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A REMARK ON PERIODIC ENTIRE FUNCTIONS

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Abstract. Periodicity of an entire function is characterized by the behavior of coefficients of its Maclaurin expansion.

We show that the periodicity of any entire function can be easily characterized by the behavior of the coefficients of its Maclaurin expansion. Namely, we have the following

Theorem 1. *Let* $f\colon \mathbb{C} \to \mathbb{C}$ *be an entire function with the expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{C}. \tag{1}$$

The function f *is periodic of a period* $p \in \mathbb{C}$, $p \neq 0$, *that is*

$$f(z+p) = f(z), \quad z \in \mathbb{C}, \tag{2}$$

if, and only if, the sequence of coefficients $(a_n)_{n=0}^{\infty}$ *satisfies the following infinite system of linear equations*

$$\sum_{n=k+1}^{\infty} a_n \binom{n}{k} p^{n-k} = 0, \quad k \in \mathbb{N} \cup \{0\}. \tag{3}$$

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Proof. Condition (2) holds if, and only if,

$$\sum_{n=0}^{\infty} a_n \left[(z+p)^n - z^n\right] = 0, \quad z \in \mathbb{C},$$

that is if, and only if,

$$\sum_{n=1}^{\infty} a_n \sum_{k=0}^{n-1} \binom{n}{k} p^{n-k} z^k = 0, \quad z \in \mathbb{C}.$$

By changing the order of summation, that is permitted since all infinite series involved are absolutely convergent (see also Remark 4), this condition is equivalent to

$$\sum_{k=0}^{\infty} \left(\sum_{n=k+1}^{\infty} a_n \binom{n}{k} p^{n-k} \right) z^k = 0, \quad z \in \mathbb{C},$$

which holds if, and only if,

$$\sum_{n=k+1}^{\infty} a_n \binom{n}{k} p^{n-k} = 0$$

for every $k \in \mathbb{N} \cup \{0\}$. □

Corollary 2. *The entire function (1) has the period* 1 *if, and only if,*

$$\sum_{n=k+1}^{\infty} a_n \binom{n}{k} = 0, \quad k = 0, 1, 2, \ldots .$$

Example 3. The entire function $f = \exp$ has a period $p = 2\pi i$, and

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbb{C}.$$

From Theorem 1 we obtain

$$\sum_{n=k+1}^{\infty} \binom{n}{k} \frac{(2\pi i)^{n-k}}{n!} = 0, \quad k = 0, 1, 2, \ldots .$$

In particular, we hence get,

$$\sum_{n=1}^{\infty} \frac{(2\pi i)^n}{n!} = \sum_{n=1}^{\infty} \binom{n}{0} \frac{(2\pi i)^n}{n!} = 0,$$

for $k = 0$, and

$$\sum_{n=2}^{\infty} \frac{(2\pi i)^{n-1}}{(n-1)!} = \sum_{n=2}^{\infty} \binom{n}{1} \frac{(2\pi i)^{n-1}}{n!} = 0$$

for $k = 1$, etc.

Remark 4. Of course, under the assumption of Theorem 1, for every $k \in \mathbb{N} \cup \{0\}$, the series $\sum_{n=k+1}^{\infty} a_n \binom{n}{k} p^{n-k}$ is absolutely convergent.

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