singular Elliptic Boundary Value Problems*, Nauka, Moscow, 1997 (in Russian).

[5] Levitan B.M., *Expansion in Fourier series and integrals over Bessel functions*, Uspekhi Mat. Nauk **6** (1951), no. 2, 102–143.

[6] Sveshnikov A.G., Bogolyubov A.N., Kravtsov V.V., *Lectures on Mathematical Physics*, Nauka, Moscow, 2004 (in Russian).

[7] Titchmarsh E.C., *Introduction to the Theory of Fourier Integrals*, Claredon, Oxford, 1948; Komkniga, Moscow, 2005.

[8] Trimèche K., *Transmutation operators and mean-periodic functions associated with differential operators*, Math. Rep. **4** (1988), no. 1, 1–282.
[9] Watson G.N., *Treatise on the Theory of Bessel Functions*, Inostrannaya Literatura, Moscow, 1949; Cambridge University Press, Cambridge, 1952.
[10] Zayed A.L., *Handbook of Function and Generalized Function Transformations*, CRC, Boca Raton, 1996.

Department of Mathematics
Faculty of Sciences Aïn Chock
University of Hassan II
Casablanca, Morocco
e-mail: ah.abouelaz@gmail.com
e-mail: rjdaher024@gmail.com
e-mail: m_elhamma@yahoo.fr