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FEJÉR-TYPE INEQUALITIES FOR STRONGLY CONVEX FUNCTIONS

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Abstract. Fejér-type inequalities as well as some refinement and a discrete version of the Hermite–Hadamard inequalities for strongly convex functions are presented.

1. Introduction

In recent years several extensions and generalizations have been considered for classical convexity, and the theory of inequalities has made essential contributions in many areas of mathematics. A significant subclass of convex functions is that of *strongly convex functions* introduced by B. T. Polyak [15]. Strongly convex functions are widely used in applied economics, as well as in nonlinear optimization and other branches of pure and applied mathematics. Our investigations are devoted to the classical results related to convex functions due to Charles Hermite [6], Jaques Hadamard [5] and Lipót Fejér [4]. The Hermite-Hadamard inequalities and Fejér inequalities have been the subject of intensive research, and many applications, generalizations and improvements of them can be found in the literature (see, for instance, [2, 3, 9, 11, 14, 20] and the references therein). In this paper we deal with these inequalities for strongly convex functions.

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Let $f\colon I \to \mathbb{R}$ be a convex function defined on an interval $I \subset \mathbb{R}$ and let $a, b \in I$ with $a < b$. The following double integral inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}\,, \tag{1}$$

is known in the literature as the Hermite–Hadamard inequality (see [3,14] for some historical notes).

In [4] Fejér generalized inequalities (1) by proving that if $g\colon [a,b] \to [0,\infty)$ is a *symmetric density function on* $[a,b]$ (that is, $g(a+b-x) = g(x)$ for all $x \in [a,b]$, and $\int_a^b g(x)\;dx = 1$), and a function $f\colon [a,b] \to \mathbb{R}$ is convex then

$$f\left(\frac{a+b}{2}\right) \le \int_a^b f(x)g(x)\,dx \le \frac{f(a)+f(b)}{2}\,. \tag{2}$$

Of course, if $g(x) = \frac{1}{b-a}$, then (2) coincides with (1).

Recall that a function $f\colon I \to \mathbb{R}$ is called *strongly convex with modulus* $c > 0$ if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2, \tag{3}$$

for all $x, y \in I$ and $t \in [0,1]$. Strongly convex functions play important role in optimization theory and mathematical economics. Many properties and applications of them can be found in the literature (see, for instance, [1,8,10,12,15–19]). In [10] the following counterpart of the classical Hermite–Hadamard inequalities for strongly convex functions was obtained:

If a function $f\colon I \to \mathbb{R}$ is strongly convex with modulus c then

$$f\left(\frac{a+b}{2}\right) + \frac{c}{12}(b-a)^2 \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} - \frac{c}{6}(b-a)^2, \tag{4}$$

for all $a, b \in I,\ a < b$.

However, the Fejér-type generalization of (4) of the form

$$f\left(\frac{a+b}{2}\right) + \frac{c}{12}(b-a)^2 \le \int_a^b f(x)g(x)\,dx \le \frac{f(a)+f(b)}{2} - \frac{c}{6}(b-a)^2, \tag{5}$$

does not hold, in general, for any symmetric density function $g\colon [a,b] \to [0,\infty)$ and a strongly convex function $f\colon I \to \mathbb{R}$ (see the example below).

Example 1. Let $f(x) = x^2$ and $[a, b] = [-1, 1]$. Clearly, f is strongly convex with modulus $c = 1$. Take the density function g on $[-1, 1]$ given by

$$g(x) = \begin{cases} 1, & \text{if } x \in [-\frac{1}{2}, \frac{1}{2}] \\ 0, & \text{if } x \in [-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]. \end{cases}$$

Then

$$\int_{-1}^{1} x^2 g(x)\, dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2\, dx = \frac{1}{12} < \frac{1}{3} = f\left(\frac{-1+1}{2}\right) + \frac{1}{12}(1+1)^2,$$

which shows that the left-hand side inequality in (5) does not hold.

Now, take the density function g on $[-1, 1]$ defined by

$$g(x) = \begin{cases} 1, & \text{if } x \in [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1] \\ 0, & \text{if } x \in (-\frac{1}{2}, \frac{1}{2}). \end{cases}$$

Then

$$\int_{-1}^{1} x^2 g(x)\, dx = 2\int_{\frac{1}{2}}^{1} x^2\, dx = \frac{7}{12} > \frac{1}{3} = \frac{f(-1) + f(1)}{2} - \frac{1}{6}(1+1)^2,$$

which shows that the right-hand side inequality in (5) does not hold.

The main aim of this paper is to derive an appropriate counterpart of the Fejér inequalities for strongly convex functions.

We present also some refinement of the Hermite–Hadamard inequalities as well as a discrete version of the Hermite–Hadamard inequalities for strongly convex functions.

2. Fejér-type inequalities

The following theorem is a counterpart of the Fejér inequalities for strongly convex functions.

Theorem 1. *Let $g\colon [a,b] \to [0,\infty)$ be a symmetric density function on $[a,b]$ and $f\colon [a,b] \to \mathbb{R}$ be a strongly convex function with modulus $c > 0$. Then*

$$\begin{aligned}(6)\quad f\Big(\frac{a+b}{2}\Big) + c\Big[\int_a^b x^2 g(x)\, dx - \Big(\frac{a+b}{2}\Big)^2\Big] &\le \int_a^b f(x)g(x)\, dx \\ &\le \frac{f(a)+f(b)}{2} - c\Big[\frac{a^2+b^2}{2} - \int_a^b x^2 g(x)\, dx\Big].\end{aligned}$$

Remark 2. Using the Fejér inequalities (2) for the function $f(x) = x^2$, we get

$$\Big(\frac{a+b}{2}\Big)^2 \le \int_a^b x^2 g(x)\, dx \le \frac{a^2+b^2}{2}$$

for every symmetric density function g on $[a,b]$. Therefore the terms

$$\int_a^b x^2 g(x)\, dx - \Big(\frac{a+b}{2}\Big)^2 \quad \text{and} \quad \frac{a^2+b^2}{2} - \int_a^b x^2 g(x)\, dx$$

on the left- and the right-hand side of (6) are nonnegative. Consequently, inequalities (6) are a strengthening of the Fejér inequalities (2). Note also that inequalities (6) generalize the Hermite–Hadamard-type inequalities (4). Indeed, for $g(x) = \frac{1}{b-a}$ we have

$$\int_a^b x^2 g(x)\, dx - \Big(\frac{a+b}{2}\Big)^2 = \frac{(b-a)^2}{12}, \qquad \frac{a^2+b^2}{2} - \int_a^b x^2 g(x)\, dx = \frac{(b-a)^2}{6},$$

and then (6) reduces to (4).

Remark 3. If g is any symmetric density function on $[a,b]$, then

$$\int_a^b xg(x)\, dx = \frac{a+b}{2}.$$

Indeed, putting $s = \frac{a+b}{2}$ and using the fact that $g(2s-x) = g(x)$, we obtain

$$\begin{aligned}\int_a^b xg(x)\, dx &= \int_a^s xg(x)\, dx + \int_s^b yg(y)\, dy \\ &= \int_a^s xg(x)\, dx + \int_a^s (2s-x)g(x)\, dx = 2s\int_a^s g(x) = s = \frac{a+b}{2}.\end{aligned}$$

Proof of Theorem 1. To prove the left-hand side of (6) put $s = \frac{a+b}{2}$, and take a function $h\colon [a,b] \to \mathbb{R}$ of the form $h(x) = c(x-s)^2 + m(x-s) + f(s)$ supporting f at s (cf. [17, p. 268]). Then

$$\begin{aligned}\int_a^b f(x)g(x)\,dx &\geq \int_a^b h(x)g(x)\,dx\\ &= c\int_a^b x^2 g(x)\ dx + (-2cs+m)\int_a^b xg(x)\ dx\\ &\quad + (cs^2 - ms + f(s))\int_a^b g(x)\ dx.\end{aligned}$$

Hence, using the integrals

$$\int_a^b g(x)\ dx = 1 \quad \text{and} \quad \int_a^b xg(x)\ dx = \frac{a+b}{2} = s, \tag{7}$$

we obtain

$$\begin{aligned}\int_a^b f(x)g(x)\,dx &\geq c\int_a^b x^2 g(x)\,dx - cs^2 + f(s)\\ &= f\Big(\frac{a+b}{2}\Big) + c\Big[\int_a^b x^2 g(x)\ dx - \Big(\frac{a+b}{2}\Big)^2\Big].\end{aligned}$$

In the proof of the right-hand side of (6) we use inequality (3).

$$\begin{aligned}&\int_a^b f(x)g(x)\,dx = \int_a^b f\Big(\frac{b-x}{b-a}a + \frac{x-a}{b-a}b\Big)g(x)\,dx\\ &\leq \int_a^b \Big(\frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b) - c\frac{(b-x)(x-a)}{(b-a)^2}(b-a)^2\Big)g(x)\,dx\\ &= \int_a^b \Big(\frac{bf(a)-af(b)}{b-a} + \frac{f(b)-f(a)}{b-a}x - c((a+b)x - ab - x^2)\Big)g(x)\,dx.\end{aligned}$$

Now, using the integrals (7), we get

$$\begin{aligned}\int_a^b f(x)g(x)\,dx &\leq \frac{bf(a)-af(b)}{b-a} + \frac{f(b)-f(a)}{b-a}\ \frac{a+b}{2}\\ &\quad - c\Big[\frac{(a+b)^2}{2} - ab - \int_a^b x^2 g(x)\ dx\Big]\\ &= \frac{f(a)+f(b)}{2} - c\Big[\frac{a^2+b^2}{2} - \int_a^b x^2 g(x)\ dx\Big].\end{aligned}$$

$\square$

Remark 4. The proof presented above is direct. However, we can also obtain inequalities (6) using the classical Fejér inequalities and the representation of strongly convex functions of the form $f(x) = h(x) + cx^2$ with a convex functions h (see [7, Prop.1.1.2]; cf. also [12]).

Rajba and Wąsowicz [16] proved recently that a function $f\colon I \to \mathbb{R}$ is strongly convex with modulus c if and only if

$$f(E[X]) \leq E[f(X)] - cD^2[X] \tag{8}$$

for any integrable random variable X taking values in I ($E[X]$ and $D^2[X]$ denote the expected value and the variance of X, respectively). Using this result we can derive, alternatively, the left-hand side inequality of (6). Indeed, if X is a random variable with values in $[a, b]$ having a symmetric density function $g\colon [a, b] \to [0, \infty)$, then

$$E[X] = \int_a^b xg(x)\, dx = \frac{a+b}{2},$$

$$E[X^2] = \int_a^b x^2 g(x)\, dx,$$

$$D^2[X] = E[X^2] - (E[X])^2 = \int_a^b x^2 g(x)\, dx - \Big(\frac{a+b}{2}\Big)^2,$$

$$E[f(X)] = \int_a^b f(x)g(x)\, dx.$$

Thus, if a function $f\colon [a, b] \to \mathbb{R}$ is strongly convex with modulus c then, substituting the above values to (8), we obtain the left-hand side of (6).

3. A refinement of the Hermite–Hadamard-type inequalities

In this section we present a refinement of the Hermite–Hadamard-type inequalities (4) for strongly convex functions. A similar result for convex functions can be found in [11, Remark 1.9.3.].

Theorem 5. *If a function $f\colon [a,b] \to \mathbb{R}$ is strongly convex function with modulus c, then*

$$\begin{aligned}(9)\qquad f\Big(\frac{a+b}{2}\Big)+\frac{c}{12}(b-a)^2 &\le \frac{1}{2}\Big[f\Big(\frac{3a+b}{4}\Big)+f\Big(\frac{a+3b}{4}\Big)\Big]+\frac{c}{48}(b-a)^2\\ &\le \frac{1}{b-a}\int_a^b f(x)dx\\ &\le \frac{1}{2}\Big[f\Big(\frac{a+b}{2}\Big)+\frac{f(a)+f(b)}{2}\Big]-\frac{c}{24}(b-a)^2\\ &\le \frac{f(a)+f(b)}{2}-\frac{c}{6}(b-a)^2.\end{aligned}$$

Proof. Applying the Hermite–Hadamard-type inequalities (4) on each of the intervals $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$ we obtain

$$f\Big(\frac{3a+b}{4}\Big)+\frac{c}{48}(b-a)^2 \le \frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(x)\,dx \le \frac{f(a)+f(\frac{a+b}{2})}{2} - \frac{c}{24}(b-a)^2$$

and

$$f\Big(\frac{a+3b}{4}\Big)+\frac{c}{48}(b-a)^2 \le \frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} f(x)\,dx \le \frac{f(\frac{a+b}{2})+f(b)}{2} - \frac{c}{24}(b-a)^2.$$

Summing up these inequalities we get

$$\begin{aligned}(10)\quad f\Big(\frac{3a+b}{4}\Big)+f\Big(\frac{a+3b}{4}\Big)+\frac{2c}{48}(b-a)^2 &\le \frac{2}{b-a}\int_a^b f(x)\,dx\\ &\le \frac{f(a)+f(b)}{2}+f\Big(\frac{a+b}{2}\Big)-\frac{2c}{24}(b-a)^2.\end{aligned}$$

Now, using the strong convexity of f and (10), we obtain

$$\begin{aligned}f\Big(\frac{a+b}{2}\Big)+\frac{c}{12}(b-a)^2 &= f\Big(\frac{\frac{3a+b}{4}+\frac{a+3b}{4}}{2}\Big)+\frac{c}{12}(b-a)^2\\ &\le \frac{1}{2}\Big[f\Big(\frac{3a+b}{4}\Big)+f\Big(\frac{a+3b}{4}\Big)\Big]-\frac{c}{4}\Big(\frac{b-a}{2}\Big)^2+\frac{c}{12}(b-a)^2\\ &= \frac{1}{2}\Big[f\Big(\frac{3a+b}{4}\Big)+f\Big(\frac{a+3b}{4}\Big)\Big]+\frac{c}{48}(b-a)^2 \le \frac{1}{b-a}\int_a^b f(x)\,dx.\end{aligned}$$

Similarly, using once more (10) and the strong convexity of f, we get

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)dx &\le \frac{1}{2}\Big[f\Big(\frac{a+b}{2}\Big)+\frac{f(a)+f(b)}{2}\Big]-\frac{c}{24}(b-a)^2\\ &\le \frac{1}{2}\Big[\frac{f(a)+f(b)}{2}+\frac{f(a)+f(b)}{2}-\frac{c}{4}(b-a)^2\Big]-\frac{c}{24}(b-a)^2\\ &= \frac{f(a)+f(b)}{2}-\frac{c}{6}(b-a)^2,\end{aligned}$$

which finishes the proof. □

Remark 6. As a consequence of the above theorem we obtain that in the Hermite–Hadamard-type inequalities (4) the left-hand side inequality is stronger than the right-hand one, that is

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)\,dx-\Big[f\Big(\frac{a+b}{2}\Big)+\frac{c}{12}(b-a)^2\Big]\\ &\qquad\le \Big[\frac{f(a)+f(b)}{2}-\frac{c}{6}(b-a)^2\Big]-\frac{1}{b-a}\int_a^b f(x)\,dx.\end{aligned}$$

It follows immediately from the third inequality in (9). For the classical Hermite–Hadamard inequalities an analogous observation is given in [14, p.140].

4. Discrete version of the Hermite–Hadamard-type inequalities

It is known (see J.E.Pečarić [13]; cf. also [14, p. 145]) that if a function $f\colon I\to\mathbb{R}$ is convex and $x_1<x_2<\ldots<x_n$ are equidistant points in I then the following discrete analogues of the Hermite-Hadamard inequalities are valid:

$$f\Big(\frac{x_1+x_n}{2}\Big)\le\frac{1}{n}\sum_{i=1}^n f(x_i)\le\frac{f(x_1)+f(x_n)}{2}.$$

In this section we present a counterpart of that result for strongly convex functions.

THEOREM 7. *Let $f\colon [a,b] \to \mathbb{R}$ be a strongly convex function with modulus c and $a = x_1 < x_2 < \ldots < x_n = b$ be equidistant points. Then*

$$
\begin{aligned}
(11) \qquad f\Big(\frac{a+b}{2}\Big) + \frac{c(n+1)}{12(n-1)}(b-a)^2 &\le \frac{1}{n}\sum_{i=1}^{n} f(x_i) \\
&\le \frac{f(a)+f(b)}{2} - \frac{c(n-2)}{6(n-1)}(b-a)^2.
\end{aligned}
$$

PROOF. Since the points $x_1, \ldots, x_n$ are equidistant, we have $\frac{1}{n}\sum\limits_{i=1}^{n} x_i = \frac{x_1+x_n}{2}$. Hence, using the Jensen-type inequality for strongly convex functions (see [1]), we get

$$
(12) \qquad f\Big(\frac{a+b}{2}\Big) = f\Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big) \le \frac{1}{n}\sum_{i=1}^{n} f(x_i) - \frac{c}{n}\sum_{i=1}^{n}(x_i - s)^2,
$$

where $s = \frac{1}{n}\sum\limits_{i=1}^{n} x_i = \frac{a+b}{2}$. To finish the left-hand side inequality in (11) we will show that

$$
\frac{1}{n}\sum_{i=1}^{n}(x_i - s)^2 = \frac{n+1}{12(n-1)}(b-a)^2.
$$

Putting $h = \frac{b-a}{n-1}$, we have $x_i = a + (i-1)h, \;\; i = 1, \ldots, n$. From here

$$
\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}(x_i - s)^2 &= \frac{1}{n}\sum_{i=1}^{n}(x_i)^2 - s^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}\big(a^2 + 2ah(i-1) + (i-1)^2h^2\big) - s^2 \\
&= a^2 + \frac{2ah}{n}\sum_{i=1}^{n}(i-1) + \frac{h^2}{n}\sum_{i=1}^{n}(i-1)^2 - s^2.
\end{aligned}
$$

Consequently, using the formulas

$$
\sum_{i=1}^{n}(i-1) = \frac{n(n-1)}{2} \quad \text{and} \quad \sum_{i=1}^{n}(i-1)^2 = \frac{(n-1)n(2n-1)}{6},
$$

we obtain

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - s)^2 = a^2 + a(b-a) + \frac{2n-1}{6(n-1)}(b-a)^2 - \Big(\frac{a+b}{2}\Big)^2$$
$$= \frac{n+1}{12(n-1)}(b-a)^2,$$

which was to be proved.

To show the right-hand side inequality in (11) note that

$$x_i = (1-q_i)a + q_i b, \quad \text{where} \quad q_i = \frac{i-1}{n-1}, \quad i = 1, \dots, n.$$

Hence, by the strong convexity of f,

$$f(x_i) = f((1-q_i)a + q_i b) \le (1-q_i)f(a) + q_i f(b) - cq_i(1-q_i)(b-a)^2.$$

Summing up the above inequalities and using the fact that the numbers $(1-q_i)f(a) + q_i f(b)$ are terms of an arithmetic sequence, we get

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) \le \frac{f(a)+f(b)}{2} - \frac{c}{n(n-1)^2}\sum_{i=1}^{n}(i-1)(n-i)(b-a)^2.$$

Now, applying the formula

$$\sum_{i=1}^{n}(i-1)(n-i) = \frac{(n-2)(n-1)n}{6},$$

we obtain

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) \le \frac{f(a)+f(b)}{2} - \frac{c(n-2)}{6(n-1)}(b-a)^2,$$

which finishes the proof. □

REMARK 8. Note that the sums $\frac{b-a}{n}\sum\limits_{i=1}^{n} f(x_i)$ are the Riemann approximate sums of the integral $\int_a^b f(x)\,dx$. Therefore, letting $n \to \infty$ in (11) , we get the Hermite–Hadamard-type inequalities (4).

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