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ORDERINGS OF HIGHER LEVEL IN MULTIFIELDS AND MULTIRINGS

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Abstract. In this note we generalize the concept of orderings of higher level to multifields and multirings. We show how part of the standard Artin-Schreier theory for orderings of higher level of fields and rings extends to multifields and multirings.

1. Introduction

The notions of multigroups, multirings, multifields, and their corresponding reduced versions were introduced by Marshall in [5] and provide a convenient framework to study the reduced theory of quadratic forms and spaces of orderings. A multigroup is a quadruple $(G, \Pi, -, 0)$, where G is a non-empty set, Π is a subset of $G \times G \times G$, $-\colon G \to G$ is a function, and $0 \in G$ is an element such that the following axioms are satisfied:

(G1) if $(a, b, c \in \Pi)$, then $(c, -b, a) \in \Pi$, and $(-a, c, b) \in \Pi$;
(G2) $(a, 0, b) \in \Pi$ if and only if $a = b$;
(G3) if there exists $e \in G$ such that $(a, b, e) \in \Pi$, and $(e, c, d) \in \Pi$, then there exists $f \in G$ such that $(b, c, f) \in \Pi$, and $(a, f, d) \in \Pi$.

Moreover, a multigroup is called commutative, if

(G4) $(a, b, c) \in \Pi$ if and only if $(b, a, c) \in \Pi$.

The set $\Pi \subset G \times G \times G$ defines a multivalued addition here, and we shall often write $c \in a + b$ to indicate that $(a, b, c) \in \Pi$. One checks that with this

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notation (G1) reads "if $c \in a + b$, then $a \in c - b$, and $b \in -a + c$", (G2) reads "$b \in a + 0$ iff. $a = b$", (G3) reads "if $\exists e \in G(e \in a + b) \wedge (d \in e + c)$ then $\exists f \in G(f \in b + c) \wedge (d \in a + f)$", and (G4) reads "$c \in a + b$ iff. $c \in b + a$".

For subsets $S, T \subset G$ it is convenient to define $S + T$ as the set $\{c \in G :$ there exist $a \in S, b \in T$ such that $c \in a + b\}$. Likewise, ΣS denotes the union of the sets $S + \ldots + S$ (k times, $k \geq 1$). We also define $S - T = S + (-T)$, for $-T = \{-a : a \in T\}$. Traditionally, S^* denotes the set $S \setminus \{0\}$.

A multiring is a system $(A, \Pi, \cdot, -, 0, 1)$ satisfying

(R1) $(A, \Pi, -, 0)$ is a commutative multigroup;
(R2) $(A, \cdot, 1)$ is a commutative monoid with 1;
(R3) $a \cdot 0 = 0$ for all $a \in A$;
(R4) if $c \in a + b$ then $cd \in ad + bd$ for all $d \in A$.

A multifield is a multiring F with $1 \neq 0$ such that every non-zero element has a multiplicative inverse.

By a submultiring of A we understand a subset S of A satisfying $S - S \subset S$, $SS \subset S$, and $1 \in S$. If, in addition, $S^{-1} \subset S$, we say that S is a submultifield. Clearly, submultirings and submultifields are multirings and multifields, respectively.

A zero divisor is a nonzero element a of A such that, for some nonzero $b \in A$, $ab = 0$. A multiring that has no zero divisors will be called an integral multidomain.

An ideal of A is a nonempty set $I \subset A$ such that $I + I \subset I$ and $AI \subset I$. For elements $a_1, \ldots, a_k \in A$, the smallest ideal of A containing $a_1, \ldots, a_k$ is $\Sigma A a_1 + \ldots + \Sigma A a_k$. If an ideal $\mathfrak{p}$ such that $1 \notin \mathfrak{p}$ satisfies the condition $ab \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}$ or $b \in \mathfrak{p}$, then we call $\mathfrak{p}$ prime. The set of all prime ideals of A, the prime spectrum of A, will be denoted by $\mathrm{Spec}\,(A)$.

Let I be an ideal in A. Denote by A/I the set of cosets $\overline{a} = a + I$, $a \in A$. We define the factor multiring $(A/I, \overline{\Pi}, \cdot, -, \overline{0}, \overline{1})$ by setting $\overline{\Pi} = \{(\overline{a}, \overline{b}, \overline{c}) : (a, b, c) \in \Pi\}$, $-\colon A/I :\to A/I$ by the formula $-\overline{a} = \overline{-a}$, and the multiplication by the formula $\overline{a} \cdot \overline{b} = \overline{ab}$. As in the case of ordinary rings, if $\mathfrak{p}$ is a prime ideal in A, the factor multiring $A/\mathfrak{p}$ becomes an integral multidomain.

If S is a multiplicative set in A, denote by $S^{-1}A$ the set of the elements $\frac{a}{s}$, $a \in A$, $s \in S$, with $\frac{a}{s} = \frac{b}{t}$ being equivalent to $atu = bsu$, for some $u \in S$. We define the localization of the multiring A at S, $(S^{-1}A, S^{-1}\Pi, \cdot, -, 0, 1)$, by setting $S^{-1}\Pi = \{(\frac{a}{s}, \frac{b}{t}, \frac{c}{u}) : (atuv, bsuv, cstv) \in \Pi$ for some $v \in S\}$, and $\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$. In the special case when A is an integral multidomain and $S = A^*$, $S^{-1}A$ becomes a multifield that will be called the multifield of fractions of A and denoted by (A). Moreover, for a multiring A and its prime ideal $\mathfrak{p}$, we define the residue multifield at $\mathfrak{p}$ to be $(A/\mathfrak{p})$, and denote it by $k(\mathfrak{p})$.

If A and B are multirings, then a function $\phi\colon A \to B$ is called a multiring homomorphism, if the following conditions are satisfied, for $a, b, c \in A$:

(H1) if $c \in a + b$ in A, then $f(c) \in f(a) + f(b)$ in B;

(H2) $f(-a) = -f(a)$;
(H3) $f(0) = 0$;
(H4) $f(ab) = f(a)f(b)$;
(H5) $f(1) = 1$.

As it has been suggested in [5], many of the standard results in the theory of orderings of higher level can be easily generalized to multifields. Following the classical presentations of [1] and [4], in Section 2 we translate the familiar theorems in the theory to this new setup. In Section 3 we extend the theory to multirings and prove a version of Positivstellensatz – methods used there mimic the ones used in [3], and the results are analogous to the ones formulated in [2] and [6].

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2. Artin-Schreier theory

Let $(F, \Pi, \cdot, -, 0, 1)$ be a multifield. An *ordering of level* n is a subset $P \subset F$ such that $P + P \subset P$, P^* is a subgroup of F^*, and F^*/P^* is a cyclic group with $|F^*/P^*| \mid 2^n$. If $|F^*/P^*| = 2^m$, we say that p has *exact level* m.

Remarks:
1. Clearly, if P has level n, then P has level $n + r$, for $r \geq 0$.

2. What is defined in Section 3 of [5] as ordering (that is, a subset $P \subset F$ such that $P + P \subset P$, $PP \subset P$, $P \cup -P = F$, and $P \cap -P = \{0\}$), will be called an *ordinary ordering* here. Note that an ordinary ordering is an ordering of exact level 1.

Proof. Let P be an ordinary ordering of F. By definition, $P + P \subset P$. Since $P \cup -P = F$ and $PP \subset P$, it follows that $F^2 \subset P$. Therefore, for $a \in P^*$, $\frac{1}{a} = a \cdot \left(\frac{1}{a}\right)^2 \in P^*$, and thus P^* is a subgroup of F^*. Finally, for $a \in F^*$, either $a \in P^*$ or $a \in -P^*$, so that P^* and $-P^*$ are the only two cosets in the group F^*/P^*. $\square$

Just as is the case with orderings of higher level of fields, to study orderings of higher level of multifields it proves to be technically important to introduce the notion of a preordering. By definition, a *preordering of level* n is a subset

$T \subset F$ such that $T + T \subset T$, $TT \subset T$, $F^{2^n} \subset T$. Readily, every ordering is a preordering.

3. T^* is a subgroup of F^*.

PROOF. Fix $a \in T^*$. Since $TT \subset T$, and $F^{2^n} \subset T$,

$$\frac{1}{a} = \left(\frac{1}{a}\right)^{2^n} \cdot a^{2^n-1} \in T^*. \qquad \square$$

Note that F^*/T^* need no longer be cyclic. We can, however, define the exact level of T: if $F^{2^m} \subset T$, but $F^{2^{m-1}} \not\subset T$, then we say that T has *exact level* m.

4. If T is a preordering of exact level m that is also an ordering (that is, F^*/T^* is cyclic), then, as an ordering, T has exact level m.

PROOF. Let $a \in F^*$ be an element such that $a^{2^{m-1}} \notin T^*$ and $a^{2^m} \in T^*$. One readily checks that $F^*/T^* = \langle aT^* \rangle$, and that $|F^*/T^*| = 2^m$. $\square$

We shall now define proper preorderings. There are a few subtle issues that we should discuss in some detail.

5. Either $T = -T$, or $T \cap -T = \{0\}$.

PROOF. Assume that $T \neq -T$. Replacing a with $-a$, if necessary, we may assume that there is $a \in T$ such that $-a \notin T$. Fix $b \in T \cap -T$. If $b \neq 0$, then $b \in T^*$, and $-b \in T^*$. Consequently, $\frac{1}{b} \in T^*$, $\frac{1}{-b} \in T^*$, and $-a = (-a) \cdot \frac{1}{-b} \cdot (-b) = a \cdot \frac{1}{b} \cdot (-b) \in T$, which is impossible. $\square$

6. $T = -T$ if and only if T is a submultifield of F.

PROOF. The implication ($\Rightarrow$) is clear. For the other one, fix $a \in T$. Since $T - T \subset T$, there is an element $b \in T$ such that $b \in a - a$. Then $-a \in b - a \subset T - T \subset T$. $\square$

7. $T \cap -T = \{0\}$ if and only if $-1 \notin T$.

PROOF. Assume $T \cap -T = \{0\}$. If $-1 \in T$, then, as $1 = 1^{2^n} \in T$, $-1 \in T \cap -T$ – a contradiction. Conversely, if $-1 \notin T$, and $a \in T^* \cap -T^*$, then $\frac{1}{a} \in T^*$, and $-1 = (-a) \cdot \frac{1}{a} \in T^*$, which, again, yields a contradiction. $\square$

A preordering T will be called *proper*, if $-1 \notin T$. Note that a proper preordering of level n is also a proper preordering of exact level m, for some $m \leq n$.

8. A preordering P of exact level m maximal subject to the condition P is proper is an ordering of exact level m. In particular, a proper preordering of level n is contained in a proper ordering of level n.

PROOF. It suffices to show F^*/P^* is cyclic with $|F^*/P^*| = 2^m$. Firstly, we claim that, for $a \in F^*$ such that $a^2 \in P$, one has $a \in P \cup -P$. Assume $a \in F^*$ with $a^2 \in P$, and $a \notin P$. Then $P - Pa$ is a preordering of exact level m with $-a \in P - Pa$, and $P \subset P - Pa$. We claim that $-1 \notin P - Pa$. For suppose $-1 \in s - at$, for $s, t \in P$. If $t = 0$, then $-1 = s \in P$, contrary to the assumption P is proper. Thus $t \neq 0$, and $at \in 1 + s$, so $a \in \frac{1}{t} + \frac{s}{t} \subset P$ – a contradiction. Now, due to the maximality of P, $P = P - Pa$, and, consequently, $-a \in P$.

Since $F^{2^{m-1}} \not\subset P$, we may choose $\omega \in F^*$ with $\omega^{2^{m-1}} \notin P^*$. We shall show that $F^*/P^* = \langle \omega P^* \rangle$, and that $|F^*/P^*| = 2^m$. Since $\omega^i P^* \neq \omega^j P^*$, for $i \neq j$, $i, j \in \{0, \dots, 2^m - 1\}$, it suffices to show that, for $a \in F^*$, $a \in \omega^k P^*$, for some $k \in \{0, \dots, 2^m - 1\}$. Clearly, $a^{2^m} \in P^*$, so, by our claim, $a^{2^{m-1}} \in P^*$ or $-a^{2^{m-1}} \in P$. If the latter is the case, as $\omega^{2^m} \in P$, and $\omega^{2^{m-1}} \notin P^*$, we see that $\omega^{2^{m-1}} \in -P^*$, and, consequently, $a^{2^{m-1}}\omega^{2^{m-1}} = (a\omega)^{2^{m-1}} \in P^*$. Repeating the argument, we get that either $a^{2^{m-2}} \in P^*$ or $(a\omega^\ell)^{2^{m-2}} \in P^*$, for some $\ell \in \{1, \dots, 2^m - 1\}$. By induction, we eventually show that $a \in P^*$ or $a\omega^k \in P^*$, for some $k \in \{1, \dots, 2^m - 1\}$, which finishes the proof. $\square$

Denote by ΣF^{2^n} the set of all sums of 2^n powers: $\Sigma F^{2^n} = \bigcup \{a_1^{2^n} + \ldots + a_k^{2^n} : a_1, \dots, a_k \in F, k \in \mathbb{N}\}$. We call F *formally $n-$real* if $-1 \notin \Sigma F^{2^n}$. In view of the above remarks we have the following:

THEOREM 1. *Let F be a multifield. The following conditions are equivalent:*

(1) *F is formally $n-$real,*
(2) *F admits an ordering of level n,*
(3) *F admits a preordering of level n.*

For a preordering T of F, denote by X_T the set of all orderings P of level n of F with $T \subset P$.

THEOREM 2. *Let F be a multifield, $T \subset F$ a preordering of level n. If T is proper, then $T = \bigcap_{P \in X_T} P$.*

Proof. Fix $a \in \bigcap_{P \in X_T} P$ and suppose that $a \notin T$. Clearly, $a \neq 0$. Replacing a with $a^{2^{n-1}}$, if necessary, we may assume $a^2 \in T$. Then $T - Ta$ is a preordering in F. Observe that $-1 \notin T - Ta$. Indeed, suppose *a contrario* that $-1 \in s - at$, for $s, t \in T$. If $t = 0$, then $-1 = s \in T$, contrary to the assumption that T is proper. Thus $t \neq 0$, and $at \in 1 + s$, so $a \in \frac{1}{t} + \frac{s}{t} \subset T$ – a contradiction. Therefore $-1 \notin T - Ta$, and we may extend $T - Ta$ to an ordering P maximal subject to the condition P is proper with $-a \in P$ and $T \subset T - Ta \subset P$. Consequently, $a \in P$, and thus $-1 = \frac{-a \cdot a}{a^2} \in P$ – a contradiction. □

Corollary 3. *Let F be a formally $n-$real multifield. Then ΣF^2 is the intersection of all orderings of level n in F.*

3. The Positivstellensatz

We now define orderings and preorderings of higher level for multirings and prove an abstract version of the Positivstellensatz known from the classical ring theory. Let $(A, \Pi, \cdot, -, 0, 1)$ be a multiring. An *ordering of level n* is a subset $P \subset A$ such that

(O1) $P + P \subset P$, $PP \subset P$, $A^{2^n} \subset P$;
(O2) $P \cap -P = \mathfrak{p}$ is a prime ideal of A;
(O3) if $ab^{2^n} \in P$, then $a \in P$ or $b \in P \cap -P$;
(O4) the set

$$\overline{P} = \bigcup \left\{ a_1^{2^n} \overline{p_1} + \ldots + a_k^{2^n} \overline{p_k} : a_1, \ldots, a_k \in k(\mathfrak{p}), p_1, \ldots, p_k \in P, k \in \mathbb{N} \right\}$$

is an ordering of level n of the multifield $k(\mathfrak{p})$. Recall that $\overline{p_i} = p_i + \mathfrak{p} \in A/\mathfrak{p}$, $i \in \{1, \ldots, k\}$.

If $\overline{P}$ is an ordering of exact level m, we say that P is of *exact level m.*

Remarks:

1. For a prime ideal $\mathfrak{p}$ in A, orderings on A having the support $\mathfrak{p}$ correspond bijectively to orderings on $k(\mathfrak{p})$: let $\pi : A \to A/\mathfrak{p} \to k(\mathfrak{p})$ denote the canonical homomorphism – then, if P is an ordering of level n in A, $P = \pi^{-1}(\overline{P})$. Conversely, if $\mathfrak{p} \in \mathrm{Spec}\,(A)$ and P' is an ordering of level n in $k(\mathfrak{p})$, then $P = \pi^{-1}(P')$ is an ordering such that $\overline{P} = P'$. Thus an ordering P of level n on A can be thought of as a pair $(\mathfrak{p}, \overline{P})$, where $\mathfrak{p} \in \mathrm{Spec}\,(A)$ and $\overline{P}$ is an ordering of level n of the multifield $k(\mathfrak{p})$.

2. If P has level n, then P has level $n+r$, for $r \geq 0$. This follows immediately from the fact that $\overline{P}$ is an ordering of level $n+r$, for $r \geq 0$.

3. What is called an ordering in Section 4 of [5], will be called an *ordinary ordering* here. Note that an ordinary ordering is an ordering of level 1 – this follows readily from the fact that the ordinary ordering $\overline{P}$ is an ordering of level 1.

A subset $T \subset A$ is called a *preordering of level* n if $T+T \subset T$, $TT \subset T$, and $A^{2^n} \subset T$. Obviously, every ordering is a preordering. If $A^{2^m} \subset T$, but $A^{2^{m-1}} \not\subset T$, we say T has *exact level* m.

4. If T is a preordering of exact level m that is also an ordering, then, as an ordering, T has exact level m.

As in the multifield case, a preordering T will be called *proper* if $-1 \notin T$. Just like in the multifield case, we shall now show how proper preorderings can be extended to orderings, and prove a theorem analogous to Theorem 1. As an application, we will prove a theorem analogous to Theorem 2 that in the multiring case is usually called a Positivstellensatz. In doing so we will utilize the notion of $T-$modules – this essentially follows the method presented in [3], although the results in the ring case can be also found in [2] and [6]. For a preordering T of level n, a $T-$*module* is a subset $M \subset A$ such that $M+M \subset M$, $T \cdot M \subset M$, and $1 \in M$. If, in addition, $-1 \notin M$, we call M a *proper* $T-$module.

5. If $I \subsetneq A$ is an ideal in A, and M is a $T-$module over a proper preordering T, denote by M/I the image of M in the factor multiring A/I via the canonical projection $A \to A/I$. M/I is a proper $T/I-$module if and only if $(1+M) \cap I = \emptyset$, in which case $M+I$ is a proper $T-$module.

If M is a proper $T-$module over a proper preordering, and I is an ideal of A, we call I $M-$*convex* if, for $m, n \in M$, $(m+n) \cap I \neq \emptyset$ implies $m, n \in I$. This roughly corresponds to the notion of $M-$compatibility in the ring case defined in [6].

6. Let M be a proper $T-$module over a proper preordering, and $I \subsetneq A$ an ideal. Then the following conditions are equivalent:
(a) I is $M-$convex,
(b) $M+I$ is a proper $T-$module with $(M+I) \cap -(M+I) = I$,
(c) M/I is a proper $T/I-$module with $M/I \cap -M/I = \{0\}$.

Proof. (b) $\Leftrightarrow$ (c) is clear. To show (a) $\Rightarrow$ (b), assume I is $M-$convex. Then $(1+M) \cap I = \emptyset$, for if $1+m \in I$, for some $m \in M$, since I is $M-$convex,

$1 \in I$, contrary to $I \subsetneq A$. Thus $-1 \notin M + I$, for if $-1 \in m + a$, for $m \in M$ and $a \in I$, then $-a \in 1 + m$ and $-a \in AI \subset I$ – therefore $M + I$ is a proper $T-$module. Clearly, $I \subset (M + I) \cap -(M + I)$, and for the other inclusion fix an $a \in (M + I) \cap -(M + I)$. Let $a \in m_1 + b_1$ and $a \in -m_2 - b_2$, for $m_1, m_2 \in M$, $b_1, b_2 \in I$. Hence $0 \in a - a \subset m_1 + b_1 + m_2 + b_2$, and, consequently, $m_1 + m_2 \cap -b_1 - b_2 \neq \emptyset$, and since I is $M-$convex, $m_1, m_2 \in I$, implying $a \in I$.

Conversely, assume (b), and let $(m_1 + m_2) \cap I \neq \emptyset$ with $m_1, m_2 \in M$. Then, for some $a \in I$, $a \in m_1 + m_2$, so that $-m_1 \in m_2 - a \subset M + I$, and $-m_2 \in m_1 - a \subset M + I$. This implies $m_1, m_2 \in (M + I) \cap -(M + I) = I$. □

7. Let M be a proper $T-$module over a proper preordering T such that $A = T - T$, let $I \subset A$ be an ideal generated by $M \cap -M$. Then $I \subsetneq A$ and I is $M-$convex.

Proof. Clearly, $I = \Sigma A(M \cap -M)$. Observe that, since $A = T - T$, $I \subset -M$. In particular, since M is proper, $1 \notin I$, so I is proper itself. To show that I is $M-$convex, take $m, n \in M$ with $(m+n) \cap I \neq \emptyset$. Consequently, $(m+n) \cap -M \neq \emptyset$, which readily implies $-m, -n \in M$, and thus $m, n \in I$. □

8. Let M be a proper $T-$module over a proper preordering T such that $A = T-T$, let $\mathfrak{p} \subset A$ an ideal maximal subject to the condition $(1+M) \cap \mathfrak{p} = \emptyset$. Then $\mathfrak{p}$ is prime and $M-$convex.

Proof. Clearly, $M+\mathfrak{p}$ is a proper $T-$module. Let $\mathfrak{q}$ be the ideal generated by $(M + \mathfrak{p}) \cap -(M + \mathfrak{p})$. By 7, $\mathfrak{q}$ is a proper $(M + \mathfrak{p})-$convex ideal, so, by 6, $(M + \mathfrak{p})/\mathfrak{q}$ is a proper $T/\mathfrak{q}-$module, and, by 5, $(1 + (M + \mathfrak{p})) \cap \mathfrak{q} = \emptyset$, and, consequently, $(1 + M) \cap \mathfrak{q} = \emptyset$. Due to maximality of $\mathfrak{p}$, $\mathfrak{p} = \mathfrak{q}$, so $\mathfrak{p}$ is $(M + \mathfrak{p})-$convex and, in particular, $M-$convex.

It remains to show that $\mathfrak{p}$ is prime. Fix $ab \in \mathfrak{p}$ with $b \notin \mathfrak{p}$. $\mathfrak{p}$ is maximal subject to the condition $(1+M) \cap \mathfrak{p} = \emptyset$, so, since $b \notin \mathfrak{p}$, there exists $c \in A$ with $c \in 1 + m$ and $c \in a_1 b + \ldots + a_k b + p$, for $m \in M$, $a_1, \ldots, a_k \in A$, and $p \in \mathfrak{p}$. Then $ca^{2^n} \in a_1 a^{2^n - 1} ab + \ldots + a_k a^{2^n - 1} ab + a^{2^n} p \subset \mathfrak{p}$, and $ca^{2^n} \in a^{2^n} + ma^{2^n}$, with $a^{2^n}, ma^{2^n} \in M$. In particular, $(a^{2^n} + ma^{2^n}) \cap \mathfrak{p} \neq \emptyset$, so $a^{2^n} \in \mathfrak{p}$, and it remains to show that if $a^2 \in \mathfrak{p}$, then $a \in \mathfrak{p}$.

Suppose there is an $a \in A$ with $a^2 \in \mathfrak{p}$ and $a \notin \mathfrak{p}$. Replacing A by $A/\mathfrak{p}$, T by $(T + \mathfrak{p})/\mathfrak{p}$, and M by $M/\mathfrak{p}$, we may assume $\mathfrak{p} = (0)$, and, consequently, $a^2 = 0$. By the definition of $\mathfrak{p}$, there exists $d \in A$ with $d \in 1 + m$ and $d \in b_1 a + \ldots b_l a$, for $m \in M$, and $b_1, \ldots, b_l \in A$. Let k be the smallest integer $k \geq 1$ such that $m^{2^k} \in M$. Since $m^{2^n} \in M$, the integer k is well-defined. We claim that there exists $d' \in A$ with $d' \in 1 + m'$ and $d' \in b'_1 a + \ldots b'_{l'} a$, for $m' \in M$, and $b'_1, \ldots, b'_{l'} \in A$ such that $m'^{2^{k-1}} \in M$.

Indeed, observe that $d^2 \in 1 + m + m + m^2$. Let $m' = -m^2$. Note that $d^2 \in \Sigma b_i b_j a^2 = \{0\}$, that is $d^2 = 0$. Then $m' = -m^2 \in 1 + m + m \subset M$, and, obviously, $m'^{2^{k-1}} \in M$. Moreover, since $m \in d - 1$, we see that $-m' = m^2 \in d^2 - d - d + 1 = 1 - d - d$. Thus there exists $d' \in d + d$ with $-m' \in 1 - d'$, that is $d' \in 1 + m'$, and clearly $d' \in b'_1 a + \ldots b'_{l'} a$, for some $b'_1, \ldots, b'_{l'} \in A$.

By induction, we eventually show that there exists $d' \in A$ with $d' \in 1 + m'$ and $d' \in b'_1 a + \ldots b'_{l'} a$, for $m' \in M$, and $b'_1, \ldots, b'_{l'} \in A$ such that $m'^2 \in M$. But then $d'^2 \in \mathfrak{p}$, and $d'^2 \in 1 + m' + m' + m'^2 \subset 1 + M$, contrary to the definition of $\mathfrak{p}$. □

Just as we defined the ordering $\overline{P}$, we now define the preordering $\overline{T}$ of the field $k(\mathfrak{p})$, and the $\overline{T}-$module $\overline{M}$:

$$\overline{T} = \bigcup \{a_1^{2^n} \bar{t}_1 + \ldots + a_k^{2^n} \bar{t}_k : a_1, \ldots, a_k \in k(\mathfrak{p}), \bar{t}_1, \ldots, \bar{t}_k \in T/\mathfrak{p}, k \in \mathbb{N}\},$$

$$\overline{M} = \bigcup \{a_1^{2^n} \overline{m}_1 + \ldots + a_k^{2^n} \overline{m}_k : a_1, \ldots, a_k \in k(\mathfrak{p}), \overline{m}_1, \ldots, \overline{m}_k \in M/\mathfrak{p}, k \in \mathbb{N}\}.$$

9. Let M be a proper $T-$module over a proper preordering T such that $A = T - T$, let $\mathfrak{p} \subset A$ be a prime ideal with $(1 + M) \cap \mathfrak{p} = \emptyset$. Then $\overline{M}$ is proper.

Proof. Let $\mathfrak{p}$ be as desired. Observe that $\overline{T} - \overline{T} = k(\mathfrak{p})$. Indeed, note that $2^n! \cap \mathfrak{p} = \emptyset$, since $(1 + M) \cap \mathfrak{p} = \emptyset$ and $2^n! - 1 \subset M$. Moreover, $2^n! \subset k(\mathfrak{p})^*$, and, in fact, $2^n! \subset \overline{T}^*$, so that if we fix $a \in k(\mathfrak{p})$, then $2^n! a \subset \overline{T} - \overline{T}$, which implies $a \in \overline{T} - \overline{T}$.

$\overline{M}$ is clearly a $\overline{T}-$module, and to show it is proper, it suffices to note that $\overline{M} \cap -\overline{M}$ is an ideal. This follows from the fact that $\overline{T}(\overline{M} \cap -\overline{M}) \subset \overline{M} \cap -\overline{M}$, and that $\overline{T} - \overline{T} = k(\mathfrak{p})$. □

10. Let M be a proper $T-$module over a proper preordering T such that $A = T - T$. Then there exists a $M-$convex $\mathfrak{p} \in \mathrm{Spec}\,(A)$ such that $\overline{M}$ is proper. This follows immediately from 8 and 9, since $(1 + M) \cap (0) = \emptyset$.

11. Let T be a proper preordering such that $A = T - T$. Then there exists a $\mathfrak{p} \in \mathrm{Spec}\,(A)$ such that $\overline{T}$ is proper. This follows from 10 and the fact that T is a proper $T-$module itself.

12. A preordering P of level n maximal subject to the condition that P is proper and $A = P - P$ is an ordering of level n.

Proof. By 11, there exists $\mathfrak{p} \in \mathrm{Spec}\,(A)$ with $\overline{P}$ proper. Now, by Theorem 1, there is an ordering P' in $k(\mathfrak{p})$ such that $\overline{P} \subset P'$. If $\pi : A \to A/\mathfrak{p} \to k(\mathfrak{p})$ is the canonical homomorphism, then $\pi^{-1}(P')$ is an ordering with $P \subset \pi^{-1}(P')$, and, by maximality of P, actually $P = \pi^{-1}(P')$. □

By summing up what we have proven so far, we obtain a result similar to Theorem 1. As before, we call A *formally $n-$real* if $-1 \notin \Sigma A^{2^n}$.

Theorem 4. *Let A be a multiring. The following conditions are equivalent:*
(1) *A is formally $n-$real with $A = \Sigma A^{2^n} - \Sigma A^{2^n}$,*
(2) *A admits an ordering of level n P such that $A = P - P$,*
(3) *A admits a preordering of level n T such that $A = T - T$.*

Just like in the multifield case, we denote by X_T the set of all orderings containing a preordering T. The following theorem, called the Positivstellensatz, is a result analogous to Theorem 2:

Theorem 5. *Let A be a multiring, $T \subset A$ a preordering of level n. If T is proper and such that $A = T - T$, then the following conditions are equivalent:*
(1) $a \in \bigcap_{P \in X_T} P$,
(2) *$at \in a^{2^{nk}} + t'$, for some $t, t' \in T$, $k \in \mathbb{N}$.*

We shall precede the proof with two lemmas.

Lemma 6. *Let A be a multiring, let T be a proper preordering such that $A = T - T$, and let $a \in P$, for all $P \in X_T$. If $M = T - aT$ is a proper $T-$module, then there exists an $M-$convex $\mathfrak{p} \in \operatorname{Spec}(A)$ with $a \in \mathfrak{p}$.*

Proof. Assume $M = T - aT$ is a proper $T-$module. By 8, the ideal $\mathfrak{p}$ maximal subject to the condition $(1 + M) \cap \mathfrak{p} = \emptyset$ is prime and $M-$convex, so it suffices to show that $a \in \mathfrak{p}$. By 9, $\overline{T}$ is proper, and, by Theorem 1, $\overline{T} = \bigcap_{P' \in X_{\overline{T}}} P'$. In particular, $\overline{a} = a + \mathfrak{p} \in \overline{T}$. Since $\overline{M}$ is a $\overline{T}-$module, this implies $\overline{a} \in \overline{M} \cap -\overline{M} = (0)$, by 7. Thus $a \in \mathfrak{p}$. □

Lemma 7. *Let A be a multiring, let T be a proper preordering such that $A = T - T$, and let $a \in A^*$ be a unit. If $a \in \bigcap_{P \in X_T} P$, then $at \in 1 + t'$, for some $t, t' \in T$.*

Proof. Since $a \in A^*$, a is not contained in any ideal of A, and thus, by Lemma 6, $M = T - aT$ is not a proper $T-$module, so that $(1+T) \cap aT \neq \emptyset$. □

We now proceed to the proof of Theorem 5.

Proof of Theorem 5. (2) $\Rightarrow$ (1) : Assume (2) and suppose $a \notin P$, for some $P \in X_T$. Identify P with the pair $(\mathfrak{p}, \overline{P})$. Then $\overline{a}^{2^{nk}} + \overline{t}' \subset \overline{P}^*$, hence $\overline{a}\overline{t} \in \overline{P}^*$, and $\overline{a} \in \overline{P}$, so that $a \in P$ – a contradiction.

$(1) \Rightarrow (2)$: Assume (1). Firstly, suppose a is nilpotent. Then $a^{2^{n\ell}+1} = 0$, for some $\ell \in \mathbb{N}$. Thus, since $a \cdot a^{2^{n\ell}} = 0 = a^{2^{n2\ell}} \in a^{2^{n2\ell}} + 0$, and $a^{2^{n\ell}}, 0 \in T$, and we are done.

Secondly, suppose a is non-nilpotent. Let A_a denote the localization of A with respect to the multiplicative set $\{1, a, a^2, a^3, \ldots\}$, and let $T_a = \{\frac{t}{a^{2^{n\ell}}} : t \in T, \ell \in \mathbb{N}\}$. Clearly, $T_a + T_a \subset T_a$, $T_a \cdot T_a \subset T_a$, $A_a^{2^n} \subset T_a$, and $A_a = T_a - T_a$. If $-1 \in T_a$, then $0 \in a^{2^{n\ell}} + t$, for some $\ell \in \mathbb{N}$, $t \in T$. If $-1 \notin T_a$, then, by Theorem 4, the set X_{T_a} of orderings of A_a is nonempty. For a fixed ordering $P' \in X_{T_a}$, consider the set $P = \{b \in A : \frac{b}{1} \in P'\}$. Again, readily $P + P \subset P$, $PP \subset P$, and $A^{2^n} \subset P$. $P \cap -P = \mathfrak{p}$ is the contraction of the ideal $P' \cap -P' = \mathfrak{p}'$, and $\overline{P} = \overline{P'}$ in the multifield $k(\mathfrak{p}) = k(\mathfrak{p}')$. Since $a \in \bigcap_{P \in X_T} P$, it follows that $\frac{a}{1} \in \bigcap_{P' \in X_{T_a}} P'$. Moreover, as a is non-nilpotent, $\frac{a}{1} \in A_a^*$. By Lemma 7, we find $t, t' \in T_a$ with $\frac{a}{1} t \in 1 + t'$. Pulling this back to A finishes the proof. $\square$

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