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A STUDY ABOUT CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

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Abstract. In the present paper, we establish some results concerning the quasi-Hadamard product for certain subclasses of analytic functions.

1. Introduction and preliminaries

The study of certain classes of analytic functions has been essential to many researchers especially to complex analysts. Then the class of univalent functions which is a subclass of analytic functions has been the prime interest to many for several decades. Though the famous Bieberbach conjecture is settled, yet many other new results are studied and solved in many ways (cf., [3, 4, 12]). In this article we study quasi-Hadamard product of finitely many functions for the two classes denoted by $S_P(\lambda,\alpha,\beta)$ and $UCV(\lambda,\alpha,\beta)$.

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Throughout this paper let

$$f(z) = a_1 z + \sum_{k=2}^{\infty} a_k z^k, \ (a_1 > 0, a_k \geq 0), \tag{1}$$

$$f_i(z) = a_{1,i} z + \sum_{k=2}^{\infty} a_{k,i} z^k, \ (a_{1,i} > 0, a_{k,i} \geq 0), \tag{2}$$

$$g(z) = b_1 z + \sum_{k=2}^{\infty} b_k z^k, \ (b_1 > 0, b_k \geq 0), \tag{3}$$

$$g_j(z) = b_{1,j} z + \sum_{k=2}^{\infty} b_{k,j} z^k, \ (b_{1,i} > 0, b_{k,j} \geq 0), \tag{4}$$

be regular and univalent in the unit disc $\mathbb{U} = \{z : |z| < 1\}$.

For $0 \leq \lambda < 1$, $0 \leq \alpha < 1$ and $\beta \geq 0$, let $S_P(\lambda, \alpha, \beta)$ denote the class of functions $f(z)$ defined by (1)(considering $a_1 = 1$) and satisfying the analytic criterion

$$\Re\left\{\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z f'(z)} - \alpha\right\} > \beta\left|\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z f'(z)} - 1\right|, \quad z \in \mathbb{U}. \tag{5}$$

Also let $UCV(\lambda, \alpha, \beta)$ denote the class of functions $f(z)$ defined by (1) (considering $a_1 = 1$) and satisfying the analytic criterion

$$\Re\left\{\frac{f'(z) + zf''(z)}{f'(z) + \lambda z f''(z)} - \alpha\right\} > \beta\left|\frac{f'(z) + zf''(z)}{f'(z) + \lambda z f''(z)} - 1\right|, \quad z \in \mathbb{U}. \tag{6}$$

Suitably specializing the parameters of the classes above we generalize the classes defined by some well known authors, see [1, 2, 13–15].

Murugusundaramoorthy and Magesh [10] proved the following results that $f(z) \in S_P(\lambda, \alpha, \beta)$ $(0 \leq \lambda < 1, 0 \leq \alpha < 1, \beta \geq 0)$ if and only if

$$\sum_{n=2}^{\infty} [n(1+\beta) - (\alpha+\beta)(1 + n\lambda - \lambda)]|a_n| \leq (1-\alpha), \tag{7}$$

and $f(z) \in UCV(\lambda, \alpha, \beta)$ $(0 \leq \lambda < 1, 0 \leq \alpha < 1, \beta \geq 0)$ if and only if

$$\sum_{n=2}^{\infty} n[n(1+\beta) - (\alpha+\beta)(1 + n\lambda - \lambda)]|a_n| \leq (1-\alpha). \tag{8}$$

We now introduce the following class of analytic functions which plays an important role in the discussion that follows.

A function f which is analytic in U belongs to the class $S_k(\lambda,\alpha,\beta)$ if and only if

$$\text{(9)}\qquad \sum_{n=2}^{\infty} n^k[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]|a_n| \leq (1-\alpha),$$

where $n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)>0$, $0\leq\lambda<1$, $0\leq\alpha<1$, $\beta\geq 0$ and k is any fixed nonnegative real number.

For $k=0$ and $k=1$, it is identical to $S_P(\lambda,\alpha,\beta)$ and $UCV(\lambda,\alpha,\beta)$ respectively (see [10]). Further, for any positive integer $h>k+1>k>\cdots>p$, we get the following inclusion relation

$$S_h(\lambda,\alpha,\beta)\subseteq S_{k+1}(\lambda,\alpha,\beta)\subseteq\cdots\subseteq S_2(\lambda,\alpha,\beta)\subseteq UCV(\lambda,\alpha,\beta)\subseteq S_P(\lambda,\alpha,\beta).$$

The class $S_k(\lambda,\alpha,\beta)$ is nonempty for any nonnegative real number k as the functions of the form

$$\text{(10)}\qquad f(z)=a_1z+\sum_{k=2}^{\infty}\frac{(1-\alpha)}{n^k[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]}\lambda_k z^k,$$

where $a_1>0$, $\lambda_k\geq 0$ and $\Sigma_2^{\infty}\lambda_k\leq 1$ satisfy the inequality given in (9).

Let us define the quasi-Hadamard product of the functions f and g by

$$\text{(11)}\qquad f*g(z)=a_1b_1z+\sum_{k=2}^{\infty}a_kb_kz^k,\ (a_1>0,b_1>0,a_k\geq 0).$$

Similarly, we can define the quasi-Hadamard product of more than two functions. The quasi-Hadamard product of two or more functions has recently been defined and used by some well known authors see [5–9, 11]. In this work we establish certain results concerning the quasi-Hadamard product of functions in the classes $S_P(\lambda,\alpha,\beta)$, $UCV(\lambda,\alpha,\beta)$ and $S_k(\lambda,\alpha,\beta)$ analogous to the results due to V. Kumar [5] and [7] as well.

2. The main results

THEOREM 2.1. *Let the functions $f_i(z)$ defined by (2) be in $UCV(\lambda,\alpha,\beta)$ for every $i=1,2,\ldots,r$ and let the functions $g_j(z)$ defined by (4) be in the class $S_P(\lambda,\alpha,\beta)$ for every $j=1,2,\ldots,q$. Then the quasi-Hadamard product $f_1*f_2*\cdots*f_r*g_1*g_2*\cdots*g_q(z)$ belongs to the class $S_{2r+q-1}(\lambda,\alpha,\beta)$.*

Proof. We denote the quasi-Hadamard product $f_1 * f_2 * \cdots * f_r * g_1 * g_2 * \cdots * g_q(z)$ by the function $G(z)$, for the sake of convenience. Clearly,

$$G(z) = \left[\prod_{i=1}^{r} |a_{1,i}|\right]\left[\prod_{j=1}^{q} |b_{1,j}|\right] z + \sum_{n=2}^{\infty}\left[\prod_{i=1}^{r} |a_{n,i}|\right]\left[\prod_{j=1}^{q} |b_{n,j}|\right] z.$$

Since $f_i(z) \in UCV(\lambda, \alpha, \beta)$, it implies

$$\sum_{n=2}^{\infty} n\big[n(1+\beta) - (\alpha+\beta)(1+n\lambda-\lambda)\big]|a_{n,i}| \leq (1-\alpha)|a_{1,i}|, \tag{12}$$

for every $i = 1, 2, \ldots, r$. Therefore

$$|a_{n,i}| \leq \frac{(1-\alpha)}{n\big[n(1+\beta) - (\alpha+\beta)(1+n\lambda-\lambda)\big]}|a_{1,i}|,$$

for every $i = 1, 2, \ldots, r$, which implies

$$|a_{n,i}| \leq n^{-2}|a_{1,i}|, \tag{13}$$

for every $i = 1, 2, \ldots r$. Similarly $g_j(z) \in S_p(\lambda, \alpha, \beta)$, it implies

$$\sum_{n=2}^{\infty} \big[n(1+\beta) - (\alpha+\beta)(1+n\lambda-\lambda)\big]|b_{n,j}| \leq (1-\alpha)|b_{1,j}|, \tag{14}$$

for every $i = 1, 2, \ldots, q$. It implies

$$|b_{n,j}| \leq n^{-1}|b_{1,j}|, \tag{15}$$

for every $j = 1, 2, \ldots q$.

Using (13), (14) and (15) for $i = 1, 2, \ldots, r$, $j = q$ and $j = 1, 2, \ldots, q-1$ respectively, we have

$$\sum_{n=2}^{\infty} n^{2r+q-1}\big[n(1+\beta) - (\alpha+\beta)(1+n\lambda-\lambda)\big]\prod_{i=1}^{r}|a_{n,i}|\prod_{j=1}^{q}|b_{n,j}|$$
$$\leq \sum_{n=2}^{\infty} n^{2r+q-1}\big[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)\big]n^{-2r}n^{-q+1}|b_{n,q}|\prod_{i=1}^{r}|a_{1,i}|\prod_{j=1}^{q-1}|b_{1,j}|$$

$$= \sum_{n=2}^{\infty} \big[n(1+\beta) - (\alpha+\beta)(1+n\lambda-\lambda)\big] |b_{n,q}| \prod_{i=1}^{r} |a_{1,i}| \prod_{j=1}^{q-1} |b_{1,j}|$$
$$\leq (1-\alpha) \prod_{i=1}^{r} |a_{1,i}| \prod_{j=1}^{q} |b_{1,j}|.$$

Therefore we have

$$\sum_{n=2}^{\infty} n^{2r+q-1} \big[n(1+\beta) - (\alpha+\beta)(1+n\lambda-\lambda)\big] \prod_{i=1}^{r} |a_{n,i}| \prod_{j=1}^{q} |b_{n,j}|$$
$$\leq (1-\alpha) \prod_{i=1}^{r} |a_{1,i}| \prod_{j=1}^{q} |b_{1,j}|.$$

Hence $f_1 * f_2 * \cdots * f_r * g_1 * g_2 * \cdots * g_q(z) \in S_{2r+q-1}(\lambda, \alpha, \beta)$. □

THEOREM 2.2. *Let the functions $f_i(z)$ defined by (2) be in $UCV(\lambda, \alpha, \beta)$ for every $i = 1, 2, \ldots, r$. Then the Hadamard product $f_1(z) * f_2(z) * \cdots * f_r(z)$ belongs to the class $S_{2r-1}(\lambda, \alpha, \beta)$.*

PROOF. To prove the theorem, we need to show that

$$\sum_{n=2}^{\infty} n^{2r-1} \big[n(1+\beta) - (\alpha+\beta)(1+n\lambda-\lambda)\big] \prod_{i=1}^{r} |a_{n,i}| \leq (1-\alpha) \prod_{i=1}^{r} |a_{1,i}|.$$

Since $f_i(z) \in UCV(\lambda, \alpha, \beta)$, therefore

$$\sum_{n=2}^{\infty} n \big[n(1+\beta) - (\alpha+\beta)(1+n\lambda-\lambda)\big] |a_{n,i}| \leq (1-\alpha)|a_{1,i}|, \tag{16}$$

for every $i = 1, 2, \ldots r$. It implies

$$|a_{n,i}| \leq \frac{(1-\alpha)}{n\big[n(1+\beta) - (\alpha+\beta)(1+n\lambda-\lambda)\big]} |a_{1,i}|,$$

for every $i = 1, 2, \ldots, r$, which implies

$$|a_{n,i}| \leq n^{-2} |a_{1,i}|, \tag{17}$$

for every $i = 1, 2, \ldots, r$.

Applying simultaneously (16) as well as (17) for $i = r$ and $i = 1, 2, \ldots, r-1$ respectively, we get

$$\sum_{n=2}^{\infty} n^{2r-1}\big[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)\big]\prod_{i=1}^{r}|a_{n,i}|$$
$$\leq \sum_{n=2}^{\infty} n^{2r-1}\big[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)\big]|a_{n,r}|n^{-2r+2}\prod_{i=1}^{r-1}|a_{1,i}|$$
$$= \sum_{n=2}^{\infty} n\big[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)\big]|a_{n,r}|\prod_{i=1}^{r-1}|a_{1,i}| = (1-\alpha)\prod_{i=1}^{r}|a_{1,i}|.$$

Hence $f_1(z) * f_2(z) * \cdots * f_r(z) \in S_{2r-1}(\lambda, \alpha, \beta)$. □

THEOREM 2.3. *Let the functions $f_i(z)$ defined by (2) be in the class $S_P(\lambda, \alpha, \beta)$ for every $i = 1, 2, \ldots, r$. Then the Hadamard product $f_1(z) * f_2(z) * \cdots * f_r(z)$ belongs to the class $S_{r-1}(\lambda, \alpha, \beta)\cdot$.*

PROOF. Since $f_i(z) \in S_p(\lambda, \alpha, \beta)$, it implies

$$\sum_{n=2}^{\infty}\big[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)\big]|a_{n,i}| \leq (1-\alpha)|a_{1,i}|, \tag{18}$$

for every $i = 1, 2, \ldots, r$. Therefore

$$\big[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)\big]|a_{n,i}| \leq (1-\alpha)|a_{1,i}|,$$

or

$$|a_{n,i}| \leq \frac{(1-\alpha)}{n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)}|a_{1,i}|,$$

for every $i = 1, 2, \ldots, r$, which implies

$$|a_{n,i}| \leq n^{-1}|a_{1,i}|, \tag{19}$$

for every $i = 1, 2, \ldots, r$.

To prove that $f_1(z) * f_2(z) * \cdots * f_r(z) \in S_{r-1}(\lambda, \alpha, \beta)$, it is enough to show that

$$\sum_{n=2}^{\infty} n^{r-1}\big[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)\big]\prod_{i=1}^{r}|a_{n,i}| \leq (1-\alpha)\prod_{i=1}^{r}|a_{1,i}|. \tag{20}$$

Using (18) and (19) for $i = r$ and $i = 1, 2, \ldots, r-1$ respectively, then we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} n^{r-1}\big[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)\big]\prod_{i=1}^{r}|a_{n,i}| \\
&\le \sum_{n=2}^{\infty} n^{r-1}\big[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)\big]|a_{n,r}|n^{-r+1}\prod_{i=1}^{r-1}|a_{1,i}| \\
&= \sum_{n=2}^{\infty}\big[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)\big]|a_{n,r}|\prod_{i=1}^{r-1}|a_{1,i}|.
\end{aligned}$$

Using (18) we get

$$\sum_{n=2}^{\infty}\big[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)\big]|a_{n,r}|\prod_{i=1}^{r-1}|a_{1,i}| = (1-\alpha)\prod_{i=1}^{r}|a_{1,i}|.$$

Hence $f_1(z) * f_2(z) * \cdots * f_r(z) \in S_{r-1}(\lambda, \alpha, \beta)$. □

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