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ALTERING DISTANCES AND FIXED POINT RESULTS FOR TANGENTIAL HYBRID PAIRS OF MAPPINGS

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Abstract. The aim of this paper is to prove a general fixed point result by altering distances for two occasionally weakly compatible (owc) pairs of hybrid mappings and to reduce the study of fixed points of the pairs of mappings satisfying a contractive condition of integral type at the study of fixed points in metric spaces by altering distances satisfying a new type of implicit relations generalizing the result recently obtained by H.K. Pathak and Naseer Shahzad (see Bull. Belg. Math. Soc. Simon Stevin, 16 (2009), 1-12) which is of Gregus type.

1. Introduction, recalls and definitions

The aim of this paper is to establish some new common fixed point results in the setting of metric spaces by using altering distances and a new class of implicit relations. In particular, our results extend the results recently obtained by Pathak and Shahzad in [25] for four single valued maps to the more general case of two hybrid pairs of maps.

We start our paper by this introduction which contains a brief comment on some new concepts used in metric common fixed point theory and several recalls and definitions.

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Let (X, d) be a metric space and $\mathcal{B}(X)$ the set of all nonempty bounded sets of X. As in [10] and [11], we define the functions $\delta(A, B)$ and $D(A, B)$ by

$$\delta(A, B) := \sup\{d(a, b) : a \in A,\ b \in B\},$$

$$D(A, B) := \inf\{d(a, b) : a \in A,\ b \in B\}.$$

If $A = \{a\}$ where a is a point in X, then we write $\delta(A, B) = \delta(a, B)$. If $B = \{b\}$ where b is a point in X, then we write $\delta(A, B) = \delta(A, b)$.

It follows immediately from the definition of $\delta(A, B)$ that

$$\delta(A, B) = \delta(B, A), \quad \forall A, B \in \mathcal{B}(X),$$

$$\forall A, B \in \mathcal{B}(X), \quad \delta(A, B) = 0 \Longleftrightarrow A = B = \{a\},$$

for some point $a \in X$, and

$$\delta(A, B) \leq \delta(A, C) + \delta(C, B), \quad \forall A, B, C \in \mathcal{B}(X).$$

Throughout this paper, $\mathbb{N}$ will be the set of non-negative integers.

Definition 1.1 ([10]). A sequence $(A_n)_{n\geq 0}$ of nonempty subsets of X is said to *converge to a subset* A of X if:

(i) each point $a \in A$ is the limit of a convergent sequence $(a_n)_{n\geq 0}$, where $a_n \in A_n$, for all $n \in \mathbb{N}$.

(ii) For arbitrary $\epsilon > 0$ there exists an integer $m > 0$ such that $A_n \subset A(\epsilon)$ for all integer $n \geq m$, where

$$A(\epsilon) := \{x \in X : \exists a \in A : d(x, a) < \epsilon\}.$$

In this case, A is said to be the limit of the sequence $(A_n)_{n\geq 0}$.

Lemma 1.1 ([10]). *If $(A_n)_{n\geq 0}$ and $(B_n)_{n\geq 0}$ are sequences in $\mathcal{B}(X)$ converging to the sets A and B respectively in $\mathcal{B}(X)$, then the sequence $(\delta(A_n, B_n))_{n\geq 0}$ converges to $\delta(A, B)$.*

Lemma 1.2 ([11]). *Let $(A_n)_{n\geq 0}$ be a sequence in $\mathcal{B}(X)$ and $y \in X$ such that $\lim_{n\to\infty} \delta(A_n, y) = 0$. Then the sequence $(A_n)_{n\geq 0}$ converges to $\{y\}$ in $\mathcal{B}(X)$.*

Let S and T be self mappings of the metric space (X, d). Jungck [13] defined S and T to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $(x_n)_{n\geq 0}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x,$$

for some $x \in X$.

In 1994, Pant [23] introduced the notion of pointwise R-weakly commuting mappings. It is proved in [24] that the notion of pointwise R-weakly commuting is equivalent to commutativity at coincidence points.

Jungck [14] defined S and T to be weakly compatible if and only if $Sx = Tx$ implies that $STx = TSx$. Thus S and T are weakly compatible if and only if S and T are pointwise R-weakly commuting.

Let S and T be two self mappings of X. We recall that a point $x \in X$ is a coincidence point of S and T if $Sx = Tx$.

We recall that a point $x \in X$ is a point of coincidence of S and T if $x = Su = Tu$ for some $u \in X$.

Definition 1.2 ([6]). The self mappings f and g of (X, d) are are said to be *occasionally weakly compatible* (owc) if there exists a point $x \in X$ which is coincidence point of f and g at which f and g commute.

Remark 1.1. Two weakly compatible mappings having a coincidence point are occasionally weakly compatible. The converse is not true as example of [6].

Weak compatibility does not imply occasional weak compatibility, as every map $f : X \to X$ and id, the identity mapping on X, are weakly compatible, while f and g are owc if and only if f has a fixed point.

Some fixed point theorems for owc mappings are proved in [4, 17, 28] and other papers.

In 2000, Sastri and Krishna Murthy [32] introduced the following notion:

A point $z \in X$ is said to be tangent to the pair (A, B) of self mappings of (X, d), if there exists a sequence $(x_n)_{n\geq 0}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$.

The pair (A, B) is called tangential if there exists a point z in X which is tangent to (A, B).

Two years later, Aamri and Moutawakil [1] rediscovered this notion and called it as property (E.A).

Definition 1.3 ([1]). Let (X, d) be a metric space and A and B be two self mappings. The pair (A, B) satisfies the *property* $(E.A)$, if there exists a sequence $(x_n)_{n\geq 0}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$, for some $z \in X$.

Recently, Liu et al. [21] defined a common property (E.A) as follows

Definition 1.4 ([21]). Let A, B, S and T be four self mappings of a metric space (X, d). We say that the pair (A, S) and (B, T) satisfy a *common property* $(E.A)$ if there exist two sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$$

for some $z \in X$.

In a recent paper, Pathak and Shahzad (see [25]) introduced the notion of weakly tangent point

Definition 1.5 ([25]). Let A, B, S and T be four self mappings of a metric space (X, d).

A point $z \in X$ is said to be *weakly tangent* to the pair (S, T) if there exist two sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = z.$$

We say also that z is a *weak tangent point* to the pair (S, T).

The pair (A, B) is called *tangential* w.r.t. the pair (S, T) if

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n = z,$$

whenever there exist two sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = z$$

for some $z \in X$.

Remark 1.2. 1) Every pair of mappings (S, T) which satisfies property $(E.A)$ (or has a tangent point) also has a weak tangent point to (S, T), but the converse is not true (see [25]). Hence, the notion of weak tangent point to the pair (S, T) is weaker than the notion of property $(E.A)$ of the pair (S, T) (and the notion of tangent point to (S, T)).

2) If $A = B$ and $S = T$, we say that the mapping A is tangential w.r.t. the mapping S.

If $S = A$ and $T = B$, we say that (A, B) is tangential with itself.

3) Obviously, every pair of mappings (S, T) satisfies property $(E.A)$ also has a point $z \in X$ which is tangent to (S, T) (to see this, just take $x_n = y_n$, but the converse need not be true (see Example 2.2 in [25]).
4) If the pair (A, B) is tangential w.r.t. the pair (S, T), then the pair (S, T) need not be tangential w.r.t. the pair (A, B). (see Example 2.3 in [25]).
5) If the pair (A, B) is tangential w.r.t. the pair (S, T) and the pair (S, T) is tangential w.r.t. the pair (A, B), then the pairs (A, S) and (B, T) satisfy the common property (E.A).

Definition 1.6. If $f \colon X \to X$ and $F \colon X \to \mathcal{B}(X)$, then

(1) a point $x \in X$ is said to be a *coincidence point* of f and F if $fx \in Fx$. We denote by $C(f, F)$ the set of all coincidence points of f and F.
(2) A point $x \in X$ is said to be a *strict coincidence point* of f and F if $\{fx\} = Fx$.
(3) A point $x \in X$ is a *fixed point* of F if $x \in Fx$.
(4) A point $x \in X$ is a *strict fixed point* of F if $\{x\} = Fx$.

Definition 1.7 ([15]). The mappings $f \colon X \to X$ and $F \colon X \to \mathcal{B}(X)$ are said to be *δ-compatible* if $\lim_{n\to\infty} \delta(Ffx_n, fFx_n) = 0$, whenever (x_n) is a sequence in X such that $fFx_n \in \mathcal{B}(X)$, $fx_n \longrightarrow t$, and $Fx_n \longrightarrow \{t\}$ for some $t \in X$.

Definition 1.8 ([16]). Let $f \colon X \to X$ and $F \colon X \to \mathcal{B}(X)$ be mappings. The hybrid pair (f, F) is said to be *weakly compatible* if for all $x \in C(f, F)$, we have $fF(x) = Ff(x)$.

If the pair (f, F) is δ-compatible, then it is weakly compatible, but the converse is not true in general (see [16]).

Recently, Djoudi and Khemis [9] introduced a generalization of pair of mappings satisfying property (E.A), named D-mappings. Some results about D-mappings are obtained in [7].

Definition 1.9 ([9]). The mappings $f \colon X \to X$ and $F \colon X \to \mathcal{B}(X)$ are said to be *D-mappings* if there exists a sequence $(x_n)_{n\geq 0}$ in X such that $\lim_{n\to\infty} fx_n = t$ and $\lim_{n\to\infty} Fx_n = \{t\}$ for some $t \in X$.

Obviously, two mappings which are not δ-compatible are D-mappings.

Definition 1.10 ([5]). The hybrid pair of mappings $f \colon X \to X$ and $F \colon X \to \mathcal{B}(X)$ is said to be *occasionally weakly compatible* (owc) if there exists $x \in C(f, F)$ such that $fFx = Ffx$.

The hybrid pair (f, F) is said to be *strict owc* if there exists a strict coincidence point x in X such that $fFx = Ffx$.

Remark 1.3. If the hybrid pair (f,F) is weakly compatible and $C(f,F) \neq \emptyset$, then the pair (f,F) is occasionally weakly compatible. There exist owc hybrid pairs which are not weakly compatible (Ex. 1.13 in [5]).

2. Preliminaries

In [8], Branciari established the following result.

Theorem 2.1 ([8]). *Let (X,d) be a complete metric space, $c \in (0,1)$ and $f\colon X \to X$ be a mapping such that for all $x,y \in X$,*

$$\int_0^{d(fx,fy)} h(t)dt \leq c \int_0^{d(x,y)} h(t)dt, \tag{2.1}$$

where $h\colon [0,\infty) \to [0,\infty)$ is a Lebesgue measurable mapping which is summable (i.e., with finite integral) on each compact subset of $[0,\infty)$, such that, for every $\epsilon > 0$ we have $\int_0^\epsilon h(t)dt > 0$. Then f has a unique fixed point $z \in X$, such that for each $x \in X$, $\lim_{n\to\infty} f^n(x) = z$.

Some fixed point theorems in metric and symmetric spaces for compatible and weakly compatible mappings satisfying a contractive condition of integral type are proved in [3, 19, 20, 22, 25, 29, 33] and other papers.

In [25], Gregus type fixed point results for tangentially mappings satisfying contractive conditions of integral type are obtained. A main result by [25] is the following.

Theorem 2.2. *Let A, B, S and T be mappings from a metric space (X,d) into itself satisfying*

$$\begin{aligned}(2.2)\quad &\Big[1+\alpha\int_0^{d(Sx,Ty)} h(t)dt\Big]\int_0^{d(Ax,By)} h(t)dt < \\ &\alpha\Big[\int_0^{d(Ax,Sx)} h(t)dt \cdot \int_0^{d(By,Ty)} h(t)\,dt + \int_0^{d(Ax,Ty)} h(t)dt \cdot \int_0^{d(Sx,By)} h(t)dt\Big] \\ &\quad + a\int_0^{d(Sx,Ty)} h(t)\,dt + (1-a)\max\Big\{\int_0^{d(Ax,Sx)} h(t)dt, \int_0^{d(By,Ty)} h(t)dt,\end{aligned}$$

$$\Big(\int_0^{d(Ax,Sx)} h(t)dt\Big)^{\frac{1}{2}} \cdot \Big(\int_0^{d(Ax,Ty)} h(t)dt\Big)^{\frac{1}{2}},$$
$$\Big(\int_0^{d(Sx,By)} h(t)dt\Big)^{\frac{1}{2}} \cdot \Big(\int_0^{d(Ax,Ty)} h(t)dt\Big)^{\frac{1}{2}}\Big\},$$

for all $x, y \in X$ for which the right-hand side of (2.2) *is positive, where* $0 < a < 1$, $\alpha > 0$ *and h is as in Theorem* 2.1.

If there exists a weak tangent point $z \in S(X) \cap T(X)$ to (S, T) and (A, B) is tangential w.r.t. (S, T), and the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

Definition 2.1. An *altering distance* is a mapping $\psi\colon [0, \infty) \to [0, \infty)$ which satisfies:

(ψ_1) : ψ is increasing and continuous.

(ψ_2) : $\psi(t) = 0$ if and only if $t = 0$.

Fixed point theorems involving altering distances have been studied in [2, 18, 27, 30, 31] and other papers.

Lemma 2.1. *The function $\psi(t) := \int_0^t h(x)\,dx$, where h is as in Theorem* 2.1, *is an altering distance.*

Proof. By definitions of ψ and h it follows that ψ is increasing and $\psi(t) = 0$ if and only if $t = 0$. By Lemma 2.5 of [22], ψ is continuous. □

In [26] a general fixed point theorem for compatible mappings satisfying implicit relations is proved. In [12] the results from [26] are improved relaxing compatibility to weak compatibility.

As it was said before, the purpose of this paper is to prove a general fixed point theorem by altering distances for two owc hybrid pairs of mappings and to reduce the study of fixed points for hybrid pairs of mappings satisfying a contractive condition of integral type to the study of fixed points in a metric space by altering distances satisfying a new type of implicit relations generalizing the result of Theorem 2.2 and extending it to the case of owc hybrid pairs of mappings.

After these two sections devoted to the introduction, recalls and preliminaries, this paper contains three other sections. In section three, we introduce a new class of implicit relations by which we define contractive conditions and give some examples. In the fourth and fifth sections we present our main results with several consequences and corollaries.

We point out that our work provides some natural continuations to the investigations started by the authors in [2] concerning tangential self mappings of metric spaces.

3. Implicit relations

For the statements of our results, we need to introduce the following class of implicit relations.

Let $\mathcal{F}_t$ be the set of all real continuous functions $F\colon \mathbb{R}^6 \to \mathbb{R}$ decreasing in the variables t_5 and t_6 satisfying the following conditions:

(F_1): $F(t,0,0,t,t,0) \le 0$ implies $t = 0$.

(F_2): $F(t,0,t,0,0,t) \le 0$ implies $t = 0$.

(F_3): $F(t,t,0,0,t,t) = 0$, for every $t > 0$.

Example 3.1. $F(t_1,\dots,t_6) = (1+\alpha t_2)t_1 - \alpha(t_3t_4 + t_5t_6) - at_2 - (1 - a)\max\{t_3, t_4, \sqrt{t_3t_6}, \sqrt{t_5t_6}\}$, where $\alpha \ge 0$ and $0 < a < 1$.

(F_1): $F(t,0,0,t,t,0) = at \le 0$ implies $t = 0$.

(F_2): $F(t,0,t,0,0,t) = at \le 0$ implies $t = 0$.

(F_3): $F(t,t,0,0,t,t) = 0$, for every $t > 0$.

Example 3.2. $F(t_1,\dots,t_6) = t_1^p - at_2^p - (1-a)\max\{t_3^p, t_4^p, (t_3t_6)^{\frac{p}{2}}, (t_5t_6)^{\frac{p}{2}}\}$, where $p > 0$ and $0 < a < 1$.

(F_1): $F(t,0,0,t,t,0) = at^p \le 0$ implies $t = 0$.

(F_2): $F(t,0,t,0,0,t) = at^p \le 0$ implies $t = 0$.

(F_3): $F(t,t,0,0,t,t) = 0$, for every $t > 0$.

Example 3.3. $F(t_1,\dots,t_6) = t_1^p - \alpha[\max\{t_2,t_3,t_4\}]^p - (1-\alpha)(at_5^p + bt_6^p)$, where $p > 0$, $a, b > 0$, $a + b = 1$ and $0 < \alpha < 1$.

(F_1): $F(t,0,0,t,t,0) = t^p(1-\alpha)(1-a) \le 0$ implies $t = 0$.

(F_2): $F(t,0,t,0,0,t) = t^p(1-\alpha)(1-b) \le 0$ implies $t = 0$.

(F_3): $F(t,t,0,0,t,t) = 0$, for every $t > 0$.

Example 3.4. $F(t_1,\dots,t_6) = t_1^p - at_2^p - b[\max\{t_3,t_4\}]^p - c[\max\{t_2,t_5,t_6\}]^p$, where $p > 0$, $a, b, c \ge 0$, $b + c < 1$ and $a + c = 1$.

(F_1): $F(t,0,0,t,t,0) = t^p(1-(b+c)) \le 0$ implies $t = 0$.

(F_2): $F(t,0,t,0,0,t) = t^p(1-(b+c)) \le 0$ implies $t = 0$.

(F_3): $F(t,t,0,0,t,t) = 0$, for every $t > 0$.

EXAMPLE 3.5. $F(t_1,\dots,t_6) = t_1 - at_2 - b\dfrac{t_5+t_6}{1+t_3+t_4}$, where $a,b \geq 0$ and $a+2b=1$.

(F_1): $F(t,0,0,t,t,0) = t(1-\frac{b}{1+t}) \leq 0$ implies $t=0$.

(F_2): $F(t,0,t,0,0,t) = t(1-\frac{b}{1+t}) \leq 0$ implies $t=0$.

(F_3): $F(t,t,0,0,t,t) = 0$, for every $t>0$.

EXAMPLE 3.6. $F(t_1,\dots,t_6) = t_1^p - at_2^p - b(t_3^p+t_4^p) - c(t_5^p+t_6^p)$, where $p>0$, $a,b,c \geq 0$, $b+c<1$ and $a+2c=1$.

(F_1): $F(t,0,0,t,t,0) = t^p(1-(b+c)) \leq 0$ implies $t=0$.

(F_2): $F(t,0,t,0,0,t) = t^p(1-(b+c)) \leq 0$ implies $t=0$.

(F_3): $F(t,t,0,0,t,t) = 0$, for every $t>0$.

EXAMPLE 3.7. $F(t_1,\dots,t_6) = t_1^{2p} - at_1^p(t_2^p+t_3^p+t_4^p) - bt_5^pt_6^p$, where $p>0$, $a,b>0$ and $a+b=1$.

(F_1): $F(t,0,0,t,t,0) = (1-a)t^{2p} \leq 0$ implies $t=0$.

(F_2): $F(t,0,t,0,0,t) = (1-a)t^{2p} \leq 0$ implies $t=0$.

(F_3): $F(t,t,0,0,t,t) = 0$, for every $t>0$.

EXAMPLE 3.8. $F(t_1,\dots,t_6) = t_1^p - t_2^p - \dfrac{t_3^qt_5+t_4^rt_6}{1+t_3+t_4}$, where $p,q,r>0$.

(F_1): $F(t,0,0,t,t,0) = t^p \leq 0$ implies $t=0$.

(F_2): $F(t,0,t,0,0,t) = t^p \leq 0$ implies $t=0$.

(F_3): $F(t,t,0,0,t,t) = 0$, for every $t>0$.

4. Altering distances and common fixed points for hybrid pairs

Before giving our main result, we need to prove the following preliminary results on strict coincidence points for two hybrid pairs of maps.

THEOREM 4.1. *Let (X,d) be a metric space, $S\colon (X,d)\to(X,d)$ and $A:(X,d)\to\mathcal{B}(X)$ be two strict o.w.c mappings. If A and S have a unique point of strict coincidence $\{z\}=\{Sx\}=Ax$, then z is the unique common fixed point for S and A which is a strict fixed point for A.*

PROOF. Since A and S are strict owc, there exists a point $x\in X$ such that $\{z\}=\{Sx\}=Ax$ and $SAx=ASx$. Then $\{Sz\}=\{SSx\}=SAx=Az=$

$\{u\}$. Hence u is a point of strict coincidence of A and S and by hypothesis, we have $u = z$. Hence $\{z\} = Az = \{Sz\}$. So, z is a common fixed point for A and S. Suppose that $v \neq z$ is another common fixed point for S and A which is a strict fixed point for A. Hence $\{v\} = \{Sv\} = Av$. Hence v is a point of strict coincidence of A and S. Therefore $z = v$. This ends the proof. □

THEOREM 4.2. *Let (X, d) be a metric space. Let $S, T\colon (X, d) \to (X, d)$ and $A, B\colon (X, d) \to \mathcal{B}(X)$ be mappings satisfying the following inequality*

$$\text{(4.1)} \quad F\Big(\psi(\delta(Ax, By)), \psi(d(Sx, Ty)), \psi(\delta(Sx, Ax)), \psi(\delta(Ty, By)), \psi(D(Sx, By)), \psi(D(Ty, Ax))\Big) < 0,$$

for all $x, y \in X$, where F satisfies property (F_3) and ψ is an altering distance.

Suppose that there exist $x_0, y_0 \in X$ such that $\{u\} = \{Sx_0\} = Ax_0$ and $\{v\} = \{Ty_0\} = By_0$, then

(a) *$u := Sx_0$ is the unique point of strict coincidence of A and S and*
(b) *$v := Ty_0$ is the unique point of strict coincidence of B and T.*
(c) *Moreover, we have $u = v$.*

PROOF. First we prove that $Sx_0 = Ty_0$. Suppose that $Sx_0 \neq Ty_0$. Then by (4.1), we obtain

$$F\Big(\psi(d(Sx_0, Ty_0)), \psi(d(Sx_0, Ty_0)), 0, 0, \psi(d(Sx_0, Ty_0)), \psi(d(Ty_0, Sx_0))\Big) < 0,$$

a contradiction of (F_3). Hence $Sx_0 = Ty_0$. Thus $\{Sx_0\} = Ax_0 = By_0 = \{Ty_0\}$.

Suppose that there exists an another point w which is of strict coincidence of A and S, $\{w\} = \{Sz\} = Az$ such that $w \neq u$. Then by (4.1) we obtain

$$F\Big(\psi(d(Sz, Ty_0)), \psi(d(Sz, Ty_0)), 0, 0, \psi(d(Sz, Ty_0)), \psi(d(Sz, Ty_0))\Big) < 0,$$

a contradiction of (F_3), hence $\{w\} = \{Sz\} = \{Ty_0\} = By = Ax_0 = \{Sx_0\} = \{u\}$. Hence $u = w$ and u is the unique point of coincidence of A and S. Similarly, v is the unique point of coincidence of B and T. This completes the proof. □

To state the main result of this paper, we need to introduce the following extension of Definition 1.5.

Definition 4.1. Let (X,d) be a metric space and let $S,T\colon X\to X$ and $A,B\colon X\to\mathcal{B}(X)$ be single valued and set valued mappings, respectively.

The pair (A,B) is called *tangential with respect to the pair* (S,T) if

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n = \{z\},$$

whenever there exist two sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = z$$

for some $z\in X$.

Now, we are ready to state and prove the main result of this paper.

Theorem 4.3. *Let (X,d) be a metric space. Let $S,T\colon (X,d)\to(X,d)$ and $A,B\colon (X,d)\to\mathcal{B}(X)$ be mappings satisfying the following inequality*

$$\begin{aligned}(4.1)\quad F\Big(&\psi(\delta(Ax,By)),\psi(d(Sx,Ty)),\psi(\delta(Sx,Ax)),\\ &\psi(\delta(Ty,By)),\psi(D(Sx,By)),\psi(D(Ty,Ax))\Big)<0,\end{aligned}$$

for all $x,y\in X$, where $F\in\mathcal{F}_t$ and ψ is an altering distance.

If there exists a weak tangent point $z\in S(X)\cap T(X)$ to the pair $\{S,T\}$ and (A,B) is tangential w.r.t. (S,T), then

a) *A and S have a coincidence point,*

b) *B and T have a coincidence point.*

Moreover, if the pairs (A,S) and (B,T) are strict owc, then A,B,S and T have a unique common fixed point which is a strict fixed point of A and B.

Proof. Since the point $z\in S(X)\cap T(X)$ is a weak tangent point to (S,T), there exist sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = z.$$

Because the pair of mappings (A,B) is tangential w.r.t. the pair (S,T), we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n = \{z\}.$$

Again, since $z \in S(X) \cap T(X)$, then $z = Su = Tv$ for some $u, v \in X$. Using (4.1) we have

$$F\Big(\psi(\delta(Ax_n, Bv)), \psi(d(Sx_n, Tv)), \psi(\delta(Sx_n, Ax_n)),$$
$$\psi(\delta(Tv, Bv)), \psi(D(Sx_n, Bv)), \psi(D(Tv, Ax_n))\Big) < 0.$$

Letting n tend to infinity, we obtain

$$F\Big(\psi(\delta(z, Bv)), 0, 0, \psi(\delta(z, Bv)), \psi(d(z, Bv)), 0)\Big) \leq 0.$$

Since ψ is non-decreasing and F is non-increasing in the fifth variable, it follows that

$$F\Big(\psi(\delta(z, Bv)), 0, 0, \psi(\delta(z, Bv)), \psi(\delta(z, Bv)), 0)\Big) \leq 0.$$

By the property (F_1), it follows that $\psi(\delta(z, Bv)) = 0$, which implies that $\delta(z, Bv) = 0$, i.e., $\{z\} = Bv$. Hence $\{z\} = Bv = \{Tv\}$ and v is a coincidence point of B and T.

Further, using (4.1) again, we get

$$F\Big(\psi(\delta(Au, By_n)), \psi(d(Su, Ty_n)), \psi(\delta(Su, Au)),$$
$$\psi(\delta(Ty_n, By_n)), \psi(D(Su, By_n)), \psi(D(Ty_n, Au))\Big) < 0.$$

Letting n tend to infinity, we obtain

$$F\Big(\psi(\delta(Au, z)), 0, \psi(\delta(z, Au)), 0, 0, \psi(d(z, Au))\Big) \leq 0.$$

Since ψ is non-decreasing and F is non-increasing in the variable t_6, it follows that

$$F\Big(\psi(\delta(Au, z)), 0, \psi(\delta(z, Au)), 0, 0, \psi(\delta(z, Au))\Big) \leq 0.$$

By (F_2), it follows that $\psi(\delta(z, Au)) = 0$ which implies $\delta(z, Au) = 0$, i.e., $\{z\} = Au$. Thus $\{z\} = Au = \{Su\}$ and u is a coincidence point of A and S.

Because F satisfies property (F_3), by Theorem 4.2, z is the unique point of coincidence of A and S and z is the unique point of coincidence of B and T.

If the pairs (A, S) and (B, T) are strict owc then by Theorem 4.1, z is the unique common fixed point of A and S and z is the unique common fixed point of B and T which is a strict fixed point for A and B.

Hence z is the unique common fixed point of S, T, A and B which is a strict fixed point for A and B. This ends the proof. $\square$

If $\psi(t) = t$ then by Theorem 4.3, we obtain

COROLLARY 4.1. *Let (X, d) be a metric space. Let $S, T\colon (X, d) \to (X, d)$ and $A, B\colon (X, d) \to \mathcal{B}(X)$ be mappings satisfying the following inequality*

$$(4.2)\quad F(\delta(Ax, By), d(Sx, Ty), \delta(Sx, Ax), \delta(Ty, By)), D(Sx, By), D(Ty, Ax)) < 0,$$

for all $x, y \in X$, where $F \in \mathcal{F}_t$.

If there exists a weak tangent point $z \in S(X) \cap T(X)$ to the pair (S, T) and (A, B) is tangential w.r.t. (S, T), then

a) *A and S have a coincidence point,*

b) *B and T have a coincidence point.*

Moreover, if the pairs (A, S) and (B, T) are strict owc, then A, B, S and T have a unique common fixed point which is a strict fixed point for A and B.

If A, B, S and T are single valued self mappings of a metric sapce (X, d), we have

COROLLARY 4.2. *Let (X, d) be a metric space. Let $A, B, S, T\colon X \to X$ be self mappings a metric sapce (X, d) such that*

$$(4.3)\quad F\Big(\psi(d(Ax, By)), \psi(d(Sx, Ty)), \psi(d(Sx, Ax)), \psi(d(Ty, By)), \psi(d(Sx, By)), \psi(d(Ty, Ax))\Big) < 0,$$

for all $x, y \in X$, where $F \in \mathcal{F}_t$ and ψ is an altering distance.

If there exists a weak tangent point $z \in S(X) \cap T(X)$ to the pair (S, T) and (A, B) is tangential w.r.t. (S, T), then

a) *A and S have a coincidence point,*

b) *B and T have a coincidence point.*

Moreover, if the pairs (A, S) and (B, T) are strict owc, then A, B, S and T have a unique common fixed point.

By Corollary 4.1 we obtain

Corollary 4.3. *Let (X, d) be a metric space. Let $A, B, S, T\colon X \to X$ be self mappings a metric sapce (X, d) such that*

$$(4.4)\quad F(d(Ax, By), d(Sx, Ty), d(Sx, Ax), d(Ty, By)), d(Sx, By), d(Ty, Ax)) < 0,$$

for all $x, y \in X$, where $F \in \mathcal{F}_t$.

If there exists a weak tangent point $z \in S(X) \cap T(X)$ to the pair (S, T) and (A, B) is tangential w.r.t. (S, T), then

a) *A and S have a coincidence point,*

b) *B and T have a coincidence point.*

Moreover, if the pairs (A, S) and (B, T) are strict owc, then A, B, S and T have a unique common fixed point.

Remark 4.1. a) By Corollary 4.3 and Example 3.1 we obtain a generalization of Corollary 3.1 of [25].

b) By Corollary 4.3 and Example 3.1 for $\alpha = 0$, we obtain a generalization of Corollary 2.9 of [25].

5. Altering distance and contractive condition of integral type

Theorem 5.1. *Let (X, d) be a metric space. Let $S, T\colon (X, d) \to (X, d)$ and $A, B\colon (X, d) \to \mathcal{B}(X)$ be mappings such that*

$$(5.1)\quad F\Big(\int_0^{\delta(Ax,By)} h(t)dt, \int_0^{d(Sx,Ty)} h(t)dt, \int_0^{\delta(Sx,Ax)} h(t)dt, \int_0^{\delta(Ty,By)} h(t)dt, \int_0^{D(Sx,By)} h(t)dt, \int_0^{D(Ty,Ax)} h(t)dt\Big) < 0,$$

for all $x, y \in X$, where $F \in \mathcal{F}_t$ and h is as in Theorem 2.1.

If there exists a weak tangent point $z \in S(X) \cap T(X)$ to the pair (S, T) and (A, B) is tangential w.r.t. (S, T), then

a) *A and S have a coincidence point,*

b) *B and T have a coincidence point.*

Moreover, if the pairs (A, S) and (B, T) are strict owc, then A, B, S and T have a unique common fixed point which is a strict fixed point of A and B.

Proof. We set $\psi(t) = \int_0^t h(t)dt$. By Lemma 2.1 we know that ψ is an altering distance. By (5.1) we obtain

$$(5.2)\quad F\Big(\psi(\delta(Ax,By)), \psi(d(Sx,Ty)), \psi(\delta(Sx,Ax)), \\ \psi(\delta(Ty,By)), \psi(D(Sx,By)), \psi(D(Ty,Ax))\Big) < 0,$$

for all $x, y \in X$, where $F \in \mathcal{F}_t$.

All the conditions of Theorem 4.3 are satisfied. Hence, the results of Theorem 5.1 follow from Theorem 4.3. □

Remark 5.1. a) If $h(t) = 1$ we obtain Corollary 4.1.

b) By Theorem 5.1 and Example 3.1 we obtain a generalization of Theorem 2.2.

Similarly, by Corollary 4.2 we obtain the following result.

Theorem 5.2. *Let (X, d) be a metric space and $S, T, A, B\colon X \to X$ be self mappings of (X, d) such that*

$$(5.3)\quad F\Big(\int_0^{d(Ax,By)} h(t)dt, \int_0^{d(Sx,Ty)} h(t)dt, \int_0^{d(Sx,Ax)} h(t)dt, \\ \int_0^{d(Ty,By)} h(t)dt, \int_0^{d(Sx,By)} h(t)dt, \int_0^{d(Ty,Ax)} h(t)dt\Big) < 0,$$

for all $x, y \in X$, where $F \in \mathcal{F}_t$ and h is as in Theorem 2.1.

If there exists a weak tangent point $z \in S(X) \cap T(X)$ to the pair (S, T) and (A, B) is tangential w.r.t. (S, T), then

a) *A and S have a coincidence point,*

b) *B and T have a coincidence point.*

Moreover, if the pairs (A, S) and (B, T) are strict owc, then A, B, S and T have a unique common fixed point.

Remark 5.2. If F is as in Example 3.2 with $p = 1$ then by Theorem 5.2, we obtain generalizations of Corollary 2.6 and Corollary 2.7 of [25].

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