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STABILITY OF THE PEXIDER FUNCTIONAL EQUATION

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Abstract. A stability result for the Pexider equation will be derived from a stability theorem published in [9] for the Cauchy functional equation. Then we discuss the quality of some constants occuring in this context; as a model case we consider functions defined on the multiplicative semigroup $\{1, 0\}$.

1. Introduction

In Theorem 1 below we describe the stability result for the Cauchy equation, which had been mentioned in the Abstract.

Let S be a groupoid, i.e., S is a set and for all $x, y \in S$ we have a product $xy \in S$. For $x \in S$ and $k = 0, 1, 2, \dots$ the powers x^{2^k} are recursively defined by

$$x^{2^0} = x^1 = x, \ x^{2^{k+1}} = x^{2^k} x^{2^k}.$$

Józef Tabor [8] pointed out the usefulness of the following condition for stability investigations:

(T) For $x, y \in S$ there always is an entire $k \geq 1$ such that

$$(xy)^{2^k} = x^{2^k} y^{2^k}. \tag{1}$$

In the present paper, groupoids S satisfying (T) are called *Tabor groupoids.* Three examples are particular cases of them; they are ordered in decreasing generality:

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1. Groupoids S with a square-symmetric operation, i.e.,

$$(xy)^2 = x^2y^2, \quad x, y \in S$$

(cf. the joint paper with Zsolt Páles and R. Duncan Luce [6]). Then (1) is true for all $x, y \in S$ with the same k, viz. $k = 1$.

2. Groupoids S with a bisymmetric operation, i.e.,

$$(xy)(\bar{x}\bar{y}) = (x\bar{x})(y\bar{y}), \quad x, y, \bar{x}, \bar{y} \in S.$$

Here $\bar{x} = x$, $\bar{y} = y$ leads to square-symmetry.

3. Commutative semigroups S.

Let us mention that Zbigniew Gajda and Zygfryd Kominek [1] considered semigroups satisfying condition (T). Inspired by Józef Tabor [8], they call them weakly commutative.

Now let E be a Banach space. A subset V of E is called *ideally convex* (E. A. Lifšic [3]), if for every bounded sequence $d_1, d_2, d_3, \ldots$ in V and for every numerical sequence $\alpha_1, \alpha_2, \alpha_3, \ldots \geq 0$ such that $\sum_{k=1}^{\infty} \alpha_k = 1$ we get $\sum_{k=1}^{\infty} \alpha_k d_k \in V$.

The following theorem is taken from [9]; in the case of a commutative semigroup S it goes back to Jacek Tabor [7].

THEOREM 1. *Let S be a Tabor groupoid, and let V be a bounded and ideally convex subset of the Banach space E. For $f\colon S \to E$ we suppose*

$$f(xy) - f(x) - f(y) \in V, \quad x, y \in S.$$

Then there exists a (unique) function $F\colon S \to E$ such that

$$F(xy) = F(x) + F(y), \quad F(x) - f(x) \in V, \quad x, y \in S.$$

2. The Pexider equation

THEOREM 2. *Let S be a Tabor groupoid having a neutral element n, i.e., $n \in S$ and*

$$nx = xn = x, \quad x \in S.$$

Let V be a symmetric, bounded, and ideally convex subset of a Banach space E (the symmetry means $-V = V$). For $f, g, h\colon S \to E$ we suppose

$$f(xy) - g(x) - h(y) \in V, \quad x, y \in S. \tag{2}$$

Then there are $F, G, H\colon S \to E$ satisfying the Pexider equation

$$F(xy) = G(x) + H(y), \quad x, y \in S, \tag{3}$$

as well as the conditions

$$F(x) - f(x) \in 3V, \quad G(x) - g(x) \in 4V, \quad H(x) - h(x) \in 4V, \quad x \in S. \tag{4}$$

Proof. With $y = n$ and with $x = n$ in (2) we get

$$f(x) - g(x) - h(n) \in V, \quad f(y) - g(n) - h(y) \in V, \tag{5}$$

hence $f(x) \in g(x) + h(n) + V$, $f(y) \in h(y) + g(n) + V$, thus

$$\begin{aligned} f(xy) - f(x) - f(y) + g(n) + h(n) &\in f(xy) - g(x) - h(y) + V + V \\ &\subseteq V + V + V = 3V, \end{aligned}$$

the last equality being true, since V is convex. For

$$\tilde{f}(x) := f(x) - g(n) - h(n), \quad x \in S, \tag{6}$$

this means

$$\tilde{f}(xy) - \tilde{f}(x) - \tilde{f}(y) \in 3V, \quad x, y \in S,$$

and by Theorem 1 there is a function $\Phi\colon S \to E$ such that

$$\Phi(xy) = \Phi(x) + \Phi(y), \quad \Phi(x) - \tilde{f}(x) \in 3V, \quad x, y \in S.$$

Now it is easily seen that for $F(x) := \Phi(x) + g(n) + h(n)$, $G(x) := \Phi(x) + g(n)$, $H(x) := \Phi(x) + h(n)$, $x \in S$, we get (3) and (4):

(3) is obvious; $F(x) - f(x) \in 3V$ follows from (6) and $\Phi(x) - \tilde{f}(x) \in 3V$; the remaining formulae in (4) are consequences of (5), (6), and $\Phi(x) - \tilde{f}(x) \in 3V$. □

Remark 1. Theorem 2 should be compared to other stability results for the Pexider equation, e.g. to those of Kazimierz Nikodem [5] and Zygfryd Kominek [2], where the target space for the functions is more general than a Banach space.

When choosing $V = \{x \mid x \in E,\ \|x\| \le \varepsilon\}$, then we get from Theorem 2 the following Corollary, which had been obtained by Nikodem [5] in the case of a commutative semigroup S.

Corollary 1. *Let S be a Tabor groupoid having a neutral element, and let E be a Banach space. For $f, g, h\colon S \to E$ we suppose*

$$\|f(xy) - g(x) - h(y)\| \le \varepsilon, \quad x, y \in S.$$

Then there are $F, G, H\colon S \to E$ such that

$$F(xy) = G(x) + H(y), \quad x, y \in S,$$

$$\|F(x)-f(x)\| \le 3\varepsilon, \quad \|G(x)-g(x)\| \le 4\varepsilon, \quad \|H(x)-h(x)\| \le 4\varepsilon, \quad x \in S. \tag{7}$$

Remark 2. If S is a commutative semigroup, then according to Zenon Moszner's survey [4], the constants 4ε in (7) can be replaced by 3ε. We do not know, wether this also holds for arbitrary Tabor groupoids.

Remark 3. In the next paragraph we shall consider the commutative semigroup $S = \{1, 0\}$. It will follow that 3ε in (7) cannot be replaced by a number less than 2ε. It also will follow that, when having in (7) the better inequality $\|F(x) - f(x)\| \le 2\varepsilon$, then the constants 4ε cannot be replaced by numbers less than $3\varepsilon/2$.

Remark 4. By calculations similar to those in the next paragraph, it can be shown that for the cyclic groups $S = Z_2$, $S = Z_3$ of two and of three elements, respectively, all the numbers 3ε, 4ε in (7) can be replaced by ε.

3. The semigroup $S = \{1, 0\}$

In $S = \{1, 0\}$ we have $1 \cdot 1 = 1,\ 1 \cdot 0 = 0 \cdot 1 = 0 \cdot 0 = 0$. It is easily seen that in this case solutions of the Pexider equation (3) necessarily are constant functions:

$$G(1) = G(0) = a, \quad H(1) = H(0) = b, \quad F(1) = F(0) = a + b.$$

Theorem 3. *Consider $S = \{1, 0\}$, let N be a normed space, and let $f, g, h\colon S \to N$ satisfy*

$$\|f(xy) - g(x) - h(y)\| \le \varepsilon, \quad x, y \in S. \tag{8}$$

Then there exist $a, b \in N$ such that

$$\|g(1) - a\| \le \frac{1}{2}\varepsilon, \quad \|h(1) - b\| \le \frac{1}{2}\varepsilon, \tag{9}$$

$$\|g(0) - a\| \le \frac{3}{2}\varepsilon, \quad \|h(0) - b\| \le \frac{3}{2}\varepsilon, \tag{10}$$

$$\|f(x) - a - b\| \le 2\varepsilon, \quad x = 0, 1. \tag{11}$$

Proof. Indeed, (8) means

$$\begin{aligned} f(1) - g(1) - h(1) &= r_1, \\ f(0) - g(1) - h(0) &= r_2, \\ f(0) - g(0) - h(1) &= r_3, \\ f(0) - g(0) - h(0) &= r_4, \end{aligned} \tag{12}$$

where

$$\|r_j\| \le \varepsilon, \quad j = 1, 2, 3, 4. \tag{13}$$

We easily get

$$g(0) - g(1) = r_2 - r_4, \tag{14}$$

$$h(0) - h(1) = r_3 - r_4, \tag{15}$$

$$f(0) - g(1) - h(1) = r_2 + r_3 - r_4. \tag{16}$$

We define

$$a = g(1) + \frac{1}{2}r_2, \quad b = h(1) + \frac{1}{2}r_3,$$

then (13) already leads to (9). From (14), (15) we now get

$$g(0) = g(1) + r_2 - r_4 = a + \frac{1}{2}r_2 - r_4, \tag{17}$$

$$h(0) = h(1) + r_3 - r_4 = b + \frac{1}{2}r_3 - r_4, \tag{18}$$

and this gives (10). From (12), (16) we finally have

$$f(1)-a-b=r_1+g(1)-a+h(1)-b=r_1-\frac{1}{2}r_2-\frac{1}{2}r_3,$$
$$f(0)-a-b=f(0)-g(1)-\frac{1}{2}r_2-h(1)-\frac{1}{2}r_3=\frac{1}{2}r_2+\frac{1}{2}r_3-r_4,$$

and these two lines prove (11). □

Example. The following example shows that

I) 2ε in (11) is best possible,

II) having 2ε in (11), then also $\frac{1}{2}\varepsilon$ in (9) and $\frac{3}{2}\varepsilon$ in (10) are best possible:

We define $f, g, h\colon S \to \mathbb{R}$ by

$$f(1)=2, \quad f(0)=-2, \quad g(1)=1, \quad g(0)=-1, \quad h(1)=0, \quad h(0)=-2.$$

Then

$$f(1)-g(1)-h(1)=2-1-0=1,$$
$$f(0)-g(1)-h(0)=-2-1+2=-1,$$
$$f(0)-g(0)-h(1)=-2+1-0=-1,$$
$$f(0)-g(0)-h(0)=-2+1+2=1,$$

hence (8) holds for

$$\varepsilon=1 \tag{19}$$

(with absolute value in $\mathbb{R}$ being the norm).

Proof of I). Suppose (11) to hold for some $a+b\in\mathbb{R}$ and with 2ε replaced by some η:

$$|2-a-b|=|f(1)-a-b|\le\eta, \quad |-2-a-b|=|f(0)-a-b|\le\eta. \tag{20}$$

Then $4\le|2-a-b|+|2+a+b|\le 2\eta$, hence (cf. (19)) $2\varepsilon=2\le\eta$.

Proof of II). Inequality (11) with $2\varepsilon=2$ leads to (20) with $\eta=2$, hence to $a+b=0$, i.e., $b=-a$. Then (10) with η instead of $\frac{3}{2}\varepsilon$ leads to

$$|-1-a|=|g(0)-a|\le\eta, \quad |-2+a|=|h(0)-b|\le\eta,$$

which implies $3=(2-a)+(1+a)\le 2\eta$, hence $\frac{3}{2}\varepsilon=\frac{3}{2}\le\eta$.

In the same way we get from (9) with $\frac{1}{2}\varepsilon$ replaced by η that

$$|1 - a| = |g(1) - a| \leq \eta, \quad |0 + a| = |h(1) - b| \leq \eta,$$

which implies $1 = (1 - a) + (0 + a) \leq 2\eta$, hence $\frac{1}{2}\varepsilon = \frac{1}{2} \leq \eta$.

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