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THE ALIENATION PHENOMENON AND ASSOCIATIVE RATIONAL OPERATIONS

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Abstract. The alienation phenomenon of ring homomorphisms may briefly be described as follows: under some reasonable assumptions, a map f between two rings satisfies the functional equation

$$f(x+y)+f(xy)=f(x)+f(y)+f(x)f(y) \tag{$*$}$$

if and only if f is both additive and multiplicative. Although this fact is surprising for itself it turns out that that kind of alienation has also deeper roots. Namely, observe that the right hand side of equation $(*)$ is of the form $Q(f(x),f(y))$ with the map $Q(u,v)=u+v+uv$ being a special *rational associative operation.* This gives rise to the following question: given an abstract rational associative operation Q does the equation

$$f(x+y)+f(xy)=Q(f(x),f(y))$$

force f to be a ring homomorphism (with the target ring being a field)?

Plainly, in general, that is not the case. Nevertheless, the 2-homogeneity of f happens to be a necessary and sufficient condition for that effect provided that the range of f is large enough.

1. Motivation

The alienation phenomenon discovered and named so by J. Dhombres [2] consists in the statement that a map f between two rings that satisfies the functional equation

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$$(*) \qquad f(x+y)+f(xy)=f(x)+f(y)+f(x)f(y)$$

establishes a homomorphism of the rings in question. In particular, from the equality of sums the equality of summands is derived. Although, with no additional assumptions upon f, that is not the case, among numerous other alienation type results, we have for instance the following one (cf. R. Ger & L. Reich's paper [5]; see also [3] and [4]):

PROPOSITION. *Let X and Y be two unitary rings and let Y be commutative with no zero divisors. Assume that a map $f\colon X \longrightarrow Y$ satisfies equation $(*)$ for all $x, y \in X$. Then either f is a ring homomorphism or f is even, $f(2x) \equiv 0$ on X and f is constant on the cosets forming the elements of the quotient ring $X/2X$.*

This fact is surprising for itself but it turns out that that kind of alienation has also deeper roots. Namely, observe that the right hand side of equation $(*)$ is of the form

$$Q(f(x), f(y))$$

with the map

$$Q(u,v)=u+v+uv$$

being a special *rational associative operation.* This gives rise to the following question: given an abstract rational associative operation Q does the equation

$$\text{(Q)} \qquad f(x+y)+f(xy)=Q(f(x),f(y))$$

force f to be a ring homomorphism (with the target ring being a field)?

As we shall see later on, in general, that is not the case. Nevertheless, the 2-homogeneity of f that kills nonhomomorphic solutions of $(*)$, will prove to be a necessary and sufficient condition for that effect provided that the range of f is large enough. As a matter of fact, we shall show (Theorem 2) that the only associative rational operations Q admitting 2-homogeneous solutions f of equation (Q) with $\operatorname{card} f(X) > 4$ are of the form

$$Q(u,v)=\kappa uv+u+v, \quad \kappa \neq 0.$$

Note also that the associative rational operation associated with $(*)$, i.e.,

$$Q(u,v)=u+v+uv$$

may be represented in the form

$$Q(u,v) = \varphi^{-1}(\varphi(u)\varphi(v))$$

where φ stands for a *homography*

$$\varphi(u) = \frac{1}{u+1}.$$

It seems worthwhile to remark at this place that $Q(u,v) = \varphi^{-1}(\varphi(u)\varphi(v))$ admits also a more convenient representation of that form with another homography, namely $\varphi(u) = u+1,$ because it exhibits two things simultaneously: there are two essentially different multiplicative generators of the same rational associative operation and one of them has a singularity whereas the other has not.

In the present paper we offer some results whose statements will be given in the next section with proofs presented in Section 3.

2. Main results

In what follows X will stand for a unitary ring with unity e and F will denote a real closed field. In particular, F is formally real, i.e a sum of squares of elements of F vanishes if and only if each of these elements is equal to zero. Moreover, for each element a of F either a or $-a$ is a square and $\operatorname{char} F = 0$. A. Cheritat [1] has shown that any nontrivial associative rational operation Q from a suitable subdomain of $F \times F$ into F admits a representation of the form

$$Q(u,v) = \varphi^{-1}\left(\frac{\varphi(u)+\varphi(v)}{1+\omega\varphi(u)\varphi(v)}\right)$$

with some constant $\omega \in F$ and with a homography

$$\varphi(u) = \frac{au+b}{cu+d} \quad \text{such that} \quad ad - bc \neq 0.$$

It is not hard to check that the following forms of an associative rational operation Q spoken of are the only possible ones:

(A) $$Q(u,v) = \varphi^{-1}(\varphi(u)+\varphi(v))$$

or

$$Q(u,v) = \varphi^{-1}(\varphi(u)\varphi(v)) \tag{M}$$

or

$$Q(u,v) = \varphi^{-1}\left(\frac{\varphi(u)+\varphi(v)}{1-\varphi(u)\varphi(v)}\right), \tag{T}$$

with a homography

$$\varphi(u) = \frac{au+b}{cu+d} \quad \text{such that} \quad ad-bc \neq 0\,. \tag{G}$$

The homography φ is then called a generator of the operation Q which, *a fortiori* is termed additively, multiplicatively or tangentially generated provided that case (A), (M) or (T) does occur, respectively.

THEOREM 1. *Given a rational associative operation Q assume that a map $f\colon X \longrightarrow F$ satisfies equation* (Q) *for all $x, y \in X$ such that the pair $(f(x), f(y))$ falls into the domain of Q and* $\operatorname{card} f(X) > 4$. *Then there exist constants λ, μ, ν and σ in F such that*

$$\begin{aligned}\sigma\left(f(x+y)+f(xy)\right)f(x)f(y)&\\ = \lambda\left(f(x+y)-f(x)-f(y)\right)&+\mu f(xy)+\nu f(x)f(y).\end{aligned}$$

More precisely, if φ given by (G) *stands for the generator of Q, then*

(i) $a \neq 0 = b \neq d$, $\lambda = \mu = 1$, $\nu = -\frac{2c}{d}$ *and* $\sigma = \left(\frac{c}{d}\right)^2$ *provided that Q is additively generated;*

(ii) $b = d \neq 0$, $\lambda = \mu = 1$, $\nu = -\frac{a+c}{b}$ *and* $\sigma = \frac{ac}{b^2}$ *provided that Q is multiplicatively generated;*

(iii) $a \neq 0 = b \neq d$, $\lambda = \mu = d^2$, $\nu = -2cd$ *and* $\sigma = a^2 + c^2$ *provided that Q is tangentially generated.*

THEOREM 2. *Given a rational associative operation Q assume that a map $f\colon X \longrightarrow F$ satisfies equation* (Q) *for all $x, y \in X$ such that the pair $(f(x), f(y))$ falls into the domain of Q and* $\operatorname{card} f(X) > 4$. *Then f is 2-homogeneous, i.e.*

$$f(2x) = 2f(x) \quad \textit{for every} \quad x \in X,$$

if and only if

$$Q(u,v) = \kappa uv + u + v, \quad u,v \in X,\ \kappa \neq 0.$$

If that is the case, then f is additive and κf is multiplicative; moreover, Q is multiplicatively generated by a generator φ given by the formula

$$\varphi(u) = \kappa u + 1,\ u \in F.$$

3. Proofs

We begin with the following

Lemma. *Given elements λ, μ, ν and σ from the field F assume that a nonzero map $f\colon X \longrightarrow F$ satisfies equation*

$$\sigma\left(f(x+y) + f(xy)\right) f(x)f(y) \\ = \lambda\left(f(x+y) - f(x) - f(y)\right) + \mu f(xy) + \nu f(x)f(y)$$

for all x, y from X. If f is $2-$homogeneous, then $\sigma = 0$.

Proof. Fix arbitrarily x, y from X and put $2x$ and $2y$ in place of x and y, respectively, in the equation considered and apply the 2-homogeneity of f to get

$$8\sigma\left(f(x+y) + 2f(xy)\right) f(x)f(y) \\ = 2\lambda\left(f(x+y) - f(x) - f(y)\right) + 4\mu f(xy) + 4\nu f(x)f(y),$$

whence due to the fact that $\mathrm{char} F = 0$ we obtain

$$4\sigma\left(f(x+y) + 2f(xy)\right) f(x)f(y) \\ = \lambda\left(f(x+y) - f(x) - f(y)\right) + 2\mu f(xy) + 2\nu f(x)f(y).$$

Subtracting the original equation side by side from the latter one we infer that

$$(**) \qquad \sigma\left(3f(x+y)f(x)f(y) + 7f(xy)f(x)f(y)\right) = \mu f(xy) + \nu f(x)f(y).$$

Replacing here again x and y by $2x$ and $2y$, respectively, and subtracting $(**)$ side by side from the resulting equation we derive the relationship

$$\sigma\left(f(x+y)f(x)f(y)+7f(xy)f(x)f(y)\right)=0$$

valid for all $x,y\in X$. Applying the duplication of arguments once again and subtracting the resulted equation from the latter one side by side we conclude that

$$\sigma f(xy)f(x)f(y)=0$$

for all $x,y\in X$. If σ were different from 0, we would obtain the equality

$$f(xy)f(x)f(y)=0 \quad \text{for all} \quad x,y\in X.$$

Now, fix arbitrarily an $x\in X$ such that $f(x)\neq 0$. Then, for every y from X we have $f(xy)f(y)=0$; in particular, setting here $y=x$ we get $f\left(x^2\right)=0$. Therefore, for every $x\in X$ either $f(x)=0$ or $f\left(x^2\right)=0$. In particular, $f(e)=0$ where e stands for the unit element of the ring X. Hence, on setting $y=e$ in $(**)$ we obtain the equality $\mu f(x)=0$ satisfied for every $x\in X$. Since $f\neq 0$ it forces μ to vanish which allows to rewrite $(**)$ in the form

$$f(x)f(y)\neq 0 \quad \text{implies} \quad 3\sigma f(x+y)+7\sigma f(xy)=\nu,$$

the implication being valid for all x and y from X. Put here $y=x$ to get

$$f(x)\neq 0 \quad \text{implies} \quad 6\sigma f(x)+7\sigma f\left(x^2\right)=6\sigma f(x)=\nu$$

which states that f assumes two values only including $0=f(e)$. Since $f\neq 0$ we must have $f(X)=\{0,c\}$ with $c=f(x_0)\neq 0$ for some $x_0\in X$. This is not possible because then $f(2x_0)=2f(x_0)=2c\notin\{0,c\}$. This condratiction shows that actually we have to have $\sigma=0$, which completes the proof. □

To proceed, for brevity, put

$$\operatorname{dom}\varphi := \begin{cases} F & \text{whenever } c=0 \\ F\setminus\{-\frac{d}{c}\} & \text{otherwise,} \end{cases}$$

and

$$\operatorname{dom}\varphi^{-1} := \begin{cases} F & \text{whenever } c=0 \\ F\setminus\{\frac{a}{c}\} & \text{otherwise.} \end{cases}$$

PROOF OF THEOREM 1 IN THE ADDITIVE CASE (A). The functional equation (Q) assumes the form

$$(\mathrm{Q_A}) \qquad f(x+y)+f(xy)=\varphi^{-1}\left(\varphi(f(x))+\varphi(f(y))\right)$$

for all $x, y \in X$ such that $f(x)$ and $f(y)$ belong to $\operatorname{dom}\varphi$ and $\varphi(f(x))+\varphi(f(y))$ belongs to $\operatorname{dom}\varphi^{-1}$.

In the case where $c=0$ one has $a \neq 0 \neq d$ and equation $(\mathrm{Q_A})$ reduces to

$$f(x+y)+f(xy)=f(x)+f(y)+\frac{b}{a}$$

for all $x, y \in F$, which forces b to vanish (take $x=y=0$). Consequently, we arrive at

$$(f(x+y)-f(x)-f(y))+f(xy)=0$$

getting (i) with $\nu=\sigma=0$.

In what follows we shall assume that $c \neq 0$. Observe first that, by a direct calculation, representation (A) with the generator φ given by (G) leads to

$$Q(u,v)=\frac{c(2ad-bc)uv+ad^2(u+v)+bd^2}{-ac^2uv-bc^2(u+v)+d(ad-2bc)}$$

which defines an associative rational operation on the set

$$\operatorname{dom} Q := \left\{(u,v) \in F \times F : \, ac^2uv+bc^2(u+v)=d(ad-2bc)\right\}.$$

In contrast to the map

$$(u,v) \longmapsto \varphi^{-1}(\varphi(u)+\varphi(v))$$

for which the arguments $(-\frac{d}{c}, v)$ and $(u, -\frac{d}{c})$ are off its domain, these singularities are fortunately apparent for the map Q itself; we have simply

$$(1) \qquad Q\left(u,-\frac{d}{c}\right)=Q\left(-\frac{d}{c},v\right)=-\frac{d}{c} \quad \text{for all} \quad u,v \in F \setminus \left\{-\frac{d}{c}\right\}$$

(a direct calculation). Note that $\alpha := f(0) \neq -\frac{d}{c}$ (i.e. $\alpha \in \operatorname{dom}\varphi$) since, otherwise, we would have

$$f(x)-\frac{d}{c}=f(x)+f(0)=Q(f(x),f(0))=Q\left(f(x),-\frac{d}{c}\right)=-\frac{d}{c}$$

provided that $f(x) \neq -\frac{d}{c}$; thus $f(x) \in \left\{0, -\frac{d}{c}\right\}$ for all $x \in X$, contradicting the assumption that f has at least 5 values.

Therefore, with $\gamma := \varphi(\alpha)$ one has

$$f(x) \neq -\frac{d}{c} \quad \text{implies} \quad f(x) + \alpha = \varphi^{-1}(\varphi(f(x)) + \gamma)$$

whenever $\varphi(f(x)) + \gamma \in \operatorname{dom} \varphi^{-1}$, i.e. $\varphi(f(x)) \neq \frac{a}{c} - \gamma$. This forces γ to vanish. In fact, suppose the contrary; then $\frac{a}{c} - \gamma \in \operatorname{dom} \varphi^{-1}$ and we would have

$$f(x) + \alpha = \varphi^{-1}(\varphi(f(x)) + \gamma) \quad \text{whenever} \quad f(x) \notin \left\{-\frac{d}{c}, \varphi^{-1}\left(\frac{a}{c} - \gamma\right)\right\},$$

whence

$$\frac{a(f(x) + \alpha) + b}{c(f(x) + \alpha) + d} = \frac{af(x) + b}{cf(x) + d} + \gamma \quad \text{whenever} \quad f(x) \notin \left\{-\frac{d}{c}, \varphi^{-1}\left(\frac{a}{c} - \gamma\right)\right\}.$$

Thus, for every $x \in X$ we have either $f(x) = -\frac{d}{c}$ or $f(x) = \varphi^{-1}\left(\frac{a}{c} - \gamma\right)$ or

$$-\gamma c^2 f(x)^2 + (ad - bc)\alpha - \gamma(2cd + c^2\alpha)f(x) - \gamma(cd\alpha + d^2) = 0$$

which forces the image $f(X)$ to be of the cardinality less or equal 4, contradicting the assumption. Thus $\gamma = 0$ and, *a fortiori*, by repeating the same reasoning (bearing $\gamma = 0$ in mind) we arrive at the equality $(ad - bc)\alpha = 0$ forcing α to be 0. Consequently, in view of the inequality $\alpha := f(0) \neq -\frac{d}{c}$, we infer that $d \neq 0$.

The equality $u = Q(u, 0)$ valid for sufficiently many $u \in f(X)$ forces b to vanish as well. Now, the map Q considered assumes the form

$$Q(u, v) = \frac{2cduv + d^2(u + v)}{-c^2uv + d^2} \quad \text{for all} \quad u, v \in F \quad \text{such that} \quad uv \neq \left(\frac{d}{c}\right)^2.$$

Equation (Q) leads now to

$$\begin{aligned}\left(\frac{c}{d}\right)^2 &(f(x + y) + f(xy))\, f(x)f(y) \\ &= f(x + y) - f(x) - f(y) + f(xy) - 2\frac{c}{d}f(x)f(y)\end{aligned}$$

which states nothing else but (i). □

Proof of Theorem 1 in the multiplicative case (M). The functional equation (Q) assumes the form

$$(\mathrm{Q_M}) \qquad f(x+y)+f(xy)=\varphi^{-1}\left(\varphi(f(x))\varphi(f(y))\right)$$

for all $x, y \in X$ such that $f(x)$ and $f(y)$ belong to $\operatorname{dom}\varphi$ and $\varphi(f(x))\varphi(f(y))$ belongs to $\operatorname{dom}\varphi^{-1}$. In the case where $c = 0$ one has $a \neq 0 \neq d$ and equation $(\mathrm{Q_M})$ reduces to

$$f(x+y)+f(xy)=\frac{a}{d}f(x)f(y)+\frac{b}{d}(f(x)+f(y))+\frac{b}{a}\left(\frac{b}{d}-1\right)$$

for all $x, y \in F$, which applied for $y = 0$ gives

$$\left(1-\frac{a\alpha}{d}-\frac{b}{d}\right)f(x)=\left(\frac{b}{d}-1\right)\left(\alpha+\frac{b}{a}\right), \quad x \in F,$$

with $\alpha := f(0)$. Since f is nonconstant we derive easily that $b = d$ getting

$$f(x+y)+f(xy)=\frac{a}{b}f(x)f(y)+f(x)+f(y) \quad \text{for all} \quad x, y \in X$$

which states nothing else but (ii) with $c = 0$.

In what follows we shall assume that $c \neq 0$. Similarly as in the previous case we show that the singularities at the points $(-\frac{d}{c}, v)$ and $(u, -\frac{d}{c})$ are apparent only, because having now

$$(2) \qquad Q(u,v)=\frac{(a^2d-bc^2)uv+bd(a-c)(u+v)+bd(b-d)}{ac(c-a)uv+ac(d-b)(u+v)+ad^2-b^2c}$$

we get (1) as well.

Now, setting $\gamma := \varphi(\alpha)$ and applying $(\mathrm{Q_M})$ for $y = 0$ we infer that

$$(3) \qquad f(x)+\alpha=Q(f(x),\alpha) \quad \text{for sufficiently many} \quad x \in X.$$

Consequently,

$$f(x) \neq -\frac{d}{c} \quad \text{implies} \quad f(x)+\alpha=\varphi^{-1}(\gamma\varphi(f(x))$$

whence

$$\frac{a(f(x)+\alpha)+b}{c(f(x)+\alpha)+d}=\gamma\frac{af(x)+b}{cf(x)+d} \quad \text{whenever} \quad f(x) \neq -\frac{d}{c}.$$

Thus,

$$\left(f(x)+\frac{d}{c}\right)\left(ac(1-\gamma)f(x)^2+(ad+ac\alpha+bc-\gamma bc-\gamma ac\alpha-\gamma ad)\,f(x)\right.$$
$$\left.+\,(ad\alpha+bd-\gamma bd-\gamma bc\alpha)\right)=0$$

for sufficiently many $x\in X$, forcing the equality $\gamma=1$ and, *a fortiori*, $(ad-bc)\alpha=0$ because of the assumption that $\operatorname{card} f(X)>4$. Hence

$$f(0)=\alpha=0 \quad \text{as well as} \quad f(x)=Q(f(x),0)=\frac{bd(a-c)f(x)+bd(b-d)}{ac(d-b)f(x)+ad^2-b^2c},$$

by means of (3) and (2). This, in turn, implies that $b=d$ whence finally

$$Q(u,v)=\frac{b(a+c)uv+b^2(u+v)}{-acuv+b^2} \quad \text{for all} \quad u,v\in F \quad \text{such that} \quad acuv\neq b^2.$$

In particular, we have obviously $b\neq 0$.

Equation (Q) leads now to the equality

$$\frac{ac}{b^2}\left(f(x+y)+f(xy)\right)f(x)f(y)$$
$$=f(x+y)-f(x)-f(y)+f(xy)-\frac{a+c}{b}f(x)f(y)$$

valid for all $x,y\in X$, which states nothing else but (ii). □

Proof of Theorem 1 in the tangential case (T). The equation (Q) assumes the form

$$(\mathrm{Q_T}) \qquad f(x+y)+f(xy)=\varphi^{-1}\left(\frac{\varphi(f(x))+\varphi(f(y))}{1-\varphi(f(x))\varphi(f(y))}\right),$$

for all $x,y\in X$ such that $f(x)$ and $f(y)$ belong to $\operatorname{dom}\varphi$ and $(\varphi(f(x))+\varphi(f(y)))/(1-\varphi(f(x))\varphi(f(y))$ belongs to $\operatorname{dom}\varphi^{-1}$. We omit somewhat tedious but basically simple calculations needed to obtain the explicite form of the corresponding two place function Q; it reads as follows:

$$(4) \qquad Q(u,v)=\frac{(2acd-bc^2+a^2b)uv+a(b^2+d^2)(u+v)+b(b^2+d^2)}{-a(a^2+c^2)uv-b(a^2+c^2)(u+v)+ad^2-ab^2-2bcd}.$$

With the aid of the notation applied in the previous cases, from equation $(\mathrm{Q_T})$ we derive the equality

$$(5)\quad (a(f(x)+\alpha)+b)\left(1-\gamma\frac{af(x)+b}{cf(x)+d}\right) = \frac{af(x)+b}{cf(x)+d}\,(c(f(x)+\alpha)+d)+\gamma\,(c(f(x)+\alpha)+d)$$

valid for all $x \in X$ such that the denominator in question does not vanish (like in the previous cases the singularities are apparent). The latter equation, after further natural rearrangements, leads easily to the conclusion stating that the image of X under f is contained in the set of zeros of a polynomial of at most second degree with the leading coefficient equal to $\gamma(a^2+c^2)$. Since the field F in question is formally real and the coefficients a and c cannot vanish simultaneously we infer that $\gamma = 0$. This, in turn, reduces formula (5) to the following one:

$$a(f(x)+\alpha)+b = \frac{af(x)+b}{cf(x)+d}\,(c(f(x)+\alpha)+d)$$

from which we derive the vanishing of the constant term:

$$(ad-bc)\alpha = 0$$

which forces α to vanish. In this situation d cannot vanish since, otherwise, (cf. (4)) we would have

$$Q(u,v) = \frac{(-bc^2+a^2b)uv+ab^2(u+v)+b^3}{-a(a^2+c^2)uv-b(a^2+c^2)(u+v)-ab^2}$$

whence for sufficiently many $x \in X$ we would get

$$f(x) = Q(f(x),0) = \frac{ab^2f(x)+b^3}{-b(a^2+c^2)f(x)-ab^2}$$

forcing the coefficient $-b(a^2+c^2)$ to vanish which implies $b = 0 = d$ and contradicts the basic assumption that $ad-bc \neq 0$. Therefore,

$$0 = \gamma = \varphi(\alpha) = \varphi(0) = b,$$

which implies that $a \neq 0$ and (cf. (5)):

$$Q(u,v) = \frac{2cd\,uv + d^2(u+v)}{-(a^2+c^2)uv + d^2}$$

whence for every $x, y \in X$ we obtain

$$f(x+y) + f(xy) = Q(f(x), f(y)) = \frac{2cd\,f(x)f(y) + d^2(f(x)+f(y))}{-(a^2+c^2)f(x)f(y) + d^2}$$

and, finally,

$$\begin{aligned}(a^2+c^2)(f(x+y)+f(xy))f(x)f(y) - d^2 f(x+y) - d^2 f(xy)&\\ = -2cdf(x)f(y) - d^2 f(x) - d^2 f(y).&\end{aligned}$$

The latter formula states nothing else but (iii) and the proof has been completed. □

Proof of Theorem 2. Assuming that Q has the form spoken of we get immediately that the map $g := \kappa f$ yields a 2-homogeneous solution to $(*)$ which on account of the Proposition forces g to be a homomorphism, i.e. g is both additive and multiplicative. Therefore, f itself is additive and κf is multiplicative, as claimed.

To prove the converse, apply Theorem 1 and the Lemma with the notation therein to state that case (iii) is excluded because we would have to have $a^2 + c^2 = \sigma = 0$ forcing a and c to vanish which contradicts the inequality $ad - bc \neq 0$. Assume (i). Then $c = 0 = \nu$, $\lambda = \mu = 1$ and $a \neq 0 = b \neq d$ which gives

$$f(x+y) - f(x) - f(y) + f(xy) = 0, \quad x, y \in X.$$

Setting here $y = e$ we infer that $f(x+e) = f(e), x \in X$, which states that f is constant, contradicting the assumption about the range of f. Thus, the only possible case is the remaining one, i.e. (ii). The equality $\sigma = 0$ implies now that either a or c vanishes. In the case where $a = 0$ we are faced to the multiplicative generator φ of the form

$$\varphi(u) = \frac{b}{cu+b} \quad \text{with} \quad b \neq 0 \neq c,$$

which implies the representation

$$Q(u,v) = \kappa uv + u + v, \quad (u,v) \in F \times F, \quad \text{with} \quad \kappa := \frac{c}{b}.$$

Finally, if $a \neq 0 = c$ and $b = d \neq 0$ as well as $\nu = -a/b$, we get

$$\varphi(u) = \kappa u + 1,\ u \in X, \quad \text{where } \kappa := \frac{a}{b} \neq 0,$$

which generates multiplicatively a rational associative operation

$$Q(u,v) = \kappa uv + u + v, \quad (u,v) \in F \times F, \quad \text{with} \quad \kappa := \frac{a}{b} \neq 0,$$

and finishes the proof. □

4. Concluding remarks

The chief point of the proof of Theorem 2 was to solve the functional equation

$$\begin{aligned}\sigma\left(f(x+y) + f(xy)\right) f(x)f(y)& \\ = \lambda\left(f(x+y) - f(x) - f(y)\right) &+ \mu f(xy) + \nu f(x)f(y)\end{aligned}$$

in the class of 2-homogeneous mappings f. It would be desirable to solve this equation with no assumptions whatsoever upon the unknown function f. It seems to be a difficult task. On the other hand such a knowledge would elucidate how far from the alienation phenomen we stay dealing with general associative rational operations occurring at the right hand side of the equation discussed.

In the present paper solutions with no more than four values have been brushed off. However, they do exist even in the very simple case of the unitary ring $X = \mathbb{Z}$ of all the integers and the real closed field $F = \mathbb{R}$ of the reals. For instance, equation $(*)$ corresponding to the canonical associative rational operation $Q(u,v) = uv + u + w$, $u, v \in \mathbb{R}$, admits an even and hence nonhomomorphic solution

$$f(x) = \begin{cases} 0 & \text{for } x \in 2\mathbb{Z}, \\ -1 & \text{for } x \in 2\mathbb{Z} + 1. \end{cases}$$

Although it might be of interest to determine all solutions of that kind, we realize that they should be disregarded while considering conditions implying the alienation phenomenon.

Finally, it should be emphasized that our strong assumptions upon the target field eliminate, for instance, the field of all complex numbers. As a matter of fact, these assumptions are essential merely in the case of tangentially generated associative rational operations. Nevertheless, the problem discussed here remains open for the fields that are not real closed.

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References

[1] Chéritat A., *Fractions rationnelles associatives et corps quadratiques*, Revue des Mathématiques de l'Enseignement Supérieur **109** (1998–1999), 1025–1040.

[2] Dhombres J., *Relations de dépendance entre équations fonctionnelles de Cauchy*, Aequationes Math. **35** (1988), 186–212.

[3] Ger R., *On an equation of ring homomorphisms*, Publ. Math. Debrecen **52** (1998), 397–417.

[4] Ger R., *Ring homomorphisms equation revisited*, Rocznik Nauk.-Dydakt. Prace Mat. **17** (2000), 101–115.

[5] Ger R., Reich L., *A generalized ring homomorphisms equation*, Monatsh. Math. **159** (2000), 225–233.

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