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Report of Meeting

The Twelfth Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities, Hajdúszoboszló (Hungary), January 25–28, 2012

The Twelfth Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities was held in Hotel Aurum, Hajdúszoboszló, Hungary, from January 25 to 28, 2012. It was organized by the Department of Analysis of the Institute of Mathematics of the University of Debrecen.

The Winter Seminar was supported by the following organizations:

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31 participants came from the University of Debrecen (Hungary) and the Silesian University of Katowice (Poland), 15 from the previous and 16 from the latter city.

Professor Zsolt Páles opened the Seminar and welcomed the participants to Hajdúszoboszló.

The scientific talks presented at the Seminar focused on the following topics: equations in a single variable and in several variables, iterative equations, equations on algebraic structures, regularity properties of the solutions of certain functional equations, functional inequalities, Hyers–Ulam stability, functional equations and inequalities involving mean values, generalized convexity. Interesting discussions were generated by the talks.

There were profitable Problem Sessions.

The social program included a Festive Dinner. Furthermore, the participants had the opportunity to take advantage of the use of the thermal bath located in the hotel.

The closing address was given by Professor Roman Ger. His invitation to the Thirteenth Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities in January 2013 in Poland was gratefully accepted.

Summaries of the talks in alphabetic order of the authors follow in Section 1, problems and remarks in chronological order in Section 2, and the list of participants in the final section.

1. Abstracts of talks

Mihály Bessenyei: *Convex and affine separation problems II*

By standard separation theorems, if a convex and a concave function are given such that one of them is "above" the other one, then there exists an affine function between them. Moreover, the existence of an affine separator between two given functions can be characterized [4], even in the more general case when the convexity notion is induced by a Chebyshev system [1, 3]. (In fact, a characterization theorem for the existence of a convex separator is also known [2].)

The notion of convexity can be extended applying Beckenbach families. The aim of the present talk is to formulate and prove analogous results to the aboves in this framework.

References

[1] Balaj M., Wąsowicz Sz., *Haar spaces and polynomial selections*, Math. Pannon. **14** (2003), no. 1, 63–70.
[2] Baron K., Matkowski J., Nikodem K., *A sandwich with convexity*, Math. Pannon. **5** (1994), no. 1, 139–144.
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Zoltán Boros: *Problems related to radial $\mathbb{Q}$-derivatives*

Let $\mathbb{Q}^+$ denote the set of positive rational numbers, I denote an open interval in $\mathbb{R}$, $f\colon I \to \mathbb{R}$, $x_0 \in I$, and $u \in \mathbb{R}$. If the finite limit

$$d_{\mathbb{Q}} f(x_0, u) = \lim_{\mathbb{Q}^+ \ni r \to 0} \frac{f(x_0 + ru) - f(x_0)}{r}$$

exists, it is called the *radial* $\mathbb{Q}$*-derivative of* f *at* x_0 *in the direction* u. We call f *radially* $\mathbb{Q}$*-differentiable* if $d_{\mathbb{Q}}f(x_0,u)\in\mathbb{R}$ exists for every $x_0\in I$ and $u\in\mathbb{R}$. It is known, for instance, that every Jensen-convex function $f\colon I\to\mathbb{R}$ is radially $\mathbb{Q}$-differentiable [1, Theorem 4.3]. Moreover, if $f,g\colon I\to\mathbb{R}$ are Jensen-convex functions and $d_{\mathbb{Q}}f(x_0,u)=d_{\mathbb{Q}}g(x_0,u)$ for all $x_0\in I$ and $u\in\mathbb{R}$, then $f-g$ is constant (for the proof one has to combine Definition 3.7 with Theorems 4.3 and 5.8 in [1]). The assumption that f and g are Jensen-convex cannot be relaxed [1, Example 4.6]. Motivated by this example, the following question has arisen. Let us suppose that $f\colon I\to\mathbb{R}$ is such that $d_{\mathbb{Q}}f(x_0,u)=0$ for every $x_0\in I$ and $u\in\mathbb{R}$. Does, for every $x\in I$, there exist a $\mathbb{Q}$-algebraically open subset $D\subset I$ such that $x\in D$ and f is constant on D? We answer this question in the negative by an appropriate modification of the cited example.

Reference

[1] Boros Z., Páles Zs., $\mathbb{Q}$*-subdifferential of Jensen-convex functions*, J. Math. Anal. Appl. **321** (2006), 99–113.

Włodzimierz Fechner: *On a composite functional equation satisfied almost everywhere*

For a nonzero $b\in\mathbb{R}$ the b-integer part of a real number a is given by $[a]_b=b[\frac{a}{b}]$ and b-decimal part of a is defined as $(a)_b=a-[a]_b$, where $[a]$ is the largest integer not greater than a. The following composite functional equation:

$$f(f(x)+y-f(y))=f(x) \tag{1}$$

is in particular satisfied by b-integer and b-decimal parts of real numbers. The general solution of this equation can be derived from a more general studies of M.H. Hooshmand and H.K. Haili [1]. We will deal with mappings $f\colon G\to G$ acting on an Abelian group $(G,+)$ which satisfy (1) for almost all pairs $(x,y)\in G\times G$.

Reference

[1] Hooshmand M.H., Haili H.K., *Decomposer and associative functional equations*, Indag. Mathem. **18** (2007), no. 4, 539–554.

Żywilla Fechner: *On some generalizations of sine-cosine equation*

Let $(G,+)$ be a locally compact Abelian group, $\mathcal{B}(G)$ the space of all Borel subsets of G and $\mu\colon \mathcal{B}(G) \to \mathbb{C}$ a bounded regular measure. The following equation

$$\int_G \{f(x+y-s) + f(x-y+s)\}\, d\mu(s) = f(x)f(y), \quad x, y \in G \tag{1}$$

where $f\colon G \to \mathbb{C}$ is essentially bounded was introduced and solved by Z. Gajda in [1]. We are going to discuss some possible generalizations of this functional equation.

Reference

[1] Gajda Z., *A generalization of D'Alembert's Functional Equation*, Funkc. Ekvacioj **33** (1990), 69–77.

Roman Ger: *The extent of the alienation phenomenon is larger than expected*

The alienation phenomenon of ring homomorphisms may briefly be described as follows: under some reasonable assumptions, a map f between two rings satisfies the functional equation

$$f(x+y) + f(xy) = f(x) + f(y) + f(x)f(y) \tag{$*$}$$

if and only if f is both additive and multiplicative. Although this fact is surprising for itself it turns out that that kind of alienation has also deeper roots. Namely, observe that the right hand side of equation $(*)$ is of the form $Q(f(x), f(y))$ with the map $Q(u,v) = u + v + uv$ being a special *rational associative operation.* This gives rise to the following question: given an abstract rational associative operation Q does the equation

$$f(x+y) + f(xy) = Q(f(x), f(y))$$

force f to be a ring homomorphism (with the target ring being a field)?

Plainly, in general, that is not the case. Nevertheless, the 2-homogeneity of f happens to be a necessary and sufficient condition for that effect provided that the range of f is large enough.

Attila Gilányi: *On supermonomial functions* (Joint work with Csaba Gábor Kézi)

Superadditive, weakly superquadratic and strongly superquadratic functions have been investigated by several authors. It is well-known that a function $f\colon \mathbb{R} \to \mathbb{R}$ is called superadditive, if it satisfies the inequality

$$f(x+y) \geq f(x) + f(y)$$

for all $x, y \in \mathbb{R}$ (cf., e.g., [4]). A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be weakly superquadratic if it fulfils the inequality

$$f(x+y) + f(x-y) \geq 2f(x) + 2f(y)$$

for $x, y \in \mathbb{R}$ ([3], [5]). A function $f\colon [0, \infty[\to \mathbb{R}$ is strongly superquadratic if, for each $x \geq 0$, there exists a constant $c_x \in \mathbb{R}$ such that the inequality

$$f(y) - f(x) \geq c_x(y-x) + f\big(|y-x|\big)$$

holds for all nonnegative y ([1], [2]).
In the present talk, we consider higher order versions of the concepts above.

References

[1] Abramovich S., Jameson G., Sinnamon G., *Refining of Jensen's inequality*, Bull. Math. Soc. Sci. Math. Roumanie **47(95)** (2004), 3–14.
[2] Abramovich S., Jameson G., Sinnamon G., *Inequalities for averages of convex and superquadratic functions*, JIPAM, J. Inequal. Pure Appl. Math. **5** (2004), Art. 91.
[3] Kominek Z., Troczka K., *Some remarks on subquadratic functions*, Demonstratio Math. **39** (2006), 751–758.
[4] Rosenbaum R.A., *Subadditive functions*, Duke Math. J. **17** (1950), 227–242.
[5] Smajdor W., *Subadditive and subquadratic set-valued functions*, Scientific Publications of the University of Silesia, vol. 889, Katowice, 1987.

Tamás Glavosits: *The general solution of a functional equation* (Joint work with Károly Lajkó)

We will give the general and the measurable solution of the following functional equation:

$$G_1\left(x(y+1)\right) + F_1(y) = G_2\left(y(x+1)\right) + F_2(x)$$

for all $x, y \in \mathbb{R}_+$ with unknown functions $G_i, F_i\colon \mathbb{R}_+ \to \mathbb{R}$ $(i = 1, 2)$, related to the characterizations of bivariate distributions.

The functions $G_i, F_i\colon \mathbb{R}_+ \to \mathbb{R}$ $(i = 1, 2)$ satisfy the above functional equation for all $x, y \in \mathbb{R}_+$ if and only if

$$G_1(x) = \mathrm{L}_1(x) + \mathrm{A}(x) + d_1, \quad F_1(x) = \mathrm{L}_2(x) - \mathrm{L}_1(x+1) + \mathrm{A}(x) + d_3,$$
$$G_2(x) = \mathrm{L}_2(x) + \mathrm{A}(x) + d_2, \quad F_2(x) = \mathrm{L}_1(x) - \mathrm{L}_2(x+1) + \mathrm{A}(x) + d_4$$

for all $x \in \mathbb{R}_+$, where $\mathrm{L}_i\colon \mathbb{R}_+ \to \mathbb{R}$ $(i = 1,2)$ are logarithmic functions, $\mathrm{A}\colon \mathbb{R} \to \mathbb{R}$ is additive function and $d_i \in \mathbb{R}$ are constants with $d_1+d_3 = d_2+d_4$.

The measurable functions $G_i, F_i\colon \mathbb{R}_+ \to \mathbb{R}$ satisfy the above functional equation for all $x, y \in \mathbb{R}_+$ if and only if

$$G_1(x) = r\log(x) + tx + d_1, \quad F_1(x) = s\log(x) - r\log(x+1) + tx + d_3,$$
$$G_2(x) = s\log(x) + tx + d_2, \quad F_2(x) = r\log(x) - s\log(x+1) + tx + d_4$$

for all $x \in \mathbb{R}_+$, where $r, s, t, d_i \in \mathbb{R}$ $(i = 1,2,3,4)$ are constants with $d_1 + d_3 = d_2 + d_4$.

Eszter Gselmann: *On the discrete version of some elliptic partial differential equations*

In a study of steady state problems (e.g. oscillations, heat conduction, diffusion) one often arises at partial differential equations of elliptic type. The most common equations of this type are

— Laplace's equation

$$\Delta u = 0$$

— the Helmholtz equation

$$\Delta u + k^2 u = f$$

— the screened Poisson equation

$$\Delta u - k^2 u = f.$$

The three most widely used numerical methods to solve PDEs are the finite element method (FEM), the finite volume method (FVM) and the finite difference method (FDM). Using the latter method, in my talk I will concern the discrete version of the above mentioned equations.

The problem of convergence and consistency will also be dealt with.

Attila Házy: *Bernstein–Doetsch type results for (α,β,a,b)-convex functions*

In our talk we investigate the (α,β,a,b)-convex functions which is a common generalization of the usual convexity, the s-convexity in first and second sense, the h-convexity, the Godunova-Levin functions and the P-functions. This notion of convexity was introduced by Maksa and Páles in [1] in the following way: an (α,β,a,b)-convex function is defined as a function $f\colon D \to \mathbb{R}$

(where D is an open, (α,β)-convex, nonempty subset of a real or complex topological vector space) which satisfies the inequality

$$f(\alpha(t)x+\beta(t)y)\le a(t)f(x)+b(t)f(y),\quad x,y\in D, t\in[0,1].$$

The main goal of the talk is to prove some regularity and Bernstein–Doetsch type results for (α,β,a,b)-convex functions.

Reference

[1] Maksa Gy., Páles Zs., *The equality case in some recent convexity inequalities*, Opuscula Math. **31** (2011), no. 2, 269–277.

Tomasz Kochanek: *Non-trivial extensions of c_0 by reflexive spaces*

F. Cabello Sánchez and J.M.F. Castillo [1] proved that $\mathrm{Ext}(c_0,\ell_2)\neq 0$, that is, there is a Banach space Z, not isomorphic to $c_0\oplus\ell_2$, producing an exact sequence of the form

$$0\to\ell_2\to Z\to c_0\to 0$$

(the space Z may be called a non-trivial *extension* of c_0 by ℓ_2). A more general result from [1] states that $\mathrm{Ext}(c_0,X)\neq 0$ for every Banach space X, complemented in X^{**}, and being of Rademacher cotype 2. However, this does not touch the class of reflexive Banach spaces not isomorphic to a Hilbert space.

We will show that the following general statement:

$$\mathrm{Ext}(c_0,X)\neq 0 \text{ for every reflexive Banach space } X$$

may be reduced to a concrete question concerning almost additive vector measures with values in ℓ_2. The reduction goes via the Dvoretzky theorem, the Diestel–Faires theorem, and some recent results on stability for vector measures.

Reference

[1] Cabello Sánchez F., Castillo J.M.F., *Uniform boundedness and twisted sums of Banach spaces*, Houston J. Math. **30** (2004), 523–536.

Michał Lewicki: *On generators of the Taylor reminder mean*

Let $I\subseteq\mathbb{R}$ be an open interval and $n\in\mathbb{N}$ be fixed. Consider a *Taylor reminder mean of degree n* given by

$$M_n^{[f]}(x,y) = \left(f^{(n)}\right)^{-1} n! \frac{f(y) - \sum\limits_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!}(y-x)^k}{(y-x)^n}, \quad x, y \in I,\ x \neq y,$$

where $f\colon I \to \mathbb{R}$ is n-times differentiable function and $f^{(n)}$, the n-th derivative of f, is one-to-one.

In the talk I will give a sketch of the proof of the following theorem:

$$M_n^{[f]} = M_n^{[g]}$$

if and only if

$$g(x) = af(x) + W_n(x), \quad x \in I,$$

where $a \neq 0$ and $W_n(\cdot)$ is a polynomial of degree at most n.

Radosław Łukasik: *The solution and the stability of the Pexiderized K-quadratic functional equation*

In the present talk, we consider the Pexiderized K-quadratic functional equation

$$\sum_{\lambda \in K} f(x + \lambda y) = |K|\, g(x) + h(y), \quad x, y \in S,$$

where $(S, +)$ is an abelian semigroup, K is a finite subgroup of the authomorphism group of S, (the action of $\lambda \in K$ on $x \in S$ is denoted by λx) and $(H, +)$ is an abelian group.

We give the form of solutions of the above functional equation, and we present stability results connected with them.

References

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[2] Ž. Djoković D., *A representation theorem for* $(X_1 - 1)(X_2 - 1)\cdots(X_n - 1)$ *and its applications*, Ann. Polon. Math. **22** (1969), 189–198.

[3] Mazur S., Orlicz W., *Grundlegende Eigenschaften der Polynomischen Operationen I.–II.*, Studia Math. **5** (1934), 50–68 & 179–189.

[4] Sibaha M.A., Bouikhalene B., Elqorachi E., *Hyers–Ulam–Rassias stability of the K-quadratic functional equation*, J. Ineq. Pure and Appl. Math. **8** (2007), article 89.

[5] Stetkær H., *Functional equation on abelian groups with involution II.*, Aequationes Math. **55** (1998), 227–240.

[6] Stetkær H., *Functional equations involving means of functions on the complex plane*, Aequationes Math. **56** (1998), 47–62.

Judit Makó: *Approximate Hermite–Hadamard type inequalities for approximately convex functions* (Joint work with Zsolt Páles)

In this talk, approximate lower and upper Hermite–Hadamard type inequalities are obtained for functions that are approximately convex with respect to a given Chebyshev system.

Gyula Maksa: *A functional equation involving comparable weighted quasi-arithmetic means* (Joint work with Zoltán Daróczy)

In this talk, we discuss the functional equation

$$f\left(M(x,y)\right) = f\left(N(x,y)\right), \quad x, y \in I,$$

where I is a nonvoid open subinterval of the set $\mathbb{R}$ of real numbers, $f\colon I \to \mathbb{R}$ is an unknown function, and M, N are weighted quasi-arithmetic means on I. We suppose additionally that f is the product of the generating functions of M and N, respectively. The complete solutions will be described in the following two particular cases:

— the generating functions of M and N are equivalent to each other on some nonvoid open subinterval of I,

— the quasi-arithmetic means M and N are comparable on some nonvoid open subinterval of I.

Fruzsina Mészáros: *Characterization of a bivariate distribution through a functional equation* (Joint work with Károly Lajkó)

The following functional equation

$$g_1\left(\frac{x - m_1 y - c_1}{\lambda_1\left(y + a_1\right)}\right)\frac{f_Y\left(y\right)}{\lambda_1\left(y + a_1\right)} = g_2\left(\frac{y - m_2 x - c_2}{\lambda_2\left(x + a_2\right)}\right)\frac{f_X\left(x\right)}{\lambda_2\left(x + a_2\right)}$$

originates in the characterization problems of conditionally specified distributions.

We determine the density function solution of this equation satisfying for almost all $(x, y) \in E_1 \times E_2$, where

$$E_1 = \left\{x \in \mathbb{R} \mid x + a_2 > 0\right\}, \quad E_2 = \left\{y \in \mathbb{R} \mid y + a_1 > 0\right\},$$

and the measurable unknown functions g_1, g_2, f_X, f_Y are non-negative, such that they are positive on some Lebesgue measurable sets of positive Lebesgue measure.

Here m_1, m_2, c_1, c_2, $a_1, a_2 \in \mathbb{R}$, λ_1, $\lambda_2 \in \mathbb{R}_+$ are constants with the conditions $K_1 := m_1 a_1 - c_1 - a_2 \geq 0$, $K_2 := m_2 a_2 - c_2 - a_1 \geq 0$.

Janusz Morawiec: *On a functional equation connected with a problem of Nicole Brillouët-Belluot*

We determine all intervals $I \subset (0, +\infty)$, parameters $\alpha \in \mathbb{R}$ and continuous bijections $f\colon I \to I$ such that

$$f(x)f^{-1}(x) = x^{\alpha} \quad \text{for every } x \in I.$$

The case where $I \subset \mathbb{R}$ will be also investigated.

Gergő Nagy: *Isometries on positive operators via identification lemmas*

In this talk, we describe the isometries of certain spaces of positive operators acting on a complex Hilbert space H. These sets consist of those positive operators on H which, for a given real number $p \geq 1$, belong to the unit sphere of the von Neumann-Schatten p-class. They are denoted by $C_p(H)_1^+$. The von Neumann-Schatten classes equipped with the p-norm are normed spaces. For a given number $p \geq 1$, we expose the structure of the isometries of $C_p(H)_1^+$.

The so-called identification lemmas play a key role in the investigation of such isometries. Let (X, d) be a metric space and A be a subset of X with the property that any $x \in X$ is uniquely determined by the function $a \mapsto d(x, a)$ $(a \in A)$. An identification lemma concerning X and A is a statement which tell us that this feature holds. We present several such lemmata concerning $C_p(H)_1^+$ $(p \geq 1)$ and its subset, the set of rank-1 projections on H.

Agata Nowak: *On a generalization of the Gołąb–Schinzel equation*

Inspired by a problem posed by J. Matkowski in [1] we investigate the equation

$$f\left(p(x,y)(xf(y)+y) + (1-p(x,y))(yf(x)+x)\right) = f(x)f(y), \quad x, y \in \mathbb{R},$$

where functions $f\colon \mathbb{R} \to \mathbb{R}$, $p\colon \mathbb{R}^2 \to \mathbb{R}$ are assumed to be continuous.

Reference

[1] Matkowski J., *A generalization of the Gołąb–Schinzel functional equation*, Aequationes Math. **80** (2010), 181–192.

Andrzej Olbryś: *On separation by P-functions*

Let X will be a real linear space, and let $D \subset X$ will be a convex set. A function $f\colon D \to \mathbb{R}$ is called P-function if

$$\bigwedge_{x,y\in D}\bigwedge_{\lambda\in[0,1]} f(\lambda x+(1-\lambda)y)\le f(x)+f(y).$$

In our talk we establish the necessary and sufficient conditions under which two functions can be separated by P-function. These results are related to the theorem on separation by convex functions presented in [1].

References

[1] Baron K., Matkowski J., Nikodem K., *A sandwich with convexity*, Math. Pannon. **5** (1994), 139–144.
[2] Nikodem K., Pales Zs., Wąsowicz S., *Abstract separation theorems of Rode type and their applications*, Ann. Polon. Math. **72** (1999), 207–217.
[3] Pearce E.M., Rubinov A.M., *P-functions, quasi-convex functions and Hadamard-type inequalities*, J. Math. Anal. Appl. **240** (1999), 92–104.

Zsolt Páles: *On cyclic inequalities*

Given a nonempty set X, a two-variable function $F\colon X^2\to\mathbb{R}$, and a natural number $n\ge 2$, we consider cyclic inequalities of the form

$$(I_n)\qquad F(x_1,x_2)+\cdots+F(x_{n-1},x_n)+F(x_n,x_1)\ge 0,\quad x_1,\dots,x_n\in X.$$

Assuming that $F(x,x)=0$ holds for all $x\in X$, one can easily see that, for all $n\ge 2$, (I_{n+1}) implies (I_n). On the other hand, in certain particular cases (I_2) also implies (I_n). In our main results, we give several necessary and sufficient conditions for (I_n) to be valid for all $n\ge 2$.

Maciej Sablik: *Functional equations in actuarial mathematics revisited*

Let u denote a utility function, X be a random variable denoting loss, $H(X)$ – a premium paid in case of loss, and, finally, let w denote the initial wealth of insurer. Then the generalized zero utility principle under the rank-dependent utility model may be expressed as the following equation

$$(*)\qquad u(w)=E_g(u(w+H(X)-X)),$$

where $g\colon[0,1]\longrightarrow[0,1]$ is so called distorted function, and E_g denotes the Choquet integral. Following M. Kaluszka and M. Krzeszowiec [1], we ask about utility functions satisfying $(*)$.

Reference

[1] Kaluszka M., Krzeszowiec M., *Pricing insurance contracts under Cumulative Prospect Theory*, Insurance: Mathematics and Economics **50** (2012), 159–166.

László Székelyhidi: *Spectral synthesis on vector modules*

Spectral synthesis in general deals with the properties of translation invariant function spaces over topological groups. Originally the problem was to find sufficiently many basic building bricks in such function spaces. Later on it turned out that the problem is equivalent to finding finite dimensional translation invariant subspaces. In this formulation the problem can be generalized to more sophisticated situations. In our talk we exhibit problems from different fields of functional analysis, which naturally can be considered spectral synthesis problems in this generalized sense.

Patrícia Szokol: *Convex and affine separation problems I* (Joint work with Mihály Bessenyei)

By standard separation theorems, if a convex and a concave function are given such that one of them is "above" the other one, then there exists an affine function between them. Moreover, the existence of an affine/convex separator between two given functions can be characterized (consult [4] and [1]).

The notion of convexity can be extended applying so-called regular pairs. The aim of the present talk is to extend the results mentioned above for this setting. In fact, such an extension is known [2] even in the more general case when the convexity notion is induced by a two-parametered Beckenbach family [3]. However, our approach is self-contained and independent of the mentioned ones.

References

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Tomasz Szostok: *Functional equations connected to numerical differentiation*

We deal with functional equations of the form

$$g(\alpha x+\beta y)(y-x)^k = a_1 f(\alpha_1 x+\beta_1 y)+\cdots+a_n f(\alpha_n x+\beta_n y). \tag{1}$$

It is shown (under some assumptions on numbers $\alpha, \beta, \alpha_i, \beta_i, i = 1,\dots,n$) that g must be continuous. This result allows us to solve particular cases of (1) which are closely connected to formulas used in numerical analysis for estimating the derivative of a given function. Moreover similarly as it was done with equations stemming from quadrature rules we are able to describe all formulas which are exact for polynomials of some fixed degree. Using our results, we also obtain some new facts concerning equation

$$g(x_1 + \cdots + x_n) = f[x_1, \ldots, x_n]$$

($f[x_1, \ldots, x_n]$ means here the n-th divided difference of f).

László Vajday: *Functional equations on special hypergroups* (Joint work with László Székelyhidi)

We present the exact form of additive and exponential functions on special (so-called *two-point support*) hypergroups. These hypergroups are defined on nonnegative reals and the convolution is given by real-pairs. Necessary regularity properties are also presented.

Peter Volkmann: *Bounded nonlinear perturbations of continuous linear operators*

The topic will be discussed within the scope of Pólya–Szegő–Hyers–Ulam-stability.

Wirginia Wyrobek-Kochanek: *Orthogonally Pexider functions modulo a discrete subgroup*

Under appropriate conditions on Abelian topological groups G and H, an orthogonality $\perp \subset G^2$ and a σ-algebra $\mathfrak{M}$ of subsets of G we prove that if at least one of functions $f, g, h\colon G \to H$, satisfying

$$f(x+y) - g(x) - h(y) \in K \qquad \text{for } x, y \in G \text{ such that } x \perp y,$$

is continuous at a point or $\mathfrak{M}$-measurable, then there exist: a continuous additive function $A\colon G \to H$, a continuous biadditive and symmetric function $B\colon G \times G \to H$ and $a, b \in H$ such that

$$\begin{cases} f(x) - B(x,x) - A(x) - a \in K, \\ g(x) - B(x,x) - A(x) - b \in K, \\ h(x) - B(x,x) - A(x) - a + b \in K \end{cases}$$

for $x \in G$, and

$$B(x,y) = 0 \qquad \text{for } x, y \in G \text{ such that } x \perp y.$$

2. Problems and Remarks

1. Problem Let $I \subset \mathbb{R}$ be a nonvoid interval, $\varphi\colon I \to \mathbb{R}$ be a continuous, increasing function and $\lambda \in]0,1[$. The weighted quasi-arithmetic mean generated by the function φ and the weight λ is defined by

$$A_\varphi(x,y;\lambda) = \varphi^{-1}\left(\lambda\varphi(x) + (1-\lambda)\varphi(y)\right) \quad (x, y \in I)\,.$$

In [1] the following statement was proved.

Theorem. *Let $I \subset \mathbb{R}$ be a nonvoid interval, $\varphi, \psi\colon I \to \mathbb{R}$ be a continuous, increasing function and $\lambda, \mu \in]0,1[$. Then inequality*

$$A_\varphi(x,y;\lambda) \le A_\psi(x,y;\mu) \quad (x, y \in I)$$

holds in, and only if $\lambda = \mu$ and the function $\psi \circ \varphi^{-1}$ is convex.

In view of this theorem, it is obvious that in case $\lambda \neq \mu$, there is no comparability between the means $A_\varphi(\cdot,\cdot,\lambda)$ and $A_\mu(\cdot,\cdot,\mu)$. Therefore, let us consider the following inequality

$$(\mathcal{C})\quad A_\varphi(x,y;\lambda) + (1-\lambda)x + \lambda y \le A_\psi(x,y;\mu) + (1-\mu)x + \mu y \quad (x, y \in I)\,.$$

Clearly, if $\lambda = \mu$, then the above theorem can be applied to give a necessary and sufficient condition to assure that $(\mathcal{C})$ holds. For the remaining case (that is, if $\lambda \neq \mu$) find a necessary and sufficient condition which guaranties $(\mathcal{C})$ to hold.

Reference

[1] Maksa Gy., Páles Zs., *The equality case in some recent convexity inequalities*, Opuscula Math. **31** (2011), no. 2, 269–277.

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2. Remark Jointly with Roman Badora and Barbara Przebieracz the result stated below has been obtained (where the notion of Tabor groupoid is as in our paper *Stability of the Pexider functional equation*, Ann. Math. Silesianae **24** (2010), 7–13.).

Theorem. *For $f\colon G \to E$, G being a Tabor groupoid, E a Banach space, and a closed, bounded subset A of E the following two statements are equivalent:*

(P) $f = a + r$, *where* $a\colon G \to E$ *is additive and* $r(x) \in A$ $(x \in G)$.
(Q) *There are bounded subsets* B, C *of* E *such that*

$$f(x+y) - f(x) - f(y) \in B \quad (x, y \in G)$$
$$2^n f(x) - f\left(x^{2^n}\right) \in 2^n A + C \quad (x \in G, n \in \mathbb{N}).$$

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3. Problem Let $I \subset \mathbb{R}$ be a nonvoid interval, $\varphi\colon I \to \mathbb{R}$ be a continuous, increasing function and $\lambda \in]0,1[$. The weighted quasi-arithmetic mean generated by the function φ and the weight λ is defined by

$$A_\varphi(x, y; \lambda) = \varphi^{-1}\left(\lambda\varphi(x) + (1-\lambda)\varphi(y)\right) \quad (x, y \in I).$$

In my talk the following theorem was proved.

Theorem. *Let* $\varphi, \psi\colon I \to \mathbb{R}$ *be differentiable functions with everywhere positive derivatives and* $\lambda, \mu \in]0,1[$. *Then*

$$\begin{aligned} A_\varphi(x_1, x_n; \lambda) + \cdots + A_\varphi(x_{n-1}, x_n; \lambda) + A_\varphi(x_n, x_1; \lambda) \\ \geq A_\psi(x_1, x_n; \mu) + \cdots + A_\psi(x_{n-1}, x_n; \mu) + A_\psi(x_n, x_1; \mu) \end{aligned}$$

for all $n \geq 2$ *and* $x_1, \ldots, x_n \in I$ *if, and only if*

$$A_\varphi(x, y; \lambda) \geq A_\psi(x, y; \mu) + (\lambda - \mu)(x - y) \quad (x, y \in I).$$

Concerning this statement my question is whether it is possible to suppose continuity for the functions φ and ψ instead of differentiability with everywhere positive derivative?

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(Compiled by Eszter Gselmann)