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FIXED POINT APPROACH TO THE STABILITY OF AN INTEGRAL EQUATION IN THE SENSE OF ULAM–HYERS–RASSIAS

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Abstract. In this paper, by using the classical Banach contraction principle, we investigate and establish the stability in the sense of Ulam–Hyers and in the sense of Ulam–Hyers–Rassias for the integral equation which defines the mild solutions of an abstract Cauchy problem in Banach spaces.

1. Introduction

1.1. In 1940, S.M. Ulam, (see [31] and [32]) was the first to introduce the notion of stability for functional equations. More precisely, he proposed the following problem:

Given a group G_1, a metric group (G_2, d) and a positive number ε, does there exist a $\delta > 0$ such that if a function $f\colon G_1 \longrightarrow G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $T\colon G_1 \longrightarrow G_2$ such that $d(f(x), T(x)) < \varepsilon$ for all $x \in G_1$?

When this problem has a solution, we say that the homomorphisms from G_1 to G_2 are stable, or that the equation defining group homomorphisms is stable (in the sense of Ulam).

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In 1941, D.H. Hyers (see [7]) gave a partial solution of *Ulam's* problem under the assumption that G_1 and G_2 are Banach spaces.

In 1950, T. Aoki (see [2]) studied the stability problem for additive mappings by using unbounded Cauchy differences (see also [16]).

In 1978, Th.M. Rassias (see [23]) studied a similar problem. The stability considered in [23] is often called the Ulam–Hyers–Rassias stability.

In 1993, M. Obloza [18] (see also [18]) has studied Hyers stability of ordinary and linear differential equations. Works along this direction were undertaken by C. Alsina and R. Ger (see [1]), S.-M. Jun (see [11], [12], [13], [14]) and by T. Miura, S. Miyajima and S.-E. Takahasi in their joint papers: [16] and [17].

In the references, at the end of this paper, we have listed other papers dealing with the stability of functional equations.

For a large amount of information on the stability of functional equations, the reader is invited to consult the books [9] and [27] (see also the papers [24], [25], [26] and others).

1.2. Recently, ([29]), Rus has studied the stability of a general ordinary differential equation in Banach spaces. Precisely, he considered the equation

$$y'(t) = A(y(t)) + f(t, y(t)), \quad t \in I \subset \mathbb{R}, \tag{1.1}$$

(i) where $I = [a, b]$ or $[a, \infty)$,
(ii) $y \in X$, X is a Banach space,
(iii) $A\colon X \longrightarrow X$, is the infinitesimal generator of a C_0-semi-group, and
(iv) $f \in \mathcal{C}(I \times X, X)$.

By using a variant of Gronwall lemma and an existence theorem of mild solutions of the equation (1.1), Rus studied various kinds of stability for the following inequalities:

$$\|v'(t) - Av(t) - f(t, v(t))\| \leq \epsilon, \quad t \in I, \tag{1.2}$$

$$\|v'(t) - Av(t) - f(t, v(t))\| \leq \phi(t), \quad t \in I, \tag{1.3}$$

$$\|v'(t) - Av(t) - f(t, v(t))\| \leq \epsilon\phi(t), \quad t \in I, \tag{1.4}$$

where $v \in \mathcal{C}^1(I, X)$.

1.3. Let $(X, \|.\|)$ be a given (real) Banach space and $I = [0, +\infty)$ or $I = [0, T]$, where $T > 0$ is a parameter. We denote by $\mathcal{L}(X)$ the set of bounded linear maps from X to X. Let $S\colon [0,\infty) \to \mathcal{L}(X)$ be a family of bounded linear operators which form a *strongly continuous semigroup* of operators, i.e.,

$$S(t+s) = S(t)S(s),\ \forall t,s \geq 0,$$
$$S(0) = \mathrm{id},\ \text{the identity mapping}$$
$$\lim_{t\to t_0} S(t)x = S(t_0)x, \quad \forall t_0 \geq 0, \forall x \in X.$$

The family $(S(t))_{t\geq 0}$ is also called a C_0-semigroup. For such semigroups it is well known (see [21]) that there exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that

$$\|S(t)\| \leq Me^{\omega t}, \quad t \geq 0. \tag{1.5}$$

(In the sequel, we shall use $\|\cdot\|$ for both the norm in X, and the norm in $\mathcal{L}(X)$.)

The *infinitesimal generator* of the C_0-semigroup $\{S(t), t \geq 0\}$ is the linear operator $A\colon D(A) \to X$ defined by

$$Ax := \lim_{t\to 0+} \frac{1}{t}\left(S(t)x - x\right), \tag{1.6}$$

where $D(A)$ denotes the set of all $x \in X$ for which that limit exists.

We know (for more details see, [21]) that the domain $D(A)$ of A is dense and that A is closed.

For a given initial state $\psi_0 \in X$, we consider the following abstract Cauchy problem:

$$(CP):\begin{cases}\dot{\psi}(t) = A\psi(t) + u(t)F(t,\psi(t)), \quad t \in I,\\ \psi(0) = \psi_0,\end{cases}$$

where $F\colon I \times X \to X$ is a a given continuous function such that, for almost all $t \in I$, we have

$$\|F(t,x) - F(t,y)\| \leq l(t)\|x - y\|, \quad \forall x, y \in X, \tag{1.7}$$

where $l\colon [0,T] \to \mathbb{R}^+$ and $u\colon [0,T] \to \mathbb{R}$ are two given measurable functions such that l, u and lu are locally integrable on I.

In general, even if u and l are continuous, the problem (CP) may have no solutions. Moreover, as in [21], one can show that if a classical solution ψ exists then it will be given by

$$\psi(t) = S(t)\psi_0 + \int_0^t u(s)\, S(t-s)F(s,\psi(s))ds, \quad \forall t \in I. \tag{1.8}$$

From [21], we recall the following definition.

DEFINITION 1.1. A continous function ψ which is a solution of the integral equation (1.8) is called a *mild solution* of problem (CP).

The purpose of this paper is to study the stability of the solutions of the integral equation (1.8) by using the classical Banach contraction principle.

Inspired by the paper [29], we introduce four definitions of stabilities which are essentially variants of the stability the sense of Hyers–Ulam and in the sense of Hyers–Ulam–Rassias or their generalizations.

This paper is organized as follows.

In the second section we present some definitions and remarks that will be used in this paper.

The third section is devoted to the study of stability of the integral equation (1.8) on a finite interval $[0, T]$. This kind of stability (see Definition 2.1) is essentially in the sense of Ulam–Hyers. The main result of this section is Theorem 3.1.

The fourth section is devoted to the study of stability of the integral equation (1.8) on a finite interval $[0, T]$ according to Definition 2.4. This kind of stability is essentially in the sense of Hyers–Ulam–Rassias. The main result of this section is Theorem 4.1. We end this section by making some comments concerning the connections between Theorem 3.1 and Theorem 4.1.

In Section 5, we prove the stability of the integral equation (1.8) on $[0, +\infty)$ according to Definition 2.1. The main result of this section is Theorem 5.1.

In Section 6, we investigate the stability of the integral equation (1.8) on $[0, +\infty)$ according to Definition 2.4. The main result of this section is Theorem 6.1.

We point out that a Fixed point method (see [5]) was used to investigate the stability of several functional equations. Works along these lines are achieved by L. Cădariu and V. Radu (see [4] and [22]). Fixed point methods were also used to study the stability of differential equations. (See [14] and other related papers).

2. Definitions and preliminaries

Let $I = [0, T]$ or $[0, +\infty)$ and let $(X, \|.\|)$ be a Banach space. The set of all continuous functions from I to X will be denoted by $\mathcal{E} := \mathcal{C}(I, X)$.

For a given $\psi_0 \in X$ and any $\psi \in \mathcal{E}$, we set

$$\Lambda(\psi)(t) := S(t)\psi_0 + \int_0^t u(s)S(t-s)F(s,\psi(s))ds$$

for all $t \in I$.

Here, l, u are measurable functions such that l, u and the product lu are locally integrable. With the assumptions made above on F, it is easy to see that the map $\psi \mapsto \Lambda(\psi)$ is a self-mapping of the space $\mathcal{E}$.

For any given $\psi_0 \in X$, $\epsilon > 0$ and $G \in \mathcal{C}(I, (0, +\infty))$, we consider the following equation

$$\psi(t) = \Lambda(\psi)(t), \quad t \in I \tag{2.1}$$

and the following inequalities:

$$\|\psi(t) - \Lambda(\psi)(t)\| \leq \epsilon, \quad t \in I, \tag{2.2}$$

$$\|\psi(t) - \Lambda(\psi)(t)\| \leq G(t), \quad t \in I, \tag{2.3}$$

where the unknown function ψ is in $\mathcal{C}(I, X)$.

As in [29], we introduce the following definitions.

Definition 2.1. The integral equation (2.1) is *Ulam–Hyers stable* if there exists a real number $c > 0$ such that for each $\epsilon > 0$ and for each solution $\psi \in \mathcal{C}(I, X)$ of (2.2) there exists a solution $v \in \mathcal{C}(I, X)$ of (2.1) such that

$$\|\psi(t) - v(t)\| \leq c\epsilon, \quad \forall t \in I.$$

Definition 2.2. The integral equation (2.1) is *generalized Ulam–Hyers stable* if there exists $\theta \in \mathcal{C}([0, +\infty), [0, +\infty))$, $\theta(0) = 0$, such that for each $\epsilon > 0$ and for each solution $\psi \in \mathcal{C}(I, X)$ of (2.2) there exists a solution $v \in \mathcal{C}(I, X)$ of (2.1) such that

$$\|\psi(t) - v(t)\| \leq \theta(\epsilon), \quad \forall t \in I.$$

Definition 2.3. The integral equation (2.1) is *generalized Ulam–Hyers–Rassias stable* with respect to $G \in \mathcal{C}([0, +\infty), (0, +\infty))$, if there exists $c_G > 0$ such that for each solution $\psi \in \mathcal{C}(I, X)$ of (2.3) there exists a solution $v \in \mathcal{C}(I, X)$ of (2.1) such that

$$\|\psi(t) - v(t)\| \leq c_G\, G(t), \quad \forall t \in I.$$

In the sequel, we are interested by the stability of the equation (2.1) in the sense of Definition 2.1 and Definition 2.3.

3. Ulam–Hyers stability of equation (2.1) on a finite interval

Let $(S(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space $(X, \|\cdot\|)$. Let $T > 0$ be a given positive real number. The set of all continuous functions from $[0, T]$ to X will be denoted by $\mathcal{E}$. For any functions $\phi, \psi \in \mathcal{E}$, we denote

$$\|\phi - \psi\|_\infty = d_\infty(\phi, \psi) := \sup\{\|\phi(t) - \psi(t)\| : t \in [0, T]\}.$$

For any continuous function $\psi\colon [0, T] \to X$, we recall that

$$\Lambda(\psi)(t) := S(t)\psi_0 + \int_0^t u(s)S(t-s)F(s, \psi(s))ds, \quad \forall t \in [0, T],$$

with a fixed $\psi_0 \in X$.

We start by providing a sufficient condition ensuring the Ulam–Hyers stability of the integral equation (2.1) on the finite interval $[0, T]$.

THEOREM 3.1. *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space $(X, \|\cdot\|)$ and let $T > 0$ be a positive real number. We set*

$$\lambda := M \int_0^T l(s)|u(s)|e^{\omega(T-s)}ds.$$

If $\lambda < 1$, then the integral equation (2.1) is stable in the sense of Ulam–Hyers.

PROOF. Suppose that $\lambda < 1$. Let $\varepsilon > 0$ be given. For any arbitrary functions $\phi, \psi \in \mathcal{E}$, we have the following inequalities:

$$\begin{aligned}
\|(\Lambda\phi)(t) - (\Lambda\psi)(t)\| &= \left\| \int_0^t u(s)S(t-s)(F(s, \phi(s)) - F(s, \psi(s)))ds \right\| \\
&\leq \int_0^t |u(s)| \|S(t-s)\| \|F(s, \phi(s)) - F(s, \psi(s))\| ds \\
&\leq \int_0^t |u(s)| M e^{\omega(t-s)} l(s) \|\phi(s) - \psi(s)\| ds \\
&\leq \left[M \int_0^T e^{\omega(T-s)} |u(s)| l(s)\, ds \right] d_\infty(\phi, \psi) \\
&\leq \lambda d_\infty(\phi, \psi),
\end{aligned}$$

for all $t \in [0, T]$. Therefore, we have

$$d_\infty(\Lambda(\phi), \Lambda(\psi)) \leq \lambda d_\infty(\phi, \psi).$$

So, Λ is a contraction.

Let $\theta \in \mathcal{E}$ be such that

$$d_\infty(\theta, \Lambda(\theta)) \leq \varepsilon.$$

Let $\phi \in \mathcal{E}$ be such that $d_\infty(\theta, \phi) \leq \frac{\varepsilon}{1-\lambda}$. Then we have

$$d_\infty(\theta, \Lambda(\phi)) \leq d_\infty(\theta, \Lambda(\theta)) + d_\infty(\Lambda(\theta), \Lambda(\phi))) \leq \varepsilon + \frac{\lambda}{1-\lambda}\varepsilon = \frac{\varepsilon}{1-\lambda}.$$

Hence the closed ball $\bar{B}_{\mathcal{E}}\left(\theta, \frac{\varepsilon}{1-\lambda}\right)$ of the Banach space $\mathcal{E}$ is invariant by the map Λ. That is

$$\Lambda\left(\bar{B}_{\mathcal{E}}\left(\theta, \frac{\varepsilon}{1-\lambda}\right)\right) \subset \bar{B}_{\mathcal{E}}\left(\theta, \frac{\varepsilon}{1-\lambda}\right).$$

By applying the Banach contraction principle to the self-mapping Λ acting in the complete subspace $\bar{B}_{\mathcal{E}}\left(\theta, \frac{\varepsilon}{1-\lambda}\right)$, we deduce that there exists a unique element $\psi \in \bar{B}_{\mathcal{E}}\left(\theta, \frac{\varepsilon}{1-\lambda}\right)$ such that $\psi = \Lambda(\psi)$. Thus, ψ is a solution of the integral equation (2.1) which satisfies

$$d_\infty(\theta, \psi) \leq \frac{\varepsilon}{1-\lambda}.$$

That is

$$\|\theta(t) - \psi(t)\| \leq \frac{1}{1-\lambda}\varepsilon = c\,\varepsilon, \quad \forall t \in [0, T],$$

where $c := \frac{1}{1-\lambda}$, which shows that the integral equation (2.1) is stable in the sense of Ulam–Hyers and completes the proof. $\square$

4. Ulam–Hyers–Rassias stability of equation (2.1) on a finite interval

In this section, we study the Ulam–Hyers–Rassias stability of equation (2.1) on a finite interval. Our second main result reads as follows.

THEOREM 4.1. *Let $(X, \|\cdot\|)$ be a (real) Banach space and let $(S(t))_{t\geq 0}$ be a C_0-semigroup on X. Let $M \geq 1, w \geq 0$ be constants such that $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. Let $\psi_0 \in X$ be fixed and let $T > 0$ be a given positive number. Let $G\colon [0,T] \to (0,\infty)$ be a continuous function.*

Suppose that a continuous function $f\colon [0,T] \to X$ satisfies

$$\left\| f(t) - S(t)\psi_0 - \int_0^t u(s)S(t-s)F(s,f(s))ds \right\| \leq G(t), \quad \forall t \in [0,T]. \tag{4.1}$$

Suppose that there exists a positive constant D such that

$$l(s)|u(s)|e^{\omega(T-s)} \leq D, \quad \text{for almost all} \quad s \in [0,T]. \tag{4.2}$$

Then there exist a constant $c_G > 0$ and a unique continuous function $v\colon [0,T] \to X$ such that

$$v(t) = S(t)\psi_0 + \int_0^t u(s)S(t-s)F(s,v(s))ds, \quad \forall t \in [0,T], \tag{4.3}$$

and

$$\|f(t) - v(t)\| \leq c_G\, G(t), \quad \forall t \in [0,T]. \tag{4.4}$$

PROOF. We recall that $\mathcal{E}$ is the set of all continuous functions from $[0,T]$ to X. Let $K > 0$ be such that

$$MKD < 1. \tag{4.5}$$

We choose a continuous function $\phi\colon [0,T] \to (0,\infty)$ such that

$$\int_0^t \phi(s)ds \leq K\,\phi(t), \quad \forall t \in [0,T]. \tag{4.6}$$

Such functions exist.

Let f and G satisfy the inequality (4.1). Let α_G and β_G be two positive numbers such that

$$\alpha_G \phi(t) \leq G(t) \leq \beta_G \phi(t), \quad \forall t \in [0,T]. \tag{4.7}$$

For all $h, g \in \mathcal{E}$, we set

$$d_\phi(h,g) := \inf\{C \in [0,\infty) : \|h(t) - g(t)\| \leq C\phi(t), \, \forall t \in [0,T]\}.$$

It is easy to see that $(\mathcal{E}, d_\phi)$ is a metric space and that $(\mathcal{E}, d_\phi)$ is complete.

Now, consider the operator $\Lambda \colon \mathcal{E} \to \mathcal{E}$ defined by

$$(\Lambda h)(t) := S(t)\psi_0 + \int_0^t u(s)S(t-s)F(s,h(s))ds, \quad \forall t \in [0,T].$$

We prove that Λ is strictly contractive on the metric space $(\mathcal{E}, d_\phi)$. Indeed, let $h, g \in \mathcal{E}$ and let $C(h,g) \in [0,\infty)$ be an arbitrary constant such that $\|h(t) - g(t)\| \leq C(h,g)\phi(t)$, $\forall t \in [0,T]$. Then, by using (1.5), (4.2) and (4.6), we have the following inequalities:

$$\begin{aligned}
\|(\Lambda h)(t) - (\Lambda g)(t)\| &= \left\| \int_0^t u(s)S(t-s)(F(s,h(s)) - F(s,g(s)))ds \right\| \\
&\leq \int_0^t |u(s)| \|S(t-s)\| \|F(s,h(s)) - F(s,g(s))\| ds \\
&\leq \int_0^t |u(s)| M e^{\omega(t-s)} l(s) \|h(s) - g(s)\| ds \\
&\leq MC(f,g) \int_0^t l(s)|u(s)|\phi(s) e^{\omega(t-s)} ds \\
&\leq MC(f,g) \int_0^t l(s)|u(s)|\phi(s) e^{\omega(T-s)} ds \\
&\leq MC(f,g)D \int_0^t \phi(s) ds \\
&\leq C(f,g)MDK\phi(t), \quad \text{for all} \quad t \in [0,T].
\end{aligned}$$

Therefore, we have $d_\phi(\Lambda(h), \Lambda(g)) \leq MDKC(h,g)$, from which we deduce that

$$d_\phi(\Lambda(h), \Lambda(g)) \leq MDK d_\phi(h,g).$$

Since $MDK < 1$, it follows that Λ is strictly contractive on the metric space $(\mathcal{E}, d_\phi)$. By the Banach fixed point principle, there exits a unique function (say) v in $\mathcal{E}$ such that $v = \Lambda(v)$.

By the triangle inequality, we have

$$d_\phi(f,v) \leq d_\phi(f,\Lambda(f)) + d_\phi(\Lambda(f),\Lambda(v))) \leq \beta_G + MDKd_\phi(f,v),$$

which implies that

$$d_\phi(f,v) \leq \frac{\beta_G}{1-MDK},$$

from which, we deduce the following inequality

$$\begin{aligned} \|f(t)-v(t)\| &\leq \frac{\beta_G}{1-MDK}\phi(t) \\ &\leq \frac{\beta_G}{1-MDK}\frac{G(t)}{\alpha_G} \leq c_G\, G(t), \quad \forall t \in [0,T], \end{aligned} \tag{4.8}$$

where

$$c_G := \frac{\beta_G}{(1-MDK)\alpha_G},$$

which is the desired inequality (4.4). □

The remainder of this section is devoted to some comments concerning connections between Theorem 3.1 and Theorem 4.1.

We start with the following immediate consequence of Theorem 4.1 which is a variant of Theorem 3.1.

Corollary 4.1. *Let $(X, \|\cdot\|)$ be a (real) Banach space and let $(S(t))_{t\geq 0}$ be a C_0-semigroup on X. Let $M \geq 1, w \geq 0$ be constants such that $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. We suppose that there exists a positive constant D such that*

$$l(s)|u(s)|e^{\omega(T-s)} \leq D, \quad \text{for almost all} \quad s \in [0,T]. \tag{4.9}$$

Then the integral equation (2.1) is stable in the sense of Ulam–Hyers.

This corollary is obtained from Theorem 4.1 by setting $G(t) = \varepsilon$, for all $t \in [0,T]$.

Remark 4.1. Suppose that (4.9) is satisfied, then we have

$$\lambda := M \int_0^T l(s)|u(s)|e^{\omega(T-s)}\, ds \le MTD.$$

If $MTD < 1$, then we can apply Theorem 3.1 to deduce a particular case of Theorem 4.1.

Remark 4.2. (i) Theorem 4.1 is based on the boundedness of the function $s \mapsto l(s)|u(s)|e^{\omega(T-s)}$ on the interval $[0,T]$ while Theorem 3.1 is based on its Lebesgue-integrability on $[0,T]$.

(ii) It turns out that from Theorem 3.1, we can derive the following variant of Theorem 4.1.

Theorem 4.2. *Let $(X, \|\cdot\|)$ be a (real) Banach space and let $(S(t))_{t\ge 0}$ be a C_0-semigroup on X. Let $M \ge 1, w \ge 0$ be constants such that $\|S(t)\| \le Me^{\omega t}$ for all $t \ge 0$. Let $\psi_0 \in X$ be fixed and let $T > 0$ be a given positive number. Let $G\colon [0,T] \to (0,\infty)$ be a continuous function.*

Suppose that a continuous function $f\colon [0,T] \to X$ satisfies

$$\left\| f(t) - S(t)\psi_0 - \int_0^t u(s)S(t-s)F(s,f(s))ds \right\| \le G(t), \quad \forall t \in [0,T]. \tag{4.10}$$

Suppose that

$$\lambda := M \int_0^T l(s)|u(s)|e^{\omega(T-s)}\, ds < 1. \tag{4.11}$$

Then there exist a constant $c_G > 0$ and a unique continuous function $v\colon [0,T] \to X$ such that

$$v(t) = S(t)\psi_0 + \int_0^t u(s)S(t-s)F(s,v(s))ds, \quad \forall t \in [0,T], \tag{4.12}$$

and

$$\|f(t) - v(t)\| \le c_G\, G(t), \quad \forall t \in [0,T]. \tag{4.13}$$

Proof. We set $\alpha := \inf G([0,T])$ and $\varepsilon := \sup G([0,T])$. We have

$$0 < \alpha \leq G(t) \leq \varepsilon, \quad \forall t \in [0,T]. \tag{4.14}$$

If (4.11) is assumed and if $f \in \mathcal{C}([0,T],X)$ satisfies (4.10), then from Theorem 3.1, it follows that there exist a unique continuous function $v\colon [0,T] \to X$ such that $(\Lambda v)(t) = v(t)$ for all $t \in [0,T]$ (i.e., (4.12) holds true) and satisfying

$$\|f(t) - v(t)\| \leq \frac{\varepsilon}{1-\lambda}, \quad \forall t \in [0,T]. \tag{4.15}$$

From (4.15), we obtain the following inequality

$$\|f(t) - v(t)\| \leq c_G G(t), \quad \forall t \in [0,T], \tag{4.16}$$

where $c_G := \dfrac{\varepsilon}{(1-\lambda)\alpha}$. This completes the proof. □

5. Ulam–Hyers stability of equation (2.1) on $[0,+\infty)$

We keep the notations and assumptions of subsection 1.3. The aim of this section is to investigate conditions ensuring the stability of equation (2.1) on $[0,+\infty)$ in the sense of Definition 2.1. The main result of this section is as follows.

Theorem 5.1. *Let $(X, \|\cdot\|)$ be a (real) Banach space and let $(S(t))_{t\geq 0}$ be a C_0-semigroup on X. Let $\psi_0 \in X$ be fixed and let $\varepsilon > 0$ be a given positive number. Suppose that a continuous function $f\colon [0,+\infty) \to X$ satisfies*

$$\left\| f(t) - S(t)\psi_0 - \int_0^t u(s)S(t-s)F(s,f(s))ds \right\| \leq \varepsilon, \quad \forall t \in [0,+\infty). \tag{5.1}$$

Suppose that

$$\lambda_\infty = \sup_{t\geq 0} \int_0^t l(s)|u(s)|\,\|S(t-s)\|\,ds < 1. \tag{5.2}$$

Then there exists a unique continuous function $v\colon [0,+\infty) \to X$ such that

$$v(t) = S(t)\psi_0 + \int_0^t u(s)S(t-s)F(s,v(s))ds, \quad \forall t \in [0,+\infty) \tag{5.3}$$

and

$$\|f(t) - v(t)\| \leq \frac{\varepsilon}{1-\lambda_\infty} \quad \forall t \in [0, +\infty). \tag{5.4}$$

PROOF. Suppose that $\lambda_\infty < 1$. Let $\varepsilon > 0$ be given. Let $f \in \mathcal{C}([0, +\infty), X)$ satisfy the inequality (5.1). We consider the set $\mathcal{E}_f$ defined by

$$\mathcal{E}_f := \{g \in \mathcal{C}([0, +\infty), X) : \sup_{t \geq 0} \|g(t) - f(t)\| < +\infty\}$$

The set $\mathcal{E}_f$ is not empty, because it contains f and $\Lambda(f)$. For any arbitrary functions $h, g \in \mathcal{E}_f$, we set

$$d_\infty(h, g) := \sup_{t \geq 0} \|h(t) - g(t)\|.$$

Then d_∞ is a distance and the metric space $(\mathcal{E}_f, d_\infty)$ is complete.

For any functions $h, g \in \mathcal{E}_f$, we have the following inequalities:

$$\begin{aligned}
\|(\Lambda h)(t) - (\Lambda g)(t)\| &= \left\| \int_0^t u(s) S(t-s)(F(s, h(s)) - F(s, g(s))) ds \right\| \\
&\leq \int_0^t |u(s)| \|S(t-s)\| \|F(s, h(s)) - F(s, g(s))\| ds \\
&\leq \int_0^t |u(s)| \|S(t-s)\| l(s) \|h(s) - g(s)\| ds \\
&\leq \left[\int_0^t l(s) |u(s)| \|S(t-s)\| \, ds \right] d_\infty(h, g) \\
&\leq \lambda_\infty d_\infty(h, g),
\end{aligned}$$

for all $t \in [0, +\infty)$. Therefore, we have

$$d_\infty(\Lambda(h), \Lambda(g)) \leq \lambda_\infty d_\infty(h, g).$$

Moreover, it is easy to show that $\Lambda(h) \in \mathcal{E}_f$ for any function $h \in \mathcal{E}_f$. So, Λ is a contraction of the complete metric space $(\mathcal{E}_f, d_\infty)$. By applying the Banach contraction principle, we deduce that there exists a unique element $v \in \mathcal{E}_f$ such that $v = \Lambda(v)$. By the triangle inequality, we have

$$d_\infty(f, v) \leq d_\infty(f, \Lambda(f)) + d_\infty(\Lambda(f), \Lambda(v))) \leq \varepsilon + \lambda_\infty d_\infty(f, v),$$

from which, we deduce the following inequality

$$d_\infty(f,v) \leq \frac{\varepsilon}{1-\lambda_\infty}.$$

That is

$$\|f(t)-v(t)\| \leq \frac{1}{1-\lambda_\infty}\varepsilon = c\,\varepsilon, \quad \forall t \in [0,+\infty), \tag{5.5}$$

where $c := \frac{1}{1-\lambda_\infty}$. The inequality (5.5) shows that the integral equation (2.1) is stable in the sense of Definition 2.1. This ends the proof. □

6. Ulam–Hyers–Rassias stability of equation (2.1) on $[0,\infty)$

The purpose of this section is to study the Ulam–Hyers–Rassias stability of equation (2.1) on $[0,\infty)$. The main result of this section reads as follows.

THEOREM 6.1. *Let X be a (real) Banach space and let $(S(t))_{t\geq 0}$ be a C_0-semigroup on X. Let $\psi_0 \in X$ be fixed. Let $K > 0$ be given and let $\phi\colon [0,\infty) \to (0,\infty)$ be a continuous function such that*

$$\int_0^t \phi(s)ds \leq K\,\phi(t), \quad \forall t \in [0,\infty). \tag{6.1}$$

Suppose that a continuous function $f\colon [0,\infty) \to X$ satisfies

$$\left\| f(t) - S(t)\psi_0 - \int_0^t u(s)S(t-s)F(s,f(s))ds \right\| \leq \phi(t), \quad \forall t \in [0,+\infty). \tag{6.2}$$

Suppose that there exists a positive constant $D > 0$ such that

$$l(s)|u(s)|\,\|S(t-s)\| \leq D, \tag{6.3}$$

for almost all $(s,t) \in [0,\infty)$ with $0 \leq s \leq t$ and suppose that

$$KD < 1. \tag{6.4}$$

Then there exists a unique continuous function $v\colon [0,\infty) \to X$ *such that*

$$v(t) = S(t)\psi_0 + \int_0^t u(s)S(t-s)F(s,v(s))ds, \quad \forall t \in [0,\infty), \tag{6.5}$$

and

$$\|f(t) - v(t)\| \le \frac{1}{1-KD}\,\phi(t), \quad \forall t \in [0,\infty). \tag{6.6}$$

Proof. We recall that $\mathcal{E}$ is the set of all continuous functions from $[0,\infty)$ to X. Let $f \in \mathcal{C}([0,+\infty))$ satisfy the inequality (6.2). We consider the set $\mathcal{E}_f$ defined by

$$\mathcal{E}_f := \{g \in \mathcal{C}([0,+\infty),X) : \exists C \ge 0 \ \|g(t) - f(t)\| \le C\phi(t),\ \forall t \in [0,+\infty)\}.$$

The set $\mathcal{E}_f$ is not empty, because it contains f and $\Lambda(f)$.

For any arbitrary functions $h, g \in \mathcal{E}_f$, we set

$$d_\phi(h,g) := \inf\{C \in [0,\infty) : \|h(t) - g(t)\| \le C\phi(t),\ \forall t \in [0,\infty)\}.$$

It is easy to see that $(\mathcal{E}, d_\phi)$ is a complete metric space satisfying $\Lambda(\mathcal{E}_f) \subset \mathcal{E}_f$, where $\Lambda\colon \mathcal{E}_f \to \mathcal{E}_f$ is defined by

$$(\Lambda h)(t) := S(t)\psi_0 + \int_0^t u(s)S(t-s)F(s,h(s))ds, \quad \forall t \in [0,\infty).$$

We prove that Λ is strictly contractive on the metric space $(\mathcal{E}_f, d_\phi)$. Indeed, let $h, g \in \mathcal{E}_f$ and let $C(h,g) \in [0,\infty)$ be an arbitrary constant such that $\|h(t) - g(t)\| \le C(h,g)\phi(t)$, $\forall t \in [0,+\infty)$. Observe that we have the following inequalities:

$$\begin{aligned}
\|(\Lambda h)(t) - (\Lambda g)(t)\| &= \left\| \int_0^t u(s)S(t-s)(F(s,h(s)) - F(s,g(s)))ds \right\| \\
&\le \int_0^t |u(s)|\|S(t-s)\|\|F(s,h(s)) - F(s,g(s))\|ds \\
&\le \int_0^t |u(s)|\|S(t-s)\|l(s)\|h(s) - g(s)\|ds \\
&\le C(h,g) \int_0^t l(s)|u(s)|\|S(t-s)\|\phi(s)ds.
\end{aligned}$$

Then, by using (6.3) and (6.1), we obtain

$$\begin{aligned}\|(\Lambda h)(t)-(\Lambda g)(t)\| &\leq C(h,g)\int_0^t l(s)|u(s)|\|S(t-s)\|\phi(s)ds \\ &\leq C(h,g)D\int_0^t \phi(s)ds \\ &\leq C(h,g)DK\phi(t), \quad \text{for all} \quad t\in[0,\infty).\end{aligned}$$

Therefore, we have $d_\phi(\Lambda(h),\Lambda(g)) \leq DKC(h,g)$, from which we deduce that

$$d_\phi(\Lambda(h),\Lambda(g)) \leq DKd_\phi(h,g).$$

By assumption (6.4), $DK < 1$. Hence, the self-mapping Λ is strictly contractive on the metric space $(\mathcal{E}_f, d_\phi)$.

By the Banach fixed point principle, there exits a unique function (say) v in $\mathcal{E}_f$ such that $v = \Lambda(v)$. By the triangle inequality, we have

$$d_\phi(f,v) \leq d_\phi(f,\Lambda(f)) + d_\phi(\Lambda(f),\Lambda(v))) \leq 1 + DKd_\phi(f,v),$$

which implies that

$$d_\phi(f,v) \leq \frac{1}{1-DK},$$

from which, we deduce the following inequality

$$\|f(t)-v(t)\| \leq \frac{1}{1-DK}\phi(t) = c_\phi\,\phi(t) \quad \forall t\in[0,\infty), \tag{6.7}$$

where

$$c_\phi := \frac{1}{1-DK}.$$

(6.7) is the desired inequality. This completes the proof. □

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References

[1] Alsina C., Ger R., *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl. **2** (1998), no. 4, 373–380.

[2] Aoki T., *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan. **2** (1950), 64–66.

[3] Bourgin D.G., *Approximately isometric and multiplicative transformations on continuous function rings*, Duke Math. J. **16** (1949), 385–397.

[4] Cădariu L., Radu V., *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math. **4** (2003), no. 1, Art. ID 4.

[5] Diaz J.B., Margolis B., *A fixed point theorem of the alternative, for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.

[6] Gajda Z., *On stability of additive mappings*, Internat. J. Math. Math. Sci. **14** (1991), 431–434.

[7] Hyers D.H., *On the stability of the linear functional equation,* Proc. Natl. Acad. Sci. USA **27** (1941), 222–224.

[8] Hyers D.H., *The stability of homomorphisms and related topics,* Global gnalysis–analysis on manifolds (T.M. Rassias ed.), Teubner-Texte Math., B. 57, Teubner, Leipzig, 1983, pp. 140–153.

[9] Hyers D.H., Isac G., Rassias Th.M., *Stability of Functional Equation in Several Variables,* Birkhäuser, Basel, 1998.

[10] Jung S.-M., *Hyers–Ulam–Rassias stability of Jensen's equation and its application*, Proc. Amer. Math. Soc. **126** (1998), 3137–3143.

[11] Jung S.-M., *Hyers–Ulam stability of linear differential equations of first order*, Appl. Math. Lett. **17** (2004), no. 10, 1135–1140.

[12] Jung S.-M., *Hyers–Ulam stability of linear differential equations of first order. II*, Appl. Math. Lett. **19** (2006), no. 9, 854–858.

[13] Jung S.-M., *Hyers–Ulam stability of a system of first order linear differential equations with constant coefficients*, J. Math. Anal. Appl. **320** (2006), no. 2, 549–561.

[14] Jung S.-M., *A fixed point approach to the stability of differential equations $y' = F(x, y)$*, Bull. Malays. Math. Sci. Soc. (2) **33** (2010), no. 1, 47–56.

[15] Kuczma M., *Functional equations in a single variable,* Monographs math., vol. 46, PWN, Warszawa, 1968.

[16] Miura T., Miyajima S., Takahasi S.-E., *A characterization of Hyers–Ulam stability of first order linear differential operators*, J. Math. Anal. Appl. **286** (2003), no. 1, 136–146.

[17] Miura T., Miyajima S., Takahasi S.-E., *Hyers–Ulam stability of linear differential operator with constant coefficients*, Math. Nachr. **258** (2003), 90–96.

[18] Obloza M., *Hyers stability of the linear differential equation*, Rocznik Nauk.-Dydakt. Prace Mat. **13** (1993), 259–270.

[19] Obloza M., *Connections between Hyers and Lyapunov stability of the ordinary differential equations*, Rocznik Nauk.-Dydakt. Prace Mat. **14** (1997), 141–146.

[20] Park C., *Hyers–Ulam–Rassias stability of homomorphisms in quasi-Banach algebras*, Bull. Sci. Math. **132** (2008), 87–96.

[21] Pazy A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[22] Radu V., *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), no. 1, 91–96.

[23] Rassias Th.M., *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[24] Rassias Th.M., *The problem of S. M. Ulam for approximately multiplicative mappings,* J. Math. Anal. Appl. **246** (2000), 352–378.

[25] Rassias Th.M., *On the stability of functional equations in Banach spaces,* J. Math. Anal. Appl. **251** (2000), 264–284.
[26] Rassias Th.M., *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62** (2000), 23–130.
[27] Rassias Th.M., *Functional Equations, Inequalities and Applications,* Kluwer Academic, Dordrecht, Boston and London, 2003.
[28] Rus I.A., *Metrical fixed point theorems,* University of Cluj-Napoca, Department of Mathematics, 1979.
[29] Rus I.A., *Ulam stabilities of ordinary differential equations in a Banach space,* Carpathian J. Math. **26** (2010), no. 1, 103–107.
[30] Takahasi S.-E., Miura T., Miyajima S., *On the Hyers–Ulam stability of the Banach space valued differential equation* $y' = \lambda y$, Bull. Korean Math. Soc. **39** (2002), no. 2, 309–315.
[31] Ulam S.M., *Problems in Modern Mathematics*, Chapter VI, Science Editions Wiley, New York, 1960.
[32] Ulam S.M., *A Collection of the Mathematical Problems,* Interscience Publ., New York, 1960.
[33] Wang J., *Some further generalizations of the Hyers–Ulam Rassias stability of functional equations*, J. Math. Anal. Appl. **263** (2001), 406–423.

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