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ON A GENERALIZED INFIMAL CONVOLUTION OF SET FUNCTIONS

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Abstract. Having in mind the ideas of J. Moreau, T. Strömberg and Á. Száz, for any function f and g of one power set $\mathcal{P}(X)$ to another $\mathcal{P}(Y)$, we define an other function $(f * g)$ of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ such that

$$(f * g)(A) = \bigcap \{f(U) \cup g(V) : A \subset U \cup V \subset X\}$$

for all $A \subset X$. Thus $(f * g)$ is a generalized infimal convolution of f and g.

We show that if f and g preserve arbitrary unions, then $(f*g)$ also preserves arbitrary unions. Moreover, if F and G are relations on X to Y such that

$$F(x) = f(\{x\}) \quad \text{and} \quad G(x) = g(\{x\})$$

for all $x \in X$, then

$$(f * g)(A) = (F \cap G)[A]$$

for all $A \subset X$.

1. Introduction

If f and g are functions of a partially ordered semigroup U to an infimum complete partially ordered semigroup V, then the function $(f * g)$ defined by

$$(f * g)(x) = \inf\{f(u) + g(v) : x \leq u + v\}$$

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for all $x \in X$ may be called the generalized infimal convolution of f and g according to Moreau [6] and Strömberg [7].

Particular cases of this infimal convolution have already been used by several mathematicians in minimization problems and regularization processes. Moreover, Á. Száz and T. Glavosits [2, 3, 4, 8] have recently shown that they can also be used to naturally prove various generalizations of the Hahn–Banach extension theorems.

In the present paper, we investigate the particular case of the generalized infimal convolution when $U = \mathcal{P}(X)$ and $V = \mathcal{P}(Y)$, for some sets X and Y, with union and inclusion as addition and inequality.

An interesting consequence of our main results states that if F and G are relations on X to Y and

$$f(A) = F[A] \quad \text{and} \quad g(A) = G[A]$$

for all $A \subset X$, then

$$(f * g)(A) = (F \cap G)[A]$$

for all $A \subset X$.

2. Union-preserving set functions

A subset F of a product set $X \times Y$ is called a relation on X to Y. For any $x \in X$ and $A \subset X$, the sets $F(x) = \{y \in Y : (x, y) \in F\}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the images of x and A under F, respectively.

Moreover, the set $D_F = \{x \in X : F(x) \neq \emptyset\}$ is called the domain of F. In particular, if $D_F = X$, then we say that F is a relation of X to Y.

A relation f on X to Y is called a function if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simple write $f(x) = y$ in place of $f(x) = \{y\}$.

In the sequel, we shall mainly be interested in functions of one power set $\mathcal{P}(X)$ to another $\mathcal{P}(Y)$.

Definition 2.1. A function f of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ is called *union-preserving* if

$$f\left(\bigcup \mathcal{A}\right) = \bigcup_{A \in \mathcal{A}} f(A)$$

holds for any family $\mathcal{A}$ of subsets of X.

REMARK 2.2. In this case, we necessarily have

$$f(\emptyset) = f\Big(\bigcup \emptyset\Big) = \bigcup_{A\in\emptyset} f(A) = \emptyset.$$

Moreover, if $A \subset B$, then we also have

$$f(A) \subset f(A) \cup f(B) = f(A \cup B) = f(B).$$

Therefore, f is in particular increasing.

In the above Remark we need only the finite union-preservingness of f.

THEOREM 2.3. *For any function f of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, the following assertions are equivalent:*
(1) *f is union-preserving;*
(2) *$f(A) = \bigcup_{x\in A} f(\{x\})$ for all $A \subset X$;*
(3) *$f(A) = F[A]$ for some relation F on X to Y and all $A \subset X$.*

PROOF. If (1) holds, then we evidently have

$$f(A) = f\Big(\bigcup_{x\in A}\{x\}\Big) = \bigcup_{x\in A} f(\{x\}),$$

for all $A \subset X$, and thus (2) also holds.

Suppose now that (2) holds and define a relation F on X to Y such that

$$F(x) = f(\{x\})$$

for all $x \in X$. Then, by (2) and the corresponding definitions, it is clear that

$$f(A) = \bigcup_{x\in A} f(\{x\}) = \bigcup_{x\in A} F(x) = F[A]$$

for all $A \subset X$. Thus, (3) also holds.

Finally, if (3) holds and $\mathcal{A} \subset \mathcal{P}(X)$, then by a well-known property of relations, it is clear that

$$f\Big(\bigcup \mathcal{A}\Big) = F\Big[\bigcup \mathcal{A}\Big] = \bigcup_{A\in\mathcal{A}} F[A] = \bigcup_{A\in\mathcal{A}} f(A).$$

Therefore, (1) also holds. □

With respect to the above theorem we mention paper [5] of Höhle and Kubiak.

3. A generalized infimal convolution of set functions

Definition 3.1. If f and g are functions of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, then for any $A \subset X$ we define

$$(f * g)(A) = \bigcap \{f(U) \cup g(V) : A \subset U \cup V \subset X\}.$$

Remark 3.2. Thus $(f * g)$ is also a function of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, which may be called a generalized infimal convolution of f and g by [9] and [10]. (See also [6] and [7].)

Theorem 3.3. *If f and g are functions of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, then $(f * g)$ is increasing.*

Proof. Let $A \subset B \subset X$. It is quite obvious, that

$$\{f(U) \cup g(V) : B \subset U \cup V \subset X\} \subset \{f(U) \cup g(V) : A \subset U \cup V \subset X\},$$

therefore

$$(f * g)(A) \subset (f * g)(B). \qquad \square$$

Theorem 3.4. *If f and g are increasing functions of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, then for any $A \subset X$ we have*

$$\begin{aligned}(f * g)(A) &= \bigcap \{f(U) \cup g(V) : A = U \cup V\} \\ &= \bigcap \{f(A \setminus V) \cup g(V) : V \subset A\}.\end{aligned}$$

Proof. We evidently have

$$\{(A \setminus V, V) : V \subset A\} \subset \{(U, V) : A = U \cup V\} \subset \{(U, V) : A \subset U \cup V \subset X\}.$$

Hence, by the corresponding definitions, it is clear that

$$\begin{aligned}(f * g)(A) &= \bigcap \{f(U) \cup g(V) : A \subset U \cup V \subset X\} \\ &\subset \bigcap \{f(U) \cup g(V) : A = U \cup V\} \\ &\subset \bigcap \{f(A \setminus V) \cup g(V) : V \subset A\}.\end{aligned}$$

To prove the converse inclusions, assume now that

$$y \in \bigcap \{ f(A \setminus V) \cup g(V) : V \subset A \}$$

and $U, V \subset X$ such that $A \subset U \cup V$. Define

$$W = A \cap V.$$

Then, because of $A \subset U \cup V$, we have

$$A \setminus W = A \setminus (A \cap V) = A \setminus V \subset U.$$

Hence, since f is increasing, it follows that

$$f(A \setminus W) \subset f(U).$$

Moreover, since $W \subset V$ and g is increasing, we also have

$$g(W) \subset g(V).$$

Now, since $W \subset A$ is also true, we can already see that

$$y \in f(A \setminus W) \cup g(W) \subset f(U) \cup g(V).$$

Hence, it is clear that

$$y \in \bigcap \{ f(U) \cup g(V) : A \subset U \cup V \subset X \} = (f * g)(A).$$

Therefore,

$$\bigcap \{ f(A \setminus V) \cup g(V) : V \subset A \} \subset (f * g)(A),$$

and thus the required equalities are also true. $\square$

Corollary 3.5. *If f and g are increasing functions of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ such that $f(\emptyset) = \emptyset$ and $g(\emptyset) = \emptyset$, then for any $x \in X$ we have*

$$(f * g)(\{x\}) = f(\{x\}) \cap g(\{x\}).$$

Proof. By Theorem 3.4 and the assumptions $f(\emptyset)=\emptyset$ and $g(\emptyset)=\emptyset$ we have

$$\begin{aligned}(f*g)(\{x\}) &= \bigcap\{f(\{x\}\setminus V)\cup g(V) : V\subset\{x\}\}\\ &= (f(\{x\})\cup g(\emptyset))\cap(f(\emptyset)\cup g(\{x\}))\\ &= f(\{x\})\cap g(\{x\}).\end{aligned}$$ □

Remark 3.6. From Theorem 3.4, we can also easily see that

$$(f*g)(\emptyset)=f(\emptyset\setminus\emptyset)\cup g(\emptyset)=f(\emptyset)\cup g(\emptyset)$$

and

$$(f*g)(A)\subset f(A\setminus\emptyset)\cup g(\emptyset)=f(A)\cup g(\emptyset).$$

4. The infimal convolution of union-preserving set functions

Theorem 4.1. *If f and g are union-preserving functions of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, then $(f*g)$ is also union-preserving.*

Proof. By Theorem 2.3, it is enough to show that for any $A\subset X$, we have

$$(f*g)(A)=\bigcup_{a\in A}(f*g)(\{a\}).$$

For this, assume first that $y\in(f*g)(A)$. Then, by Theorem 3.4, we have

$$y\in\bigcap\{f(A\setminus V)\cup g(V) : V\subset A\},$$

and thus $y\in f(A\setminus V)\cup g(V)$ for all $V\subset A$. Define

$$V=\{x\in A : y\in f(\{x\})\}.$$

Then, $V\subset A$ such that for any $x\in A\setminus V$, we have $y\notin f(\{x\})$. Hence, by using Theorem 2.3, we can see that

$$y\notin\bigcup_{x\in A\setminus V}f(\{x\})=f(A\setminus V).$$

Therefore, since $y \in f(A \setminus V) \cup g(V)$, we necessarily have

$$y \in g(V) = \bigcup_{x \in V} g(\{x\}).$$

Thus, there exists $x \in V$ such that $y \in g(\{x\})$. Now, by the definition of V, we can see that $y \in f(\{x\})$. Therefore, by Remark 2.2 and Corollary 3.5 we also have

$$y \in f(\{x\}) \cap g(\{x\}) = (f * g)(\{x\}) \subset \bigcup_{a \in A} (f * g)(\{a\}).$$

Consequently,

$$(f * g)(A) \subset \bigcup_{a \in A} (f * g)(\{a\}).$$

The converse inclusion is quite obvious, because by Theorem 3.3 we have that $(f * g)$ is also increasing. Hence,

$$(f * g)(\{a\}) \subset (f * g)(A)$$

for all $a \in A$, therefore

$$\bigcup_{a \in A} (f * g)(\{a\}) \subset (f * g)(A). \qquad \square$$

Corollary 4.2. *If f and g are union-preserving functions of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, and F and G are relations on X to Y such that*

$$F(x) = f(\{x\}) \quad \textit{and} \quad G(x) = g(\{x\})$$

for all $x \in X$, then for any $A \subset X$ we have

$$(f * g)(A) = (F \cap G)[A].$$

Proof. By Theorem 4.1, Remark 2.2, Corollary 3.5 and the corresponding definitions, we can see that

$$\begin{aligned}(f * g)(A) &= \bigcup_{x \in A} (f * g)(\{x\}) = \bigcup_{x \in A} (f(\{x\}) \cap g(\{x\})) \\ &= \bigcup_{x \in A} (F(x) \cap G(x)) = \bigcup_{x \in A} (F \cap G)(x) = (F \cap G)[A]. \qquad \square\end{aligned}$$

Remark 4.3. Thus, in particular for any $x \in X$ we have

$$(f * g)(\{x\}) = (F \cap G)(x).$$

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