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ON CONTINUOUS INVOLUTIONS AND HAMEL BASES

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Abstract. We provide the form of all continuous involutions f of a locally convex linear topological space X such that $H \cap f(H) = \emptyset$ for any basis H of the vector space X over the rationals.

1. The case of reals

By a Hamel basis of $\mathbb{R}$ we mean (see [1, p. 82]) a basis of the vector space $\mathbb{R}$ over the field $\mathbb{Q}$ of rationals. Answering a (private) question of Bartłomiej Ulewicz we provide the form of all continuous involutions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$H \cap f(H) = \emptyset \quad \text{for any Hamel basis } H \text{ of } \mathbb{R}. \tag{1}$$

We start with a more general theorem.

Theorem 1. *If $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is continuous, then* (1) *holds if and only if there are $a, b \in \mathbb{Q} \setminus \{1\}$ such that*

$$f(x) = \begin{cases} ax & \text{for } x \in (-\infty, 0), \\ bx & \text{for } x \in (0, \infty). \end{cases} \tag{2}$$

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Proof. First we will observe that if f is not of the form (2) with $a, b \in \mathbb{Q} \setminus \{1\}$, then either

$$\text{there is an } x_0 \in \mathbb{R} \setminus \{0\} \text{ such that } \frac{f(x_0)}{x_0} \notin \mathbb{Q} \tag{3}$$

or

$$f(x) = x \quad \text{for } x \in (-\infty, 0) \quad \text{or} \quad f(x) = x \quad \text{for } x \in (0, \infty). \tag{4}$$

In fact, the negation of (3) and the continuity of f gives (2) with some rationals a, b and, consequently, $a = 1$ or $b = 1$.

Assume (3). Then x_0 and $f(x_0)$ are linearly independent over $\mathbb{Q}$ for an $x_0 \in \mathbb{R} \setminus \{0\}$ and so there is a Hamel basis H of $\mathbb{R}$ containing both x_0 and $f(x_0)$. In particular $f(x_0) \in H \cap f(H)$ and (1) does not hold.

Clearly also (4) implies that (1) does not hold. Consequently (1) gives (2) with some $a, b \in \mathbb{Q} \setminus \{1\}$.

Assume now (2) with some $a, b \in \mathbb{Q} \setminus \{1\}$ and suppose $H \cap f(H) \neq \emptyset$ for a Hamel basis H of $\mathbb{R}$. Then $f(h) \in H$ for an $h \in H$, whence, according to (2), $ch \in H$ for a $c \in \mathbb{Q} \setminus \{1\}$, a contradiction. □

Corollary 1. *If $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and*

$$f(f(x)) = x \quad \textit{for } x \in \mathbb{R}, \tag{5}$$

then (1) *holds if and only if there is a $c \in (-\infty, 0) \cap \mathbb{Q}$ such that*

$$f(x) = \begin{cases} cx & \textit{for } x \in (-\infty, 0], \\ \frac{1}{c}x & \textit{for } x \in [0, \infty). \end{cases}$$

Proof. Assume (2) with $a, b \in \mathbb{Q} \setminus \{1\}$. Since (5) implies $f(\mathbb{R}) = \mathbb{R}$, we have $ab > 0$. If $a > 0$, then

$$a = f(f(a)) = f(ba) = b^2 a,$$

whence $b = 1$, a contradiction. If $a < 0$, then

$$a = f(f(a)) = f(a^2) = ba^2,$$

and so $ab = 1$. □

2. The case of higher dimensions

The above results were presented at the Eleventh Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities (Wisła–Malinka, Poland, February, 2–5, 2011). After the presentation Professor Lajos Molnár asked the author about a vector case. The next two results answer this question.

Theorem 2. *Assume X is a real locally convex linear topological space with* $\dim X \geq 2$. *If $f\colon X\setminus\{0\}\to X$ is continuous, then the condition*

$$H\cap f(H)=\emptyset \quad \textit{for any basis } H \textit{ of the vector space } X \textit{ over the field } \mathbb{Q} \tag{6}$$

holds if and only if there is a $c\in\mathbb{Q}\setminus\{1\}$ such that

$$f(x)=cx \qquad \textit{for } x\in X\setminus\{0\}. \tag{7}$$

Proof. The point is to show that if

$$\text{for every } x\in X\setminus\{0\} \text{ there is an } r\in\mathbb{Q} \text{ such that } f(x)=rx, \tag{8}$$

then (7) holds with a $c\in\mathbb{Q}$.

Assume (8) and let $r\colon X\setminus\{0\}\to\mathbb{Q}$ be the function such that

$$f(x)=r(x)x \quad \text{for } x\in X\setminus\{0\}. \tag{9}$$

We have to prove that r is constant.

If $x^*\in X^*$, then the set $P(x^*)$ given by

$$P(x^*)=\{x\in X:\ x^*x>0\}$$

is convex, and so connected, and according to (9) we have

$$r(x)=\frac{x^*f(x)}{x^*x} \quad \text{for } x\in P(x^*).$$

Consequently, for any $x^*\in X^*$ the continuous function

$$x\to\frac{x^*f(x)}{x^*x}, \quad x\in P(x^*),$$

i.e., the function $r|_{P(x^*)}$, is constant:

$$r(x) = c(x^*) \quad \text{for } x \in P(x^*).$$

If $x_1, x_2 \in X \setminus \{0\}$ are linearly independent (over the reals), then (see [2, Theorem 3.6]) there is an $x^* \in X^*$ such that $x^*x_1 = x^*x_2 = 1$ and $r(x_1) = c(x^*) = r(x_2)$. If $x_1, x_2 \in X \setminus \{0\}$ are linearly dependent, then, since $\dim X \geq 2$, we can find an $x_3 \in X \setminus \{0\}$ such that x_1, x_3 are linearly independent, and making use of the previous part we see that $r(x_1) = r(x_3) = r(x_2)$. This proves that r is a constant function.

To finish the proof of the theorem it is now enough to observe that (6) implies (8). □

Corollary 2. *Assume X is a real locally convex linear topological space with* $\dim X \geq 2$. *If $f\colon X \to X$ is continuous and*

$$f(f(x)) = x \quad \text{for } x \in X,$$

then (6) *holds if and only if*

$$f(x) = -x \quad \text{for } x \in X.$$

Remark 1. *In the above theorems and corollaries one can replace the field $\mathbb{Q}$ by any proper subfield of $\mathbb{R}$.*

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References

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